CBMS Conference: Gaussian Random Fields, Fractals, SPDEs, and Extremes

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Introduction to the Lectures

Random fields are stochastic processes indexed by vectors in multidimensional Euclidean space $\mathbb{R}^N$ or its subsets such as the N-dimensional integer lattice $\mathbb{Z}^N$, the unit sphere $\mathbb{S}^{N-1}$ or more general manifolds. The study of random fields was originated from the work of Kolmogorov (1941) in statistical theory of turbulence and Lévy's work on multiparameter Brownian motion [cf. Lévy (1948, 1963)].

In Mathematics, random fields form an important class of stochastic processes. They play significant roles in studying stochastic partial differential equations (their solutions, when exist, are random fields).

The studies of random fields and stochastic partial differential equations have recently entered a period of rapid growth. The enormous amount of current, as well as past, interest in random fields has been motivated in part by the vast number of their applications to sciences outside mathematics ranging from astronomy to fluid mechanics, and to finance. These connections have, in turn, motivated and generated a large number of exciting and novel mathematical questions. The list of references further information.

The lectures will put the latest developments on random fields from the subjects of probability theory, stochastic partial differential equations, fractal geometry, and extreme value theory in a nutshell and present to young researchers and graduate students in Probability and related areas. We will identify open problems in the theory of random fields that are motivated by statistics, mathematical physics, among other disciplines.

The following are the main topics that will be covered by the lectures.

1 Gaussian random fields and their regularity properties

Random fields arise naturally in probability theory, stochastic partial differential equations and in studies of Markov processes. In recent years, there has been an increased interest in investigating various properties of random fields. However, compared with the rich theory of

Brownian motion and Markov processes, many aspects of the theory on random fields are still under development. One of the main difficulties is the lack of powerful technical tools such as the Markov property and stopping times.

The theory on Gaussian processes and random fields has played one of the leading roles in modern Probability. The books by Ibragimov and Rozanov (1978), Adler (1981), Ledoux and Talagrand (1991), Khoshnevisan (2002), Marcus and Rosen (2006), Adler and Taylor (2007), Talagrand (2005, 2014) provide systematic accounts on fundamental methods for studying Gaussian processes and their applications to other areas of mathematics and statistics. We also refer to Yaglom (1987), Guyon (1995) for more applications of Gaussian random fields to statistical physics, to Rasmussen and Williams (2006) for applications in machine learning, and to van der Vaart and van Zanten (2008, 2009, 2011) for applications in Bayesian statistics.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field, which takes values in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $\omega \in \Omega$, the function $t \mapsto X(t, \omega)$ is called a sample path or a sample function of X. Following the convention, we usually suppress ω from the notation.

Regularity properties such as continuity and differentiability of the sample functions of Gaussian processes have been studied by many authors. The seminal works of Dudley (1967), Fernique (1975), and Talagrand (1987) are not only of fundamental importance in probability theory, but also in statistical applications [cf. e.g., Cressie (1993), Stein (1999), van der Vaart and Wellner (1996), Banerjee and Gelfand (2003), Kosorok (2008)]. The methods based on metric entropy or majorizing measures are very powerful for determining the continuity, differentiability, as well as upper bounds for the uniform modulus of continuity of Gaussian random fields. See Marcus and Rosen (2006), Talagrand (2005, 2014), Adler and Taylor (2007).

More recently, many fine properties of the sample functions of Gaussian random fields have been proved. These include small ball probabilities, Chung's laws of the iterated logarithm, and small deviations [Keulbs and Li (1993), Keulbs, Li and Talagrand (1994), Li and Linde (1999), Li and Shao (2001, 2005)]; exact Hausdorff and packing measure functions [Talagrand (1995, 1998), Xiao (1996, 1997, 2003), Luan and Xiao (2012)]; exact uniform modulus of continuity [Meerschaert, Wang and Xiao (2013), Li, Wang and Xiao (2015)]; local times [Berman (1972, 1973), Geman and Horowitz (1980), Xiao (1997, 2007), Baraka and Mountford (2008, 2011), Baraka, *et al* (2009), Lee (2021)], multiple points and self-intersection local times [Rosen (1984), Wu and Xiao (2010), Chen, *et al* (2011), Dalang, *et al* (2012), Dalang and Mueller (2015), Dalang, Lee, Mueller and Xiao (2021)]. These results have revealed deep and subtle structures of Gaussian random fields, and the developed methods based on general Gaussian principles are important for further investigating Gaussian and other random fields.

In this part of the lectures, we will present general methods for establishing exact uniform and local modulus of continuity results for Gaussian random fields. In particular, we prove four types of limit theorems: the law of the iterated logarithm, uniform modulus of continuity, Chung's law of the iterated logarithm, and the modulus of nondifferentiability, under a general framework that will be convenient for studying the solutions of stochastic partial differential equations. An important tool is the properties of strong local nondeterminism.

2 Geometry of random fields

The sample functions of a random field $X = \{X(t), t \in \mathbb{R}^N\}$ generate various interesting geometric objects such as the image set $X(E) = \{X(t) : t \in E\}$, the graph set $\mathrm{Gr}X(E) = \{(t, X(t)) : t \in E\}$, where $E \subseteq \mathbb{R}^N$ is a Borel set; the level set $X^{-1}(x) = \{t \in \mathbb{R}^N : X(t) = x\}$, where $x \in \mathbb{R}^d$; and the excursion set $X^{-1}(F) = \{t \in \mathbb{R}^N : X(t) \in F\}$, where $F \subseteq \mathbb{R}^d$; the set of multiple points; just to mention a few. The geometric and topological properties of these random sets contain a lot of information about the random field X and are intrinsically related to potential theoretical properties and extreme value properties of X.

If the sample function $t \mapsto X(t)$ is non-smooth (i.e., not differentiable), then tools from fractal geometry will be needed and there has been a lot of research in this direction. If the sample function $t \mapsto X(t)$ is smooth (i.e., continuously differentiable), one can apply tools from integral/differential geometry to characterize the geometric and topological structures of the random sets generated by X. It is important to remark that there is a deep connection between the mean Euler characteristic of the excursion set and the tail probability of the supremum of the Gaussian random field. See the books by Adler and Taylor (2007), Azaïs and Wschebor (2009).

At this CBMS conference, we will focus mostly on fractal properties of Gaussian random fields and the solutions of stochastic partial differential equations.

For Gaussian random fields with non-differentiable sample functions, fractal dimensions (e.g., Hausdorff dimension, box dimension, packing dimension, etc) are important measures of their roughness and have been studied extensively. We refer to Falconer (1990), Mattila (1995), Bishop and Peres (2016) for general information on fractal geometry, and to Taylor (1986), Lawler (1999), Xiao (2004, 2009, 2013) for overviews on random fractals generated by Markov processes and random fields.

In recent years many authors have studied fractal properties of anisotropic Gaussian random fields, which arise naturally as scaling limits of discrete stochastic systems or as solutions of stochastic differential equations, or as stochastic models in spatial statistics and hydrology. See e.g., Bonami and Estrade (2003), Ayache and Xiao (2005), Khoshnevisan, Wu and Xiao (2006), Biermé, Meerschaert and Scheffler (2007), Wu and Xiao (2007, 2011), Xiao (2009), Xue and Xiao (2011), Luan and Xiao (2012), Lee (2021). These works show that, compared with isotropic Gaussian fields such as fractional Brownian motion, the probabilistic and geometric properties of anisotropic Gaussian random fields are much richer and more difficult to study. As pointed out in Xiao (2013), analogous problems on exact Hausdorff measure functions for the trajectory and multiple points in Talagrand (1995, 1998), and on local times and level sets in Xiao (1997), Baraka and Mountford (2011), remain open for anisotropic Gaussian fields.

2.1 Fractal dimension and exact Hausdorff measure functions

Hausdorff and packing dimensions of the range $X([0,1]^N)$, graph $\mathrm{Gr}X([0,1]^N)$ and level sets are obtained for time-anisotropic Gaussian random fields by Ayache and Xiao (2005) and Xiao (2009). It is a natural question to determine exact Hausdorff and packing measure functions for these random sets. Recall that a measure function $\varphi : (0,1) \to \mathbb{R}_+$ is called an exact Hausdorff measure function for a set $F \subseteq \mathbb{R}^d$ if $0 < \varphi\text{-}m(F) < \infty$. Here φ-m denotes the φ-Hausdorff measure. We will also consider φ-packing measure for some random sets such as the range $X\big([0,1]^N\big)$. A measure function φ is called an exact packing measure function for F if $0 < \varphi\text{-}p(F) < \infty$.

Investigating exact Hausdorff and packing measure functions for the random sets generated by a random field X not only provides more precise information about the fractal properties of the sample functions of X, but also stimulates deep understanding of the probability properties such as small ball probabilities, large deviations and dependence structures of X. These latter questions have proved to be significant and sometimes challenging.

The problems on finding exact Hausdorff measure functions for the range and graph of the Brownian sheet and fractional Brownian motion have been considered by Ehm (1981), Talagrand (1995, 1998), Xiao (1997a, 1997b), Luan and Xiao (2012), and Lee (2021). Most of these references either consider special Gaussian random fields or assume stationarity on the increments. We will investigate these problems under a more general setting that is convenient for applications to the solutions of SPDEs.

2.2 Local times of Gaussian random fields

The roughness or irregularity of sample functions of X can be reflected in the regularity (or smoothness) of the local times of X. This was first observed by Berman (1972) who developed Fourier analytic method for studying the existence and continuity of local times of Gaussian processes. Berman (1973) introduced the notion of "local nondeterminism" for Gaussian processes to overcome many difficulties caused by the lack of Markov property and to unify his methods for studying local times. Berman's work has been extended and strengthened in various ways. See Geman and Horowitz (1980) and Xiao (2007, 2009) for more information.

The existence and joint continuity of local times of a fractional Brownian sheet $W^{\vec{H}}$ with values in $\mathbb{R}^d$ and index $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$ were studied by Xiao and Zhang (2002). Ayache, Wu and Xiao (2008) proved that the optimal condition for the joint continuity of the local times of $W^{\vec{H}}$ is $\sum_{j=1}^N H_j^{-1} > d$. Xiao (2009) proved similar results for a class of Gaussian random fields with stationary increments under certain general conditions.Wu and Xiao (2011) provided a unified treatment by applying sectorial local nondeterminism to estimate high moments of local times and improved significantly the results in Ayache, Wu and Xiao (2008) and Xiao (2009). Their results have been improved recently by Lee (2021b).

In the lecture on local times, we establish optimal regularity results on the local times by applying the properties of strong local nondeterminism.

2.3 Hitting probabilities of Gaussian random fields

Fractal properties (e.g., Hausdorff dimension) of random fields are closely related to potential theory, which is a topic of independent interest. While potential theory for Markov processes is a mature subject, it is still at a developing stage for Gaussian random fields. The most significant result in this aspect is Khoshnevisan and Shi (1999), who studied the hitting probabilities of the range of the Brownian sheet. More specifically, they proved that if $X = \{X(t), t \in \mathbb{R}_+^N\}$ is the Brownian sheet with values in $\mathbb{R}^d$, then for any closed interval $I \subset (0,\infty)^N$ and any compact set $F \subset \mathbb{R}^d$,

$$\mathbb{P}\big\{X(I) \cap F \neq \varnothing\big\} > 0 \Longleftrightarrow \text{Cap}_{d-2N}(F) > 0, \tag{1}$$

where Cap_β denotes the Riesz-Bessel capacity of order $\beta \geq 0$ [cf. e.g., Khoshnevisan (2002)]. An analogous necessary and sufficient condition on $E \subset (0,\infty)^N$ for $\mathbb{P}\{X^{-1}(x) \cap E \neq \varnothing\} > 0$ was proved by Khoshnevisan (1999) for $N = 2$, Khoshnevisan and Xiao (2007) for $N \geq 3$, provided X is the Brownian sheet. However, it has been an open problem to prove (1) for a general Gaussian random field such as a fractional Brownian motion.

Various partial extensions of (1) have been proved for several classes of Gaussian random fields by Xiao (1999), Khoshnevisan (2002), Biermé, Lacaux and Xiao (2009), Xiao (2009), and Chen and Xiao (2012). In particular, Chen and Xiao (2012) proved necessary conditions (in terms of Hausdorff measure) and sufficient conditions (in terms of a capacity) on compact sets $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$ for $\mathbb{P}\{X\big(E\big) \cap F \neq \varnothing\} > 0$. In the special case when X is Brownian motion, Watson (1976, 1978) showed that the last statement holds if and only of $E \times F$ has positive "thermal capacity". See Khoshnevisan and Xiao (2015) for a probabilistic treatment and results on Hausdorff dimension of the intersection $X\big(E\big) \cap F$ when it is not empty.

Results analogous to (1) on hitting properties have been established for solutions of linear SPDEs, such as the stochastic heat and wave equations in various spatial dimensions by Mueller and Tribe (2002), Dalang and Nualart (2004). Extensions of these results to nonlinear SPDEs have been obtained recently by Dalang, Khoshnevisan, and Nualart (2007, 2009, 2013), Dalang and Sanz-Solé (2010, 2015). Dalang and Pu (2020, 2021), Hinojosa-Calleja and Sanz-Solé (2020, 2021). These extensions make extensive use of Malliavin's calculus [Nualart (2006, 2010)] in order to establish heat-kernel-type bounds for probability density functions connected to the solution of SPDEs. Those bounds are nonlinear analogues of covariance-function estimates for Gaussian random fields.

3 Stochastic partial differential equations

Typically, solutions to linear SPDEs driven by Gaussian noise are Gaussian random fields, or at least, generalized Gaussian random fields. For instance, the solution to a linear stochastic heat or wave equation driven by spatially homogeneous Gaussian noise is a generalized Gaussian random field, and one can determine necessary and sufficient conditions for the solution to be an ordinary random field [cf. Walsh (1986), Dalang (1999)]. Therefore, well-established techniques from the theory of Gaussian processes can be used to study many of the analytic and geometric properties of the solution to linear SPDEs. This gives an indication of what sort of local result can be expected to hold also for nonlinear SPDEs, though it is often a substantial challenge to establish a corresponding result for nonlinear SPDEs.

For the nonlinear stochastic heat or wave equations, this has been addressed in works of Dalang (1999), Peszat and Zabczyk (2007), Nualart and Viens (2009), Foondun, Khoshnevisan and Nualart (2011) and Eisenbaum, Foondun, Khoshnevisan (2011). The last two references also contain an unexpected connection with existence of local times of Markov processes.

An important issue on regularity of the solution concerns Hölder continuity. For a linear SPDE, the exact uniform and local moduli of the solution can be determined by applying the methods from the theory of Gaussian fields in Section 1 [see, e.g., Meerschaert, Wang and Xiao (2013), Tudor and Xiao (2017), Allouba and Xiao (2017), Herrell *et al* (2020)]. This again indicates what are likely to be the best possible results for nonlinear SPDEs. Establishing such Hölder continuity results may still pose many challenges, as can be seen in the AMS Memoir of Dalang and Sanz-Solé (2009). For nonlinear parabolic SPDEs, this problem has recently been studied by Khoshnevisan *et al* (2018). We refer to Dalang *et al* (2009), Foondun, and Khoshnevisan (2009), Khoshnevisan (2014), Mijena and Nane (2015), Khoshnevisan, Kim and Xiao (2017, 2018) for recent advances on other topics of SPDEs such as intermittency and macroscopic multifractality.

In the lectures, we study fractal properties, level sets, local times, small-ball properties of the solutions of stochastic heat and wave equations.

We remar that, while the solutions of stochastic heat or wave equations are not differentiable in the space and time variables, solutions of some higher order SPDEs with Gaussian white noise may be differentiable in the space variable x. This has been proved by Allouba (2015), Allouba and Xiao (2017) for the fourth order L-Kuramoto-Sivashinsky (L-KS) SPDEs and time-fractional stochastic partial integro-differential equations. See Mijena and Nane (2015, 2016) for related work on fractional SPDEs. We expect that the tools from integral/differential geometry can be employed to study solutions of these higher order and/or fractional SPDEs.

4 Extremes

Excursion probability (or the tail probability of the extreme value) of a real-valued continuous-time random field has been studied extensively and there is an enormous literature. See the books of Adler (1981), Piterbarg (1996), Adler and Taylor (2007, 2011), Azaïs and Wschebor (2009), Yakir (2013), and their combined references.

Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field, where T is the index set, which can be a subset of $\mathbb{R}^N$ such as the unit cube $T = [0,1]^N$, or the unit sphere $T = \mathbb{S}^{N-1}$ or, more generally, a manifold.

The study of the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ is a very important problem in probability and has many applications in statistics and other scientific areas. Various methods for precise approximation of $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ have been developed. These include the double sum method [Pickands (1969), Qualls and Watanabe (1973), Piterbarg (1996), Dębicki, Hashorva and Ji (2016)], the tube method [Knowles and Siegmund (1989), Johansen and Johnstone (1990), Sun (1993), Siegmund and Worsley (1995), Sun and Loader (1994)], the

Euler characteristic method [Worsley (1994, 1995), Adler (1981, 2000), Taylor, Takemura and Adler (2005), Adler and Taylor (2007)] and the Rice method [Azaïs and Wschebor (2009)].

So far, most of the research on excursion probabilities has been focused on real-valued Gaussian random fields, the development of extreme value theory of multivariate continuous-time random fields is still in an evolutionary stage [see Ashin (2005), Hashorva and Ji (2014), Dębicki *et al* (2015)] and the range of its applications is growing.

Zhou and Xiao (2017) considered an $\mathbb{R}^2$-valued continuous locally stationary Gaussian random field $\{X(t) = (X_1(t), X_2(t))^T,\ t \in \mathbb{R}^N\}$ with $\mathbb{E}[X(t)] = \mathbf{0}$ and proved a Pickands-type approximation for the bivariate excursion probability

$$\mathbb{P}\bigg(\max_{s \in T_1} X_1(s) > u,\ \max_{t \in T_2} X_2(t) > u \bigg), \quad \text{as} \ \ u \to \infty, \tag{2}$$

where $T_1, T_2 \subset \mathbb{R}^N$ are compact sets. Their results show explicitly that the excursion probabilities depend not only on the smoothness parameters of the coordinate fields X_1 and X_2, but also on their maximum correlation ρ. More precise approximations for the excursion probability in (2) has been obtained by Cheng and Xiao (2021) for smooth bivariate Gaussian random fields by applying the Euler characteristic method.

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