NSF-CBMS Conference
The Cahn–Hilliard equation: recent advances and applications

Expanded account of the lectures

Lecture 1: The Cahn–Hilliard equation

In this lecture, we introduce the equation and its (phenomenological) derivation.

The Cahn–Hilliard equation was proposed by J.W. Cahn (who died on March 14, 2016) and J.E. Hilliard in order to describe phase separation processes in binary alloys. More precisely, when an homogeneous binary alloy is cooled down sufficiently, one can observe a total phase separation (called spinodal decomposition) and one obtains a fine-grained structure. This, in turn, can have essential consequences on the mechanical properties of the system, e.g., strength or aging.

This equation is particularly popular among material scientists, engineers and physicists. Indeed, it is simple to state (it is a fourth-order in space parabolic equation), easy to implement numerically and gives very good and precise simulations (in 2 and even 3D). It reads

$$\frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) = 0,\ \alpha,\ \kappa > 0.$$

Here, one usually considers a cubic nonlinear term,

$$f(s) = s^3 - s,$$

but such a nonlinear term is an approximation of thermodynamically relevant logarithmic nonlinear terms of the form

$$f(s) = -\theta_c s + \frac{\theta}{2}\ln\frac{1+s}{1-s},\ 0 < \theta < \theta_c,\ s \in (-1,1).$$

From a mathematical point of view, a lot has been done (from a theoretical and also a numerical/simulation point of view); it suffices to type Cahn on mathscinet to see how popular this equation was and still is. Nevertheless, the equation still is very much studied (one can again see mathscinet) and indeed a lot is still to be done.

We also discuss in details the boundary conditions.

In general, one considers Neumann boundary conditions, namely,

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma,$$

where Γ is the boundary of the domain Ω occupied by the material and ν is the unit outer normal vector to the boundary. One can also consider periodic boundary conditions, but not Dirichlet boundary conditions. Indeed, one important property of the Cahn–Hilliard equation is the conservation of mass, i.e., of the spatial average of the order parameter.

In particular, we can note that the usual Neumann boundary conditions yield a contact angle (when the interface between the two components meets the boundary/walls (e.g., in a confined system)) of $\frac{\pi}{2}$. Now, in several situations (e.g., for the study of immiscible binary mixtures), it is necessary to have a contact angle which moves from its equilibrium state. Hence the necessity to introduce dynamic boundary conditions. One can consider several approaches to define such boundary conditions, based on energy principles and (different versions of) mass conservation. Two types of dynamic boundary conditions read

$$\frac{1}{d}\frac{\partial u}{\partial t} - \alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma,\ d > 0,$$

$$\frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma,$$

where

$$\mu = -\alpha \Delta u + f(u)$$

is the chemical potential, and

$$\mu = -\alpha_\Gamma \Delta_\Gamma u + g(u) + \alpha \frac{\partial u}{\partial \nu} \text{ on } \Gamma,\ \alpha_\Gamma \geq 0,$$

$$\frac{\partial u}{\partial t} = \beta_\Gamma \Delta_\Gamma \mu - \kappa \frac{\partial \mu}{\partial \nu} \text{ on } \Gamma,\ \beta_\Gamma \geq 0.$$

Here, Δ_Γ is the Laplace–Beltrami operator. Note that the first set of boundary conditions again yields the conservation of mass in the bulk, while the second one yields a total mass conservation, in the bulk and on the boundary.

It is also interesting and important to note that the Cahn–Hilliard equation, or some of its variants, has applications in other areas/contexts, in which phase separation and/or coarsening/clustering processes can be observed or come into play. We can mention, for instance, population dynamics, bacterial films, wound healing and tumor growth, thin films, image processing (image denoising and inpainting in particular) and even the rings of Saturn and the clustering of mussels. In particular, several interesting variants of the equation (with applications, e.g., to tumor growth and image processing) can be written in the form

$$\frac{\partial u}{\partial t} + \alpha\kappa \Delta^2 u - \kappa \Delta f(u) + g(x, u) = 0.$$

Essential (mathematical) difficulties arise from the fact that, when endowing the equation with Neumann boundary conditions, one no longer has the conservation of the average of the order parameter, contrary to the original Cahn–Hilliard equation.

We finally introduce several important variants and generalizations of the Cahn–Hilliard equation. We mention in particular:

• Higher-order Cahn–Hilliard equations,

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{M} (-1)^i \sum_{|k|=i} a_k \mathcal{D}^{2k} u - \Delta f(u) = 0,$$

where (we consider here the case $n = 3$), for $k = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$,

$$|k| = k_1 + k_2 + k_3$$

and

$$\mathcal{D}^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}.$$

Such equations account for anisotropic interfaces.

• The viscous Cahn–Hilliard equation,

$$-\beta \Delta \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) = 0, \ \beta > 0,$$

proposed by A. Novick–Cohen to account for viscosity effects in the phase separation of polymer/polymer systems.

• The hyperbolic Cahn–Hilliard equation,

$$\beta \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \alpha\kappa\Delta^2 u - \kappa\Delta f(u) = 0, \ \beta > 0,$$

proposed to model the early stages of spinodal decomposition in certain glasses.

• The Cahn–Hilliard–Navier–Stokes equations,

$$\frac{\partial u}{\partial t} - \xi \mathrm{div} D(u) + (u \cdot \nabla) u + \nabla p = \varepsilon \mu \nabla \rho,$$

$$\mathrm{div} u = 0,$$

$$\frac{\partial \rho}{\partial t} + (u \cdot \nabla)\rho = \kappa \Delta \mu,$$

$$\mu = -\varepsilon \Delta \rho + \frac{1}{\varepsilon} f(\rho).$$

Here, $u = (u_1, u_2)$ or $u = (u_1, u_2, u_3)$ is the velocity of the mixture, $D(u) = \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the deformation tensor, p is the pressure, ρ is the order parameter, defining the phase/fluid (it takes the value 1 in one of the fluids, the value -1 in the other one and varies continuously in the (diffuse) interface between the two fluids) and μ is the chemical potential. Furthermore, $\xi > 0$ is the kinematic viscosity of the mixture, $\kappa > 0$ is the mobility of the mixture (we assume here that ξ and κ are constants), $\varepsilon > 0$ is related to the thickness of the interface and the nonlinear term f is the derivative of a double-well potential F. Finally, the term $\varepsilon\mu\nabla\rho$ is called the Korteweg force.

These equations model mixtures of immiscible fluids.

Two major issues are uniqueness (for logarithmic nonlinear terms) and proper boundary conditions. In particular, uniqueness has only been proved recently for weak solutions in 2D. Furthermore, the usual no slip boundary condition is not realistic and one needs to define proper dynamic boundary conditions. Possible boundary conditions are the following:

$$u \cdot \nu = 0,\ \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$\xi(D(u) \cdot \nu)_\Gamma + \beta u_\Gamma = \mathcal{L}(\rho)\nabla_\Gamma \rho \text{ on } \Gamma,$$
$$\frac{\partial \rho}{\partial t} + u_\Gamma \cdot \nabla_\Gamma \rho = -l_0 \mathcal{L}(\rho) \text{ on } \Gamma,$$

where

$$\mathcal{L}(\rho) = -\gamma\Delta_\Gamma \rho + \varepsilon\frac{\partial \rho}{\partial \nu} + \zeta\rho + g(\rho).$$

Here, l_0, β, ζ and $\gamma > 0$ are four phenomenological parameters, β being a slip coefficient. Furthermore, for any vector $v : \Gamma \to \mathbb{R}^n$, $v_\nu = v \cdot \nu$ is the normal component of the vector field, while $v_\Gamma = v - (v_\nu)\nu$ is the tangential component of v. Finally, the function g is a nonlinear function of the local composition which accounts for the interfacial energy at the mixture-wall interface.

Note that, while the existence of weak solutions are known, both for regular and logarithmic nonlinear terms, uniqueness and additional regularity are still open problems.

Lectures 2 and 3: The Cahn–Hilliard equation with regular nonlinear terms

In these lecture, we discuss the mathematical analysis of the equation. Here, we consider regular nonlinear terms (a typical choice being the usual cubic nonlinear term $f(s) = s^3 - s$) and standard (Neumann) boundary conditions.

We first present the functional framework (linear operators, linear problem, functional form of the equation).

In particular, we define the operators $A = -\Delta$ and A^2, associated with Neumann boundary conditions and acting on functions with vanishing spatial average. We can note that, setting

$$\langle u \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u\, dx,\ \ u \in L^1(\Omega),$$
$$\langle u \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle u, 1 \rangle,\ \ u \in H^{-1}(\Omega),$$
$$H = \dot{L}^2(\Omega) = \{u \in L^2(\Omega),\ \langle u \rangle = 0\},$$
$$V = \dot{H}^1(\Omega) = H^1(\Omega) \cap H,$$

we can see that

$$V' = \{u \in H^{-1}(\Omega),\ \langle u \rangle = 0\},$$
$$D(A) = \{u \in H^2(\Omega) \cap V,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\},$$
$$D(A^2) = \{u \in H^4(\Omega) \cap V,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

Furthermore, we can see that $Au = f$, $u \in D(A)$ and $f \in H$, is equivalent to

$$-\Delta u = f \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

and $A^2 u = f$, $u \in D(A^2)$ and $f \in H$, is equivalent to

$$\Delta^2 u = f \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma.$$

This allows to write the Cahn–Hilliard equation in functional form, namely,

$$\frac{d\overline{u}}{dt} + A^2 \overline{u} + A\overline{f(u)} = 0,$$

where $\overline{v} = v - \langle v \rangle$. We also consider the weaker formulation

$$A^{-1} \frac{d\overline{u}}{dt} + A\overline{u} + \overline{f(u)} = 0.$$

We then discuss the well-posedness and regularity of solutions, as well as the asymptotic behavior of the system (existence of global attractors).

As far as the well-posedness and regularity are concerned, we prove the following results, for the cubic nonlinear term.

Theorem 1. *We assume that* $u_0 \in H^1(\Omega)$*, i.e.,* $\overline{u}_0 \in V$*. Then, the problem possesses a unique weak solution* $u = \overline{u} + \kappa$ *such that* $\overline{u} \in L^\infty(\mathbb{R}^+; V) \cap \mathcal{C}([0,T]; V) \cap L^2(0,T; D(A^{\frac{3}{2}}))$ *and* $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}^+; V')$*,* $\forall T > 0$*.*

Theorem 2. *We assume that* $u_0 \in H^2(\Omega)$*, with* $\frac{\partial u_0}{\partial \nu} = 0$ *on* Γ*, i.e.,* $\overline{u}_0 \in D(A)$*. Then, the solution* $u = \overline{u} + \kappa$ *satisfies* $\overline{u} \in \mathcal{C}([0,T]; D(A)) \cap L^2(0,T; D(A^2))$ *and* $\frac{\partial u}{\partial t} \in L^2(0,T;H)$*,* $\forall T > 0$*. Furthermore, it is a strong solution, i.e.,*

$$\frac{d\overline{u}}{dt} + A^2\overline{u} + A\overline{f(u)} = 0 \text{ in } L^2(0,T;H).$$

We then prove the existence of the global attractor which is the unique compact subset of the phase space which is invariant and attracts all bounded subsets of initial data as time goes to infinity; as it is the smallest closed set enjoying these properties, it appears as a suitable object in view of the study of the large time behavior of the system.

Theorem 3. *The semigroup* $S(t)$ *associated with the problem possesses the global attractor* $\mathcal{A}_0$ *on* V *which is bounded in* $H^2(\Omega)$*.*

Here, the spatial average is set, for simplicity, equal to 0. Furthermore, the global attractor has finite fractal dimension, meaning, roughly speaking, that, even though the phase space is infinite-dimensional, the asymptotic behavior of the system can be described by a finite number of parameters.

Theorem 4. *The global attractor* $\mathcal{A}_0$ *has finite fractal dimension for the topology of* V'*.*

We finally discuss the viscous Cahn–Hilliard equation for which one has higher regularity in time, allowing to employ the comparison principle for second-order in space parabolic equations.

In the second part, we give an improved regularity result, allowing to address polynomials with arbitrary odd degree (with a strictly positive leading coefficient) in 3D.

More precisely, we make the following assumptions on f:

$$f \text{ is of class } \mathcal{C}^2,\ f(0) = 0,$$
$$f'(s) \geq -c_0,\ c_0 \geq 0,\ s \in \mathbb{R},$$
$$f(s)s \geq c_1 F(s) - c_2,\ F(s) \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R},$$
$$|f(s)| \leq \epsilon F(s) + c_\varepsilon,\ \forall \varepsilon > 0,\ s \in \mathbb{R},$$

where $F(s) = \int_0^s f(\xi)\, d\xi$.

In particular, these assumptions are satisfied by polynomials of degree $2p+1$ with a positive leading coefficient, $p \geq 1$ (and, of course, by the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, the last assumption says that the growth of f cannot be exponential at infinity. Actually, it is not needed when the spatial average of the order parameter κ vanishes; it is however needed when $\kappa \neq 0$ in order to handle the spatial average of $f(u)$.

We have the following result.

Theorem 5. *We assume that $u_0 \in H^2(\Omega)$, with $\frac{\partial u_0}{\partial \nu} = 0$ on Γ. Then, the problem possesses a unique solution u such that $u(t) \in H^2(\Omega)$, $\forall t \geq 0$.*

This result is proved by a more careful treatment of the nonlinear term; we use here in an essential way the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$.

We next discuss the higher-order Cahn–Hilliard models introduced in the first lecture. The strategy essentially follows the above one, with simplifications due to the fact that $H^k(\Omega) \subset \mathcal{C}(\overline{\Omega})$ for $k \geq 0$. Note however that we cannot consider here Neumann boundary conditions in general.

We finally discuss the numerical analysis of the equation. In particular, we stress that, in view of the numerical analysis of the problem, as well as the construction of efficient algorithms, it is more convenient to rewrite the equation as an equivalent system of two second-order in space equations which is easier to handle.

Lectures 4 and 5: The Cahn–Hilliard equation with logarithmic nonlinear terms

In these lectures, we address the case of logarithmic nonlinear terms. Indeed, the usual cubic/regular nonlinear terms are approximations of thermodynamically relevant logarithmic nonlinear terms which follow from a mean-field model. Such nonlinear terms induce additional mathematical questions and an essential issue (not completely solved in 3D) is the separation from the pure states/singular points of the nonlinear term, namely, an estimate of the form

$$\|u(t)\|_{L^\infty} \leq 1 - \delta, \ \delta \in (0,1),$$

at least after some transient time (δ possibly depending on a final time T).

We thus take

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \frac{1+s}{1-s}, \ 0 < \theta < \theta_c, \ s \in (-1,1),$$

and we set $c_0 = \theta_c$. This function satisfies

$$f' \geq -c_0,$$

$$f(s) = f_1(s) - c_0 s,\ f_1' \geq 0,$$
$$f(s)s \geq F(s) - c_1 \geq -c_2,\ c_1,\ c_2 \geq 0,$$
$$f(s)s \geq c_3|f(s)| - c_4,\ c_3 > 0,\ c_4 \geq 0,$$
$$f(s)(s-m) \geq F(s) - c_m,\ c_m \geq 0,$$
$$f(s)(s-m) \geq c_m|f(s)| - c_m',\ c_m > 0,\ c_m' \geq 0,$$

where the constants c_m and c_m' depend continuously on m.

Next, we define, for $N \in \mathbb{N}$, the approximated functions $f_N \in \mathcal{C}^1(\mathbb{R})$ as

$$f_N(s) = f(-1+\frac{1}{N}) + f'(-1+\frac{1}{N})(s+1-\frac{1}{N}),\ s < -1+\frac{1}{N},$$
$$f_N(s) = f(s),\ |s| \leq 1-\frac{1}{N},$$
$$f_N(s) = f(1-\frac{1}{N}) + f'(1-\frac{1}{N})(s-1+\frac{1}{N}),\ s > 1-\frac{1}{N}.$$

We easily see that f_N is odd and

$$f_N' \geq -c_0.$$

Furthermore, we can write $f_N(s) = f_{1,N}(s) - c_0 s$, where $f_{1,N}' \geq 0$.

The function f_N also enjoys the following properties, for N large enough:

$$f_N(s)s \geq F_N(s) - c_5 \geq -c_6,\ c_5,\ c_6 \geq 0,\ s \in \mathbb{R},$$
$$f_N(s)s \geq c_7|f_N(s)| - c_8,\ c_7 > 0,\ c_8 \geq 0,\ s \in \mathbb{R},$$

where the constants c_i, $i = 5,\ \cdots,\ 8$, are independent of N, and, more generally:

$$f_N(s)(s-m) \geq c_m(|f_N(s)| + F_N(s)) - c_m',$$
$$c_m > 0,\ c_m' \geq 0,\ s \in \mathbb{R},\ m \in (-1,1),$$

where the constants c_m and c_m' are independent of N and depend continuously on m.

We finally introduce the approximated problems

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) = 0,$$
$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$u_N|_{t=0} = u_0.$$

Noting that f_N has a linear growth at infinity, it is easy to adapt what is done in the regular case and prove the existence and uniqueness of the solutions to the approximated problems (note that we still have the conservation of the spatial average of the order parameter here).

We then obtain a priori estimates on the solutions u_N which are independent of N. This allows us to pass to the limit $N \to +\infty$ in the approximated problems. The crucial step, to prove the existence of a solution to the original problem, consists in deriving an a priori estimate independent of N on $f_N(u_N)$ in $L^2(\Omega \times (0,T))$, $T > 0$.

This allows to prove the following result.

Theorem 6. *We assume that u_0 is given such that $u_0 \in H^1(\Omega)$ and $-1 < u_0(x) < 1$, a.e. x. Then, the problem possesses a unique (weak) solution such that, $\forall T > 0$,*

$$u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)').$$

Furthermore, $-1 < u(x,t) < 1$, a.e. (x,t).

We thus see that such nonlinear terms force the order parameter to stay in the physically relevant interval; this does not hold in the case of regular nonlinear terms (we can give very simple counterexamples).

We can also prove the existence of global attractors; note however that the finite fractal dimensionality is a very delicate issue which necessitates sophisticated tools, due to the lack of strict separation for the pure states at this stage (such a strict separation does not hold in 3D though).

We next study the strict separation property from the pure states, as well as additional regularity on the solutions. More precisely, we have the following results.

Proposition 7. *The solution u satisfies*

$$\frac{\partial u}{\partial t} \in L^\infty(r, +\infty; H^{-1}(\Omega)) \cap L^2(r, T; H^1(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Proposition 8. *We assume that $2 \le p < +\infty$, when $n = 2$, and $2 \le p \le 6$, when $n = 3$. Then, the solution u further satisfies*

$$\|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \le c,$$
$$\|u(t)\|_{W^{2,p}(\Omega)} \le c,$$

$\forall t \ge r$, $r > 0$ given, where the constant c depends on the $H^1(\Omega)$-norm of u_0.

Proposition 9. *We assume that* $n = 1$*. Then, there exists* $\delta \in (0, 1)$ *depending on the* $H^1(\Omega)$*-norm of* u_0 *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le 1 - \delta,\ \ t \ge r,$$

$r > 0$ *given.*

Proposition 10. *We assume that* $n = 2$*. Then, the following holds for every* $t \ge r$*,* $r > 0$ *given, and for every* $p \in \mathbb{N}$*:*

$$\|f'(u)\|_{L^p(\Omega\times(r,t))} \le c,$$

where the constant c *depends on* p*.*

Proposition 11. *We assume that* $n = 2$*. Then, the weak solution* u *further satisfies*

$$\frac{\partial u}{\partial t} \in L^\infty(r, +\infty; H) \cap L^2(r, T; H^2(\Omega)),$$

$\forall r < T$*,* $r > 0$ *and* $T > 0$ *given.*

Theorem 12. *We assume that* $n = 2$*. Then, there exists* $\delta \in (0, 1)$ *depending on the* $H^1(\Omega)$*-norm of* u_0 *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le 1 - \delta,\ \ t \ge r,$$

$r > 0$ *given.*

We again mention that the strict separation is not known in three space dimensions. Furthermore, the strict separation allows to prove in a simple way the finite-dimensionality of the global attractors.

In the second part, we discuss the viscous Cahn–Hilliard equation. In particular, for this equation, one has the strict separation from the pure states, even in 3D. This is due to the fact that we can now use the comparison principle for second-order in space parabolic equations, owing to a higher regularity in time of the solutions.

We finally consider higher-order Cahn–Hilliard models. In that case however, we are not able to derive a uniform estimate on $f_N(u_N)$ and thus cannot pass to the limit in the approximated problems, meaning that we are not able to prove the existence of a classical solution. We can however prove the existence of weaker solutions, based on variational inequalities.

Lecture 6: The Cahn–Hilliard equation with dynamic boundary conditions

In this lecture, we address the Cahn-Hilliard equation with dynamic boundary conditions. As already mentioned, such boundary conditions are important to account for the interactions with the walls, in particular, to account for dynamic contact angles.

We start by the study of the following problem:

$$\frac{\partial u}{\partial t} = \Delta\mu,$$
$$\mu = -\Delta u + f(u),$$
$$\frac{\partial u}{\partial t} = \Delta_\Gamma \mu - \frac{\partial \mu}{\partial \nu} \text{ on } \Gamma,$$
$$\mu = -\Delta_\Gamma u + \frac{\partial u}{\partial \nu} + g(u) \text{ on } \Gamma,$$
$$u|_{t=0} = u_0,$$

where Δ_Γ denotes the Laplace–Beltrami operator. Note that we have the total (in the bulk and on the boundary) mass conservation

$$\frac{d}{dt}(\int_\Omega u\,dx \int_\Gamma u\,d\Sigma) = 0.$$

We assume that f is the usual cubic nonlinear term, $f(s) = s^3 - s$, and that g is affine, $g(s) = as + b$, $a > 0$.

We start by introducing proper linear operators. To do so, we set

$$H = L^2(\Omega),\ H_\Gamma = L^2(\Gamma),\ \mathcal{H} = H \times H_\Gamma,$$
$$V = H^1(\Omega),\ V_\Gamma = H^1(\Gamma)$$
$$\mathcal{V} = \{\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in V \times V_\Gamma,\ \varphi|_\Gamma = \psi\},$$
$$\dot{\mathcal{H}} = \{\phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathcal{H},\ \langle\phi\rangle = 0\},$$
$$\dot{\mathcal{V}} = \mathcal{V} \cap \dot{\mathcal{H}},$$
$$\langle\phi\rangle = \frac{1}{\mathrm{Vol}(\Omega) + |\Gamma|}(\int_\Omega \varphi\,dx + \int_\Gamma \psi\,d\Sigma).$$

We then introduce the linear operator

$$A\phi = \begin{pmatrix} -\Delta\varphi \\ -\Delta_\Gamma\varphi + \frac{\partial\varphi}{\partial\nu} \end{pmatrix},\ \phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}.$$

We can prove that $D(A) = \dot{\mathcal{V}} \cap (H^2(\Omega) \times H^2(\Gamma))$ and the norm $\|A\cdot\|_{\mathcal{H}}$ is equivalent to the usual $H^2(\Omega) \times H^2(\Gamma)$-one on $D(A)$. Furthermore, for $k \in \mathbb{N}$, the embedding $D(A^k) \subset H^{2k}(\Omega) \times H^{2k}(\Gamma)$ is continuous. Furthermore, the norm $\|A^k\cdot\|_{\mathcal{H}}$ is equivalent to the usual $H^{2k}(\Omega) \times H^{2k}(\Gamma)$-one on $D(A^k)$. Note that we can also consider the operator A as an operator from $\mathcal{V}$ onto $\mathcal{V}'$.

Having this, we can rewrite the equations in the following functional form:

$$\frac{dU}{dt} = -AW,$$
$$W = AU + \mathcal{F}(U),$$
$$U|_{t=0} = U_0 \text{ in } \mathcal{V}',$$

for $T > 0$ given, where $U = \begin{pmatrix} u \\ u \end{pmatrix}$, $U_0 = \begin{pmatrix} u_0 \\ u_0 \end{pmatrix}$, $W = \begin{pmatrix} \mu \\ \mu \end{pmatrix}$ and $\mathcal{F}(U) = \begin{pmatrix} f(u) \\ g(u) \end{pmatrix}$. We aslo set

$$\overline{\phi} = \phi - \begin{pmatrix} \langle\phi\rangle \\ \langle\phi\rangle \end{pmatrix},$$

so that

$$A^{-1}\frac{dU}{dt} = -\overline{W},$$

where $\langle W\rangle = \langle \mathcal{F}(U)\rangle$. We also have the equivalent formulations

$$\frac{d\overline{U}}{dt} + A^2\overline{U} + A\overline{\mathcal{F}(U)} = 0$$

and

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \overline{\mathcal{F}(U)} = 0.$$

We can prove the following.

Theorem 13. *We assume that $U_0 \in \mathcal{V}$. Then, the problem possesses a unique weak solution $U = \overline{U} + \kappa$ such that $\overline{U} \in L^\infty(\mathbb{R}^+; \dot{\mathcal{V}}) \cap \mathcal{C}([0,T]; \dot{\mathcal{V}}') \cap L^2(0,T; D(A))$ and $\frac{\partial U}{\partial t} \in L^2(\mathbb{R}^+; \dot{\mathcal{V}}')$, $\forall T > 0$. Furthermore,*

$$A^{-1}\frac{d\overline{U}}{dt} + A\overline{U} + \overline{\mathcal{F}(U)} = 0 \text{ in } L^2(0,T; \dot{\mathcal{H}}).$$

Having this, we can obtain, by bootstrap arguments, additional regularity when $t > 0$. This allows to recover the original problem, for $t > 0$. We can also prove the existence of global attractors.

We then consider a logarithmic nonlinear term,

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \frac{1+s}{1-s},\ 0 < \theta < \theta_c,\ s \in (-1, 1).$$

In that case, we again approximate f as in the previous talk. We then consider the approximated problems

$$A^{-1} \frac{d\overline{U}_N}{dt} + A\overline{U}_N + \overline{\mathcal{F}_N(U_N)} = 0,$$
$$U_N(0) = U_0,$$

where

$$\mathcal{F}_N(U_N) = \begin{pmatrix} f_N(u_N) \\ g(u_N) \end{pmatrix}.$$

However, we are not able to derive uniform estimates on $f_N(u_N)$ and thus cannot pass to the limit in the nonlinear term. We can however prove the existence of a very weak solutions by duality arguments.

We can also introduce a weaker notion of a solution which satisfies the following variational inequality:

$$((A^{-1} \frac{d\overline{U}}{dt}, U - W))_{\mathcal{H}} + ((\overline{U}, U - W))_{\dot{\mathcal{V}}} + ((f_1(w), u - w))$$
$$\leq c_0((u, u - w)) - ((g(u), u - w))_\Gamma,$$

for almost every $t > 0$ and for every test function $W = W(x)$ such that $W \in \mathcal{V}$, $f(w) \in L^1(\Omega)$ and $\langle W \rangle = \langle W_0 \rangle$.

We have the following result.

Theorem 14. *We assume that $U_0 \in \mathcal{V}$. Then, the problem possesses at most one variational solution U such that $U(0) = U_0$. In particular, such a solution solution satisfies, $\forall T > 0$,*

(i) $U \in \mathcal{C}([0, T]; \mathcal{V}') \cap L^\infty(0, T; \mathcal{V})$.

(ii) $\frac{\partial U}{\partial t} \in L^2(0, T; \dot{\mathcal{V}}')$.

(iii) $f(u) \in L^1(\Omega \times (0, T))$.

(iv) $-1 < u(x, t) < 1,$ a.e. (x, t).

(v) $U(0) = U_0$.

(vi) $\langle U(t) \rangle = \langle U_0 \rangle,\ \forall t \geq 0$.

(vii) The variational inequality is satisfied for almost every $t > 0$ and for every test function $W = W(x)$ such that $W \in \mathcal{V}$, $f(w) \in L^1(\Omega)$ and $\langle W \rangle = \langle W_0 \rangle$.

Unfortunately, we are not able to prove the existence of a variational solution.

We also discuss the following dynamic boundary condition:

$$\frac{\partial u}{\partial t} - \Delta_\Gamma u + g(u) + \frac{\partial u}{\partial \nu} = 0 \text{ on}\Gamma,$$

together with

$$\frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma.$$

In the case of logarithmic nonlinear terms, we can now prove the existence and uniqueness of variational solutions.

Furthermore, we can be more precise and give sufficient conditions ensuring the existence of classical solutions. More precisely, we have the following.

Theorem 15. *We assume that*

$$\lim_{s \to \pm 1} F_1(s) = +\infty,$$

where F_1 is any antiderivative of f_1. Then, a variational solution is a classical one.

Note that this cannot hold for the logarithmic nonlinear terms.

Theorem 16. *We assume that*

$$\pm g(\pm 1) > 0.$$

Then, a variational solution is a classical one.

In particular, when the sign conditions are not satisfied, the order parameter can reach the pure states on parts of, or even on the whole, boundary. We show numerical simulations illustrating this.

Lecture 7: The Cahn–Hilliard–Oono equation

The Cahn-Hilliard-Oono equation,

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \beta u = 0, \ \beta > 0,$$

was introduced in order to account for long-ranged (i.e., nonlocal) effects in the phase separation process.

We show in this lecture that the additional simple linear term already leads to several additional difficulties, especially when considering logarithmic nonlinear terms.

First, in the case of the usual cubic nonlinear term $f(s) = s^3 - s$, we can prove the existence and uniqueness of solutions and the existence of the global attractor for the associated semigroup $S_\beta(t)$ on

$$\Phi_M = \{v \in L^2(\Omega),\ |\langle v\rangle| \le M\},\ M \ge 0.$$

Theorem 17. *The semigroup $S_\beta(t)$ possesses the finite-dimensional (for the $H^{-1}(\Omega)$-topology) global attractor $\mathcal{A}_\beta^M$ on the phase space Φ_M which is compact in $L^2(\Omega)$ and bounded in $H^1(\Omega)$.*

We also discuss the dynamics of the equation when β goes to 0. More precisely, we can construct robust, as β goes to 0^+, exponential attractors (an exponential attractor is a compact subset of the phase space which is only positively invariant by the flow, but has finite fractal dimension, contains the global attractor and attracts exponentially fast all bounded sets of initial data). Indeed, setting

$$\tilde{\Phi}_M = \{v \in H^{-1}(\Omega),\ |\langle v\rangle| \le M\},\ M \ge 0,$$

we have the following result.

Theorem 18. *For every $\beta \in [0,\beta_0]$, $\beta_0 > 0$ given, the semigroup $S_\beta(t)$ acting on $\tilde{\Phi}_M$ possesses an exponential attractor $\mathcal{M}_\beta^M$ on $\tilde{\Phi}_M$ such that*

1. The set $\mathcal{M}_\epsilon^M$ has finite fractal dimension in $H^{-1}(\Omega)$,

$$\dim_{\mathrm{F}} \mathcal{M}_\beta^M \le c.$$

2. The set $\mathcal{M}_\beta^M$ is positively invariant by $S_\beta(t)$,

$$S_\beta(t)\mathcal{M}_\beta^M \subset \mathcal{M}_\beta^M,\ t \ge 0.$$

3. The set $\mathcal{M}_\beta^M$ attracts all bounded subsets of $\tilde{\Phi}_M$ exponentially fast, i.e., for every bounded subset B of $\tilde{\Phi}_M$, there exists a constant $c = c(B)$ such that

$$\mathrm{dist}_{H^{-1}(\Omega)}(S_\beta(t)B, \mathcal{M}_\beta^M) \le ce^{-c't},\ t \ge 0,\ c' > 0.$$

4. The family of sets $\mathcal{M}_\beta^M$ is Hölder continuous at 0,

$$\mathrm{dist}_{\mathrm{sym}}(\mathcal{M}_\beta^M, \mathcal{M}_0^M) \le c\beta^{c'},\ c' \in (0,1).$$

Furthermore, all constants are independent of β and can be computed explicitly.

This shows that the dynamics of the Cahn–Hilliard–Oono equation is, in some proper sense, close to that of the limit Cahn–Hilliard equation when β is small.

In the case of logarithmic nonlinear terms, we recover the results obtained for the Cahn–Hilliard equation. As already mentioned, the proofs are however more involved. More precisely, we have the following results.

Theorem 19. *We assume that u_0 is given such that $u_0 \in H^1(\Omega)$ and $-1 < u_0(x) < 1$, a.e. x, with $|\langle u_0 \rangle| < 1$. Then, the problem possesses a unique (weak) solution such that, $\forall T > 0$,*

$$u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)'),$$

$$\mu \in L^2(0,T; H^1(\Omega)).$$

Furthermore, $-1 < u(x,t) < 1$, a.e. (x,t).

Proposition 20. *The solution u satisfies*

$$\frac{\partial u}{\partial t} \in L^\infty(r,+\infty; H^{-1}(\Omega)) \cap L^2(r,T; H^1(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Proposition 21. *We assume that $2 \leq p < +\infty$, when $n = 2$, and $2 \leq p \leq 6$, when $n = 3$. Then, the solution u further satisfies*

$$\|f(u)\|_{L^\infty(r,t;L^p(\Omega))} \leq c,$$
$$\|u(t)\|_{W^{2,p}(\Omega)} \leq c,$$

$\forall t \geq r$, $r > 0$ given, where the constant c depends on the $H^1(\Omega)$-norm of u_0.

Proposition 22. *We assume that $n = 1$. Then there exists $\delta \in (0,1)$ depending on the $H^1(\Omega)$-norm of u_0 such that*

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \ t \geq r,$$

$r > 0$ given.

Proposition 23. *We assume that $n = 2$. Then, the following holds for every $t \geq r$, $r > 0$ given, and for every $p \in \mathbb{N}$:*

$$\|f'(u)\|_{L^p(\Omega \times (r,t))} \leq c,$$

where the constant c depends on p.

Proposition 24. *We assume that $n = 2$. Then, the weak solution u further satisfies*

$$\frac{\partial u}{\partial t} \in L^\infty(r,+\infty; H) \cap L^2(r,T; H^2(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given.

Theorem 25. *We assume that $n = 2$. Then, there exists $\delta \in (0,1)$ depending on the $H^1(\Omega)$-norm of u_0 such that*

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1-\delta,\ t \geq r,$$

$r > 0$ *given.*

Lecture 8: The Cahn–Hilliard equation in image inpainting

In this lecture, we explain how the Cahn–Hilliard equation can be used in image restoration. We present in particular a Cahn–Hilliard model proposed by A. Bertozzi, S. Esedoglu and A. Gillette in view of binary (i.e., black and white) image inpainting which reads

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(x,u) = 0,$$

where

$$g(x,s) = \lambda_0 \chi_{\Omega\setminus D}(x)(s-h(x)),\ \lambda_0 > 0,\ D \subset \Omega,\ h \in L^2(\Omega),$$

χ denoting the indicator function. Here, h is the damaged image and D is the damaged region.

We then discuss the mathematical analysis of the models.

We first consider the usual cubic nonlinear term.

We have, in that case, the following.

Theorem 26. *We assume that $u_0 \in L^2(\Omega)$. Then, the problem possesses a unique weak solution u such that $u \in \mathcal{C}([0,T]; L^2(\Omega)_w) \cap L^2(0,T; H^2(\Omega))$, $\forall T > 0$.*

The main difficulty here is to deal with the equation for the spatial average of the order parameter which reads

$$\frac{d\langle u\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} u\, dx = 0.$$

To do so, we set

$$u = \langle u\rangle + v$$

and have (taking for simplicity $h = 0$ and $\lambda_0 = 1$)

$$\frac{d\langle u\rangle}{dt} + c_0\langle u\rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} v\, dx,\ c_0 = \frac{\mathrm{Vol}(\Omega\setminus D)}{\mathrm{Vol}(\Omega)},$$

where v is solution to

$$\frac{\partial}{\partial t}(-\Delta)^{-1}v - \Delta v + f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle$$
$$+(-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)u - \langle \chi_{\Omega\setminus D}(x)u\rangle) = 0.$$

We then prove that v is globally bounded and can thus obtain global in time and dissipative estimates. We also use in a crucial way the following coercivity relation:

$$((f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle, v))$$
$$= ((f(\langle u\rangle + v) - f(\langle u\rangle), v))$$
$$\geq \frac{c_0}{2}\int_\Omega (v^4 + v^2\langle u\rangle^2)\,dx - \|v\|^2.$$

We also have the following.

Theorem 27. *The semigroup $S(t)$ possesses the finite-dimensional (for the $H^{-1}(\Omega)$-topology) global attractor $\mathcal{A}$ such that $\mathcal{A}$ is compact in $L^2(\Omega)$ and bounded in $H^4(\Omega)$.*

An important open problem is the convergence of trajectories to steady states. Indeed, the final inpainting result is expected to be an equilibrium.

We then consider a logarithmic nonlinear term. Note that such nonlinear terms give better numerical results.

However, since we need the coercivity relation employed in the case of regular nonlinear terms, we need to approximate the logarithmic nonlinear term more carefully. More precisely, we write $F(s) = \frac{\theta_c}{2}(1 - s^2) + F_1(s)$ and set $f_1 = F_1' \geq 0$. We then introduce, for $N \in \mathbb{N}$, the approximated functions $F_{1,N} \in \mathcal{C}^4(\mathbb{R})$ defined as

$$F_{1,N}^{(4)}(s) = F_1^{(4)}(1 - \frac{1}{N}),\ s > 1 - \frac{1}{N},$$
$$F_{1,N}^{(4)}(s) = F_1^{(4)}(s),\ |s| \leq 1 - \frac{1}{N},$$
$$F_{1,N}^{(4)}(s) = F_1^{(4)}(-1 + \frac{1}{N}),\ s < -1 + \frac{1}{N},$$
$$F_{1,N}^{(k)}(0) = F_1^{(k)}(0),\ k = 0,\ 1,\ 2,\ 3,$$

so that

$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!}F_1^{(k)}(1 - \frac{1}{N})(s - 1 + \frac{1}{N})^k,\ s > 1 - \frac{1}{N},$$
$$F_{1,N}(s) = F_1(s),\ |s| \leq 1 - \frac{1}{N},$$
$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!}F_1^{(k)}(-1 + \frac{1}{N})(s + 1 - \frac{1}{N})^k,\ s < -1 + \frac{1}{N}.$$

Setting $F_N(s) = \frac{\theta_c}{2}(1 - s^2) + F_{1,N}(s)$, $f_{1,N} = F_{1,N}'$ and $f_N = F_N'$, we can prove that

$$f'_{1,N} \geq 0,\ f'_N \geq -c_0,\ c_0 = \theta_c,$$

$$F_N \geq -c_1,\ c_1 \geq 0,$$

$$f_N(s)(s-m) \geq c_2(m)(F_N(s) + |f_N(s)|) - c_3(m),$$

$$c_2 > 0,\ c_3 \geq 0,\ s \in \mathbb{R},\ m \in (-1,1),$$

where the constants c_i, $i = 1, 2, 3$, are independent of N, for N large enough.

We further have the following coercivity relation which is essential to derive proper a priori estimates.

Proposition 28. *The following holds for N large enough:*

$$(f_N(s+a) - f_N(a))s \geq c_4(s^4 + a^2 s^2) - c_5,\ c_4 > 0,\ c_5 \geq 0,\ s,\ a \in \mathbb{R},$$

where the constants c_4 and c_5 are independent of N.

In that case, we once more need to obtain a uniform estimate on $f_N(u_N)$. In particular, this necessitates to have a strict separation property on $\langle u_N \rangle$. However, due to the fact that we no longer have the conservation of mass, this can only be proved locally in time. This then yields the following.

Theorem 29. *We assume that $u_0 \in H^1(\Omega)$, $|\langle u_0 \rangle| < 1$ and $-1 < u_0(x) < 1$,* a.e. *$x \in \Omega$. Then, there exists $T_0 = T_0(u_0) > 0$ and a solution u to the problem on $[0, T_0]$ such that $u \in \mathcal{C}([0,T_0]; H^1(\Omega)_w) \cap L^2(0,T_0; H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T_0; H^{-1}(\Omega))$. Furthermore, $-1 < u(x,t) < 1$, a.e. $(x,t) \in \Omega \times (0,T_0)$.*

Having this, we can now rewrite the equation in the following equivalent form (we again take $h = 0$ and $\lambda_0 = 1$):

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + u - \chi_D(x)u = 0$$

and prove the

Theorem 30. *The solution u is global in time.*

We note that uniqueness and additional regularity are open problems.

We also discuss several extensions of this model (for multicolor images and grayscale images).

We finally discuss the numerical analysis and present several numerical simulations which show that such models are efficient.

Lecture 9: The Cahn–Hilliard equation with a proliferation term

In this lecture, we introduce variants of the Cahn–Hilliard equation in wiew of biological and medical applications. More precisely, we consider the generalized equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(u) = 0,$$

where g is a polynomial; typically, $g(s) = s^2 - s$. This equation models wound healing and tumor growth in 1D and the clustering of malignant brain tumor cells in 2D.

We can note that, here, we can have blow up in finite time. This can easily be seen, when $u_0 < 0$, by considering spatially homogeneous solutions. Of course, such an initial datum is not biologically relevant, but numerical simulations show that we can also have blow up in finite for biologically relevant initial data, i.e., which belong to $[0, 1]$. This shows that the choice of the nonlinear terms is crucial, as one may have blow up in finite time if one is not careful enough; we will see that such a blow up can be avoided by considering logarithmic nonlinear terms f.

Again one difficulty, as far as the mathematical analysis is concerned, is to handle the equation for the order parameter which reads

$$\frac{d\langle u\rangle}{dt} + \langle g(u)\rangle = 0.$$

Setting $u = \langle u\rangle + \overline{u}$, we have

$$\begin{aligned}\langle g(u)\rangle &= \langle \overline{u}^2 + 2\langle u\rangle\overline{u} + \langle u\rangle^2 - \langle u\rangle - \overline{u}\rangle \\ &= g(\langle u\rangle) + \langle \overline{u}^2\rangle + 2\langle u\rangle\langle\overline{u}\rangle - \langle\overline{u}\rangle \\ &= g(\langle u\rangle) + \langle\overline{u}^2\rangle,\end{aligned}$$

so that

$$\frac{d\langle u\rangle}{dt} + g(\langle u\rangle) = -\langle\overline{u}^2\rangle.$$

We can prove a local well-posedness result.

Proposition 31. *For every $u_0 \in L^2(\Omega)$, there exists $T_0 = T_0(\|u_0\|) > 0$ and a unique solution u to the problem such that $u \in \mathcal{C}([0, T_0); L^2(\Omega)_w) \cap L^4(\Omega \times (0, T)) \cap L^2(0, T; H^2(\Omega))$, $\forall T < T_0$.*

Next, we show that either $\langle u\rangle$ is positive (and is thus uniformly (in time) bounded) or $\langle u\rangle$ tends to $-\infty$ as time goes to $+\infty$, at least exponentially fast (if it exists globally in time). Actually, we can do better and prove that, in the second case, $\langle u\rangle$ (and also u) blows up in finite time.

We first see that the function $\overline{u} = u - \langle u \rangle$ satisfies

$$\frac{\partial \overline{u}}{\partial t} + \Delta^2 \overline{u} - \Delta f(\overline{u} + \langle u \rangle) + g(\overline{u} + \langle u \rangle) - \langle g(\overline{u} + \langle u \rangle) \rangle = 0,$$
$$\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial \Delta \overline{u}}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$\overline{u}|_{t=0} = \overline{u}_0 (= u_0 - \langle u_0 \rangle).$$

Proceeding as in the previous talk, we can prove that $\overline{u}$ is globally bounded.

Considering then the Riccati ODE $y' + y^2 - y = 0$, whose solution reads

$$y(t) = \frac{1}{\frac{y(T)}{y(T)-1} e^{t-T} - 1} + 1, \ t \geq T,$$

we deduce the following, employing the comparison principle.

Theorem 32. *If $\langle u(T) \rangle < 0$, for some $T \geq 0$ (and, in particular, if $\langle u_0 \rangle < 0$), then the solution to the problem blows up in finite time. Furthermore, the blow up time T^+ satisfies*

$$T^+ \leq T + \ln \frac{\langle u(T) \rangle - 1}{\langle u(T) \rangle}.$$

We deduce from this the following corollaries.

Corollary 33. *Let u be a solution to the problem. Then, either u blows up in finite time or it exists globally in time and $0 \leq \langle u(t) \rangle \leq 1$, $\forall t \geq 0$.*

Corollary 34. *Let u be a global in time solution to the problem. Then, u is dissipative in $L^2(\Omega)$.*

We finally have the following result.

Theorem 35. *Let u be a nonvanishing solution to the problem such that $u(t) \in [0, 1]$, $\forall t \geq 0$. Then, $u(t)$ tends to 1 in $H^1(\Omega)$ as $t \to +\infty$.*

Next, we consider a logarithmic nonlinear term. We approximate the singular nonlinear term as in the previous talk and again prove the existence of a local in time solution, due to the fact that the solutions to the approximated problems may blow up in finite time. We thus have the following.

Theorem 36. *We assume that $u_0 \in H^1(\Omega)$, $|\langle u_0 \rangle| < 1$ and $-1 < u_0(x) < 1$,* a.e. *$x \in \Omega$. Then, there exists $T = T(u_0) > 0$ and a solution u to the problem on $[0, T]$ such that $u \in \mathcal{C}([0, T]; H^1(\Omega)_w) \cap L^2(0, T; H^2(\Omega)) \cap L^4(\Omega \times (0, T))$ and $\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$. Furthermore, $-1 < u(x, t) < 1$, a.e. $(x, t) \in \Omega \times (0, T)$.*

Proceeding as in the previous talk, we also have the following.

Theorem 37. *The solution u is global in time.*

Again, uniqueness and further regularity are open problems.

We finally discuss how to account for nutrients in tumor growth models based on the Cahn-Hilliard equation.

Lecture 10: Further generalizations of the Cahn–Hilliard equation

In this last lecture, we discuss the approach due to M. Gurtin and based on a separate balance law for internal microforces. The corresponding models read

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(Z)\frac{\partial u}{\partial t}) = \operatorname{div}(B(Z)\nabla\mu),$$
$$\mu - b(Z)\cdot\nabla\mu = \beta(Z)\frac{\partial u}{\partial t} - \alpha\Delta u + f(u),$$

where

$$\beta(Z)(\frac{\partial u}{\partial t})^2 + (a(Z)+b(Z))\cdot\nabla\mu\frac{\partial u}{\partial t} + B(Z)\nabla\mu\cdot\nabla\mu \geq 0$$

for all fields. These models contain the original and viscous Cahn–Hilliard equations.

This approach also allows to generalize the Cahn-Hilliard equation to account for important effects such as deformations and heat transfers. In particular, if we account for heat transfers, we have the following models:

$$\frac{\partial u}{\partial t} = \operatorname{div}(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}),$$
$$\frac{\partial e}{\partial t} = -\operatorname{div}(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta} - \alpha(u,\theta)\frac{\partial u}{\partial t}\nabla u),\ \alpha > 0,$$
$$\mu = 2c(\theta-\theta_c)u + \frac{1}{2}\partial_u\alpha(u,\theta)|\nabla u|^2 - \operatorname{div}(\alpha(u,\theta)\nabla u) + f(u),\ c,\ \theta_c > 0,$$
$$e = c_V\theta - c\theta_c u^2 + \frac{1}{2}(\alpha(u,\theta) - \partial_\theta\alpha(u,\theta))|\nabla u|^2 + F(u),\ c_V > 0,$$

where θ is the absolute temperature and e is the internal energy and for proper constitutive moduli A, B, C and D (four matrices such that A and D are positive semidefinite).

These equations seem particularly difficult to study. Let us indeed consider the simplest case $A = D = I$, $B = C = 0$, $\alpha = 1$ constant, $\theta_c = 0$, $c_V = 1$ and $c = 0$. In that case, the equations read

$$\frac{\partial u}{\partial t} = \Delta\frac{\mu}{\theta},$$
$$\mu = -\Delta u + f(u),$$

$$\frac{\partial\theta}{\partial t} + \Delta\frac{1}{\theta} = -f(u)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t}\Delta u.$$

We can then prove that we have the conservation of the energy,

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) + \theta)\, dx = 0,$$

and that is all.

We then discuss, in a simple case (and without thermal effects), the difficulties related to defining and studying dynamic boundary conditions in the models of M. Gurtin.

We finally discuss the hyperbolic relaxation of the Cahn–Hilliard equation,

$$\beta\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0,\ \beta > 0,$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$u|_{t=0} = u_0,\ \frac{\partial u}{\partial t}|_{t=0} = u_1.$$

Here, we no longer have the conservation of mass; more precisely, the following holds:

$$\langle u(t)\rangle = \langle u_0\rangle e^{-\frac{t}{\beta}} + \langle \beta u_1 + u_0\rangle(1 - e^{-\frac{t}{\beta}}),\ t \geq 0.$$

Note that, for a fixed $t \geq 0$,

$$\lim_{\beta\to 0^+}\langle u(t)\rangle = \langle u_0\rangle.$$

A major difficulty is to define a good notion of a solution, ensuring well-posedness in 2 and 3D. One possibility is to consider bounded energy solutions, i.e., solutions such that the total free energy belongs to $L^\infty(0,T)$, $T > 0$, and whose existence can be proved by implementation of a standard Galerkin scheme. Furthermore, in two space dimensions, one can also prove the uniqueness. However, in three space dimensions, the uniqueness of such solutions is still an open problem.

We can note that the case of logarithmic nonlinear terms is still to be addressed. To illustrate the difficulties we have to face, we discuss the existence of solutions for the damped wave equation (i.e., the hyperbolic relaxation of the Allen–Cahn equation). More precisely, we can only prove the existence of strong solutions, when β is small and the initial datum is not too large. This result is obtained via a perturbation argument, based on the fact that, when β is small, the solution is close to that of the limit equation, and the dissipativity provided by the equation.