J Fourier Anal Appl (2012) 18:118–145
DOI 10.1007/s00041-011-9193-2

Spectral Conditions for Strong Local Nondeterminism and Exact Hausdorff Measure of Ranges of Gaussian Random Fields

Nana Luan · Yimin Xiao

Received: 20 October 2010 / Revised: 24 August 2011 / Published online: 23 September 2011

Abstract Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \dots, X_d(t)), \quad t \in \mathbb{R}^N,$$

where $X_1, \dots, X_d$ are independent copies of a real-valued, centered, anisotropic Gaussian random field X_0 which has stationary increments and the property of strong local nondeterminism. In this paper we determine the exact Hausdorff measure function for the range $X([0, 1]^N)$.

We also provide a sufficient condition for a Gaussian random field with stationary increments to be strongly locally nondeterministic. This condition is given in terms of the spectral measures of the Gaussian random fields which may contain either an absolutely continuous or discrete part. This result strengthens and extends significantly the related theorems of Berman (Indiana Univ. Math. J. 23:69–94, 1973, Stochast. Process. Appl. 27:73–84, 1988), Pitt (Indiana Univ. Math. J. 27:309–330, 1978) and Xiao (Asymptotic Theory in Probability and Statistics with Applications, pp. 136–176, 2007, A Minicourse on Stochastic Partial Differential Equations, Lecture Notes in Math, vol. 1962, pp. 145–212, 2009), and will have wider applicability beyond the scope of the present paper.

Communicated by Christian Houdre.

Research partially supported by NSF grant DMS-1006903.

N. Luan
School of Insurance and Economics, University of International Business and Economics, Beijing 100029, China
e-mail: luannana318@gmail.com

Y. Xiao (✉)
Department of Statistics and Probability, Michigan State University, A-413 Wells Hall, East Lansing, MI 48824, USA
e-mail: xiaoyimi@stt.msu.edu
url: http://www.stt.msu.edu/~xiaoyimi

Keywords Gaussian random fields · Strong local nondeterminism · Spectral condition · Anisotropy · Hausdorff dimension · Hausdorff measure

Mathematics Subject Classification (2010) 60G15 · 60G17 · 60G60 · 28A80

1 Introduction

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$, where

$$X(t) = (X_1(t), \dots, X_d(t)), \quad t \in \mathbb{R}^N. \tag{1.1}$$

For brevity we call X an (N, d)-Gaussian random field. Sample path properties of X such as the Hausdorff dimensions of the range $X([0,1]^N) = \{X(t) : t \in [0,1]^N\}$, the graph $\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\}$ and the level set $X^{-1}(x) = \{t \in \mathbb{R}^N : X(t) = x\}$ $(x \in \mathbb{R}^d)$ have been studied by many authors under various assumptions on the coordinate processes $X_1, \dots, X_d$. We refer to Adler [1], Kahane [13] and Xiao [31, 32] for further information.

In the cases when $X_1, \dots, X_d$ are independent copies of an approximately *isotropic* Gaussian random field X_0 [a typical example is fractional Brownian motion], the problems for finding the exact Hausdorff measure functions for $X([0,1]^N)$, $\mathrm{Gr}X([0,1]^N)$ and $X^{-1}(x)$ have been investigated by Talagrand [23, 24], Xiao [28–30], Baraka and Mountford [4, 5].

The main objective of this paper is to study the exact Hausdorff measure of the range of Gaussian random fields which are anisotropic in the time-variable. More specifically, we consider an (N, d)-Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ whose coordinate processes $X_1, \dots, X_d$ in (1.1) are independent copies of a centered, real-valued Gaussian field X_0 with stationary increments and $X_0(0) = 0$ almost surely; and we assume there exists a constant vector $H = (H_1, \dots, H_N) \in (0,1)^N$ such that the following conditions hold:

(C1) There exists a positive constant $c_{1,1} \ge 1$ such that

$$c_{1,1}^{-1}\, \rho(s,t)^2 \le \mathbb{E}(X_0(s) - X_0(t))^2 \le c_{1,1}\, \rho(s,t)^2 \quad \text{for all } s, t \in [0,1]^N,$$

where $\rho(s,t)$ is the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \quad \forall s, t \in \mathbb{R}^N.$$

(C2) There exists a positive constant $c_{1,2}$ such that for all integers $n \ge 1$ and all $u, t^1, \dots, t^n \in [0,1]^N$, we have

$$\mathrm{Var}(X_0(u) | X_0(t^1), \dots, X_0(t^n)) \ge c_{1,2} \min_{0 \le k \le n} \rho(u, t^k)^2 \quad (t^0 = 0).$$

Section 2 below provides a way to construct a large class of Gaussian random fields with stationary increments that satisfy (C1) and (C2). Further examples can be found in Xiao [32] and Luan and Xiao [15]. Under Condition (C1), the (N,d)-Gaussian random field X has a version which has continuous sample functions on $[0,1]^N$ almost surely. Henceforth we will assume without loss of generality that the Gaussian random field X has continuous sample paths. When $\{X_0(t), t\in\mathbb{R}^N\}$ satisfies (C2), we say that X_0 has the property of strong local nondeterminism in metric ρ on $[0,1]^N$.

Xiao [32] proved that, if Condition (C1) holds, then with probability 1,

$$\dim_{\mathrm{H}} X([0,1]^N)=\min\left\{d;\ \sum_{j=1}^{N}\frac{1}{H_j}\right\}, \tag{1.2}$$

where $\sum_{j=1}^{0}\frac{1}{H_j}:=0$. In the above, $\dim_{\mathrm{H}}$ denotes Hausdorff dimension [cf. Kahane [13] or Falconer [11]]. Further analytic and fractal properties of Gaussian random fields which satisfy Conditions (C1) and (C2) have been studied by Xiao [32], Biermé et al. [10], Luan and Xiao [15], Meerschaert et al. [16] (see also Benassi et al. [6], Ayache and Xiao [3], Wu and Xiao [26, 27] for related results).

The first objective of this paper is to refine (1.2) by determining the exact Hausdorff measure function for the range $X([0,1]^N)$.

Theorem 1.1 *Let $X=\{X(t), t\in\mathbb{R}^N\}$ be an (N,d)-Gaussian random field with stationary increments defined by* (1.1), *where $X_1,\dots,X_d$ are independent copies of a centered, real-valued Gaussian field X_0 with stationary increments and $X_0(0)=0$. We assume that X_0 satisfies Conditions* (C1) *and* (C2). *If $d>\sum_{j=1}^N H_j^{-1}$, then we have*

$$0<\varphi_1\text{-}m(X([0,1]^N))<\infty \quad a.s.,$$

where φ_1 is the function

$$\varphi_1(r)=r^{\sum_{j=1}^N H_j^{-1}}\log\log\frac{1}{r}$$

and φ_1-m is the corresponding Hausdorff measure.

The following remark is concerned with the cases not covered by Theorem 1.1.

Remark 1.2

- If $d<\sum_{j=1}^N H_j^{-1}$, then Theorem 8.2 in [32] implies that $X([0,1]^N)$ a.s. has interior points and hence has positive d-dimensional Lebesgue measure. In this case, Wu and Xiao [27] showed that X has a jointly continuous local time and provides a lower bound for the exact Hausdorff measure (in the metric ρ) of the level set $X^{-1}(x)$. For fractional Brownian motion and some other isotropic Gaussian random fields, the exact Hausdorff measure function for $X^{-1}(x)$ has been determined by Xiao [30] and Baraka and Mountford [5]. However, no such result has been established for *anisotropic* Gaussian random fields.

- If $d = \sum_{j=1}^{N} H_j^{-1}$, then $\dim X([0,1]^N) = d$ a.s. The problem to determine the exact Hausdorff measure function for $X([0,1]^N)$ in this "critical case" is open and is certainly a deeper question.

It will become clear that the proof of Theorem 1.1 relies crucially on Condition (C2)—the property of strong local nondeterminism, which is useful for studying many other sample path and statistical properties of Gaussian random fields (cf. [32, 33]). The second objective of this paper is to provide a rather general condition for a Gaussian random field with stationary increments to satisfy both Conditions (C1) and (C2). This condition is given in terms of the spectral measures of the Gaussian random fields which may contain either an absolutely continuous or a discrete part. Theorem 2.4 extends the related theorems of Berman [8, 9], Pitt [20] and Xiao [31, 32], which will have wider applicability beyond the scope of the present paper. For example, we can apply this theorem to prove that the solution of a fractional stochastic heat equation on the circle $\mathbb{S}_1$ (see [18, 25]) has the property of strong local nondeterminism in the space variable (at fixed time t). Hence fine properties of the sample functions of the solution can be obtained by using the results in [15, 17, 32], and [16]. Similarly, we can show that the spherical fractional Brownian motion on $\mathbb{S}_1$ introduced by Istas [12] is also strongly locally nondeterministic. Both of these processes share local properties with ordinary fractional Brownian motion with appropriate Hurst indices. Details of these results will be given elsewhere.

The rest of this paper is organized as follows. Section 2 gives a sufficient condition for a Gaussian random field with stationary increments to be strongly locally nondeterministic. Section 3 is concerned with the exact Hausdorff measure function for the range of X. After recalling the definition of Hausdorff measure and its basic properties, and establishing some estimates, we prove Theorem 1.1.

We end the Introduction with some notation. The inner product of $s, t \in \mathbb{R}^N$ is denoted by $\langle s, t\rangle$ and the Euclidean norm of $t \in \mathbb{R}^N$ is denoted $\|t\|$. Given two points $s = (s_1, \ldots, s_N) \in \mathbb{R}^N$ and $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$, $s \le t$ (resp. $s < t$) means that $s_i \le t_i$ (resp. $s_i < t_i$) for all $1 \le i \le N$. When $s \le t$, we use $[s, t]$ to denote the N-dimensional interval (or rectangle) $[s, t] = \prod_{i=1}^{N} [s_i, t_i]$. For any $T \subseteq \mathbb{R}^N$, $f(s) \asymp g(s)$ means the ratio $f(s)/g(s)$ is bounded from below and above by positive and finite constants which are independent of $s \in T$.

Throughout this paper we will use c to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

2 Spectral Condition for Strong Local Nondeterminism of Gaussian Fields with Stationary Increments

One of the major difficulties in studying the probabilistic, analytic or statistical properties of Gaussian random fields is the complexity of their dependence structures. In many circumstances, the properties of local nondeterminism can help us to overcome this difficulty so that many elegant and deep results for Brownian motion can be extended to Gaussian random fields; see [8, 9, 20] and [31, 32] for further information.

Hence, for a given Gaussian random field, it is an interesting question to determine whether it satisfies certain forms of local nondeterminism. In this section we provide a general sufficient condition for a Gaussian random field with stationary increments to satisfy Conditions (C1) and (C2).

Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with stationary increments and $X_0(0) = 0$. We assume that X_0 has continuous covariance function $R(s,t) = \mathbb{E}[X(s)X(t)]$. According to Yaglom [34], $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1) F(d\lambda) + \langle s, Mt\rangle, \tag{2.1}$$

where M is an $N \times N$ non-negative definite matrix and $F(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1+\|\lambda\|^2} F(d\lambda) < \infty. \tag{2.2}$$

In analogy to the stationary case, the measure F is called the spectral measure of X_0. If F is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$, its density f will be called the spectral density of X_0.

It follows from (2.1) that X_0 has the following stochastic integral representation:

$$X_0(t) \overset{d}{=} \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1) W(d\lambda) + \langle Y, t\rangle, \tag{2.3}$$

where $\overset{d}{=}$ means equality of all finite dimensional distributions, Y is an N-dimensional Gaussian random vector with mean 0 and covariance matrix M, $W(d\lambda)$ is a centered complex-valued Gaussian random measure which is independent of Y and satisfies

$$\mathbb{E}(W(A)\overline{W(B)}) = F(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}$$

for all Borel sets $A, B \subseteq \mathbb{R}^N$ with finite F-measure. The above properties of $W(d\lambda)$ ensures that the stochastic integral in (2.3) is real-valued. The spectral measure F is called the control measure of W. Since the linear term $\langle Y, t\rangle$ in (2.3) will not have any effect on the problems considered in this paper, we will from now on assume $Y = 0$. This is equivalent to assuming $M = 0$ in (2.1). Consequently, for any $h \in \mathbb{R}^N$ we have

$$\sigma^2(h) \triangleq \mathbb{E}(X_0(t+h) - X_0(t))^2 = 2\int_{\mathbb{R}^N} (1 - \cos\langle h, \lambda\rangle) F(d\lambda). \tag{2.4}$$

It is important to note that $\sigma^2(h)$ is a negative definite function in the sense of I.J. Schoenberg, which is determined by the spectral measure F. See Berg and Forst [7] for more information on negative definite functions. If the function $\sigma^2(h)$ depends only on $\|h\|$, then X_0 is called an isotropic random field. More generally, if $\sigma^2(h) \asymp \phi(\|h\|)$ in a neighborhood of $h = 0$ for some nonnegative function ϕ, then X_0 is called approximately isotropic.

Various centered Gaussian random fields with stationary increments can be constructed by choosing appropriate spectral measures F. For the well known fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ of Hurst index $H \in (0, 1)$, its spectral measure has a density function

$$f_H(\lambda) = c(H, N)\frac{1}{\|\lambda\|^{2H+N}}, \tag{2.5}$$

where $c(H, N) > 0$ is a normalizing constant such that $\sigma^2(h) = \|h\|^{2H}$. Since $\sigma^2(h)$ depends on $\|h\|$ only, the increments of B^H are isotropic and stationary. Examples of approximately isotropic Gaussian fields with stationary increments can be found in [31].

A typical example of anisotropic Gaussian random field with stationary increments can be constructed by choosing the spectral density

$$f(\lambda) = \frac{1}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{2+Q}}, \quad \forall \lambda \in \mathbb{R}^N \backslash \{0\}, \tag{2.6}$$

where the constants $H_j \in (0, 1)$ for $j = 1, \dots, N$ and $Q = \sum_{j=1}^N H_j^{-1}$. This notation will be fixed throughout the rest of the paper.

It can be verified that $f(\lambda)$ in (2.6) satisfies (2.2) and the corresponding Gaussian random field X_0 has stationary increments. In the special case when $H_1 = \cdots = H_N = H$, (2.6) is very similar to (2.5). Consequently, X_0 shares many properties with fractional Brownian motion.

In general, X_0 with spectral density (2.6) is anisotropic in the sense that the sample function $X_0(t)$ has different geometric and probabilistic characteristics along different directions. This gives more flexibility from modeling point of view. Moreover, X_0 is operator-self-similar with exponent $A = (a_{ij})$, where $a_{ii} = H_i^{-1}$ and $a_{ij} = 0$ if $i \neq j$. The latter means that for any constant $c > 0$,

$$\{X_0(c^A t),\, t \in \mathbb{R}^N\} \stackrel{d}{=} \{c\, X_0(t),\, t \in \mathbb{R}^N\}, \tag{2.7}$$

where c^A is the linear operator defined by $c^A = \sum_{n=0}^{\infty} \frac{(\ln c)^n A^n}{n!}$. Xiao [32] proved that the Gaussian random field X_0 satisfies Conditions (C1) and (C2), and characterized many sample path properties of the corresponding (N, d)-Gaussian field X in terms of $(H_1, \dots, H_N)$ explicitly.

We remark that all centered stationary Gaussian random fields can also be treated using the above framework. In fact, if $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a centered, real-valued stationary Gaussian random field, it can be represented as $Y(t) = \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle}\, W(d\lambda)$. Thus the random field X_0 defined by

$$X_0(t) = Y(t) - Y(0) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1)\, W(d\lambda), \quad \forall t \in \mathbb{R}^N$$

is Gaussian with stationary increments and $X_0(0) = 0$. Note that the spectral measure F of X_0 in the sense of (2.4) is the same as the spectral measure [in the ordinary sense] of the stationary random field Y.

The main purpose of this section is to prove a sufficient condition for a general Gaussian random field X_0 with stationary increments to satisfy Conditions (C1) and (C2). In particular, this condition implies that X_0 is strongly locally nondeterministic in metric ρ.

To this end we first introduce some notation and state several lemmas. For any $\lambda \in \mathbb{R}^N$ and $h > 0$, we denote by $C(\lambda, h)$ the cube with side-length $2h$ and center λ, i.e.,

$$C(\lambda, h) = \{x \in \mathbb{R}^N : |x_j - \lambda_j| \le h,\ j = 1, \dots, N\}.$$

For any $g \in L^2(\mathbb{R}^N)$, let $\widehat{g}(\lambda) = \int_{\mathbb{R}^N} e^{i\langle \lambda, x\rangle} g(x)dx$ be the Fourier transform of g and let $L^2(C(0,T))$ denote the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0,T)$. In the following, Lemma 2.1 is Proposition 4 of [19]. Lemma 2.2 is taken from [31], which is an extension of a result of [20, p. 326].

Lemma 2.1 *Let $\widetilde{\Delta}(d\lambda)$ be a positive measure on $\mathbb{R}^N$. If, for some constant $h > 0$, $\widetilde{\Delta}(d\lambda)$ satisfies*

$$0 < \liminf_{\|\lambda\| \to \infty} \widetilde{\Delta}(C(\lambda, h)) \le \limsup_{\|\lambda\| \to \infty} \widetilde{\Delta}(C(\lambda, h)) < \infty, \tag{2.8}$$

then, for every $T > 0$ satisfying $ThN < \log 2$, there exist positive and finite constants $c_{2,2}$ and $c_{2,3}$ such that

$$c_{2,2} \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 d\lambda \le \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 \widetilde{\Delta}(d\lambda) \le c_{2,3} \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 d\lambda \tag{2.9}$$

for all $\psi \in L^2(C(0,T))$.

Lemma 2.2 *Let $\Delta_1(d\lambda)$ be a positive measure on $\mathbb{R}^N$ with density function $\Delta_1(\lambda)$. If there exist constants $c_{2,4} > 0$ and $\eta > 0$ such that*

$$\Delta_1(\lambda) \ge \frac{c_{2,4}}{\|\lambda\|^\eta} \quad \text{for all } \lambda \in \mathbb{R}^N \text{ with } \|\lambda\| \text{ large}. \tag{2.10}$$

Then for any constants $T > 0$ and $c_{2,5}$, there exists a positive and finite constant $c_{2,6}$ such that for all functions g of the form

$$g(\lambda) = \sum_{j=1}^n a_j (e^{i\langle s^j, \lambda\rangle} - 1), \tag{2.11}$$

where $a_j \in \mathbb{R}$ and $s^j \in C(0,T)$, we have

$$|g(\lambda)| \le c_{2,6} \|\lambda\| \cdot \left(\int_{\mathbb{R}^N} |g(\xi)|^2\, \Delta_1(\xi) d\xi \right)^{1/2}$$

for all $\lambda \in \mathbb{R}^N$ with $\|\lambda\| \le c_{2,5}$.

Lemma 2.3 below is an extension of Proposition 8.4 of [20]. It allows us to connect the property of strong local nondeterminism of a Gaussian random field with a general spectral measure to that of a Gaussian random field with an absolutely continuous spectral measure, which has been studied in [31, 32].

Lemma 2.3 *Let $\Delta_2(d\lambda)$ be a positive measure on $\mathbb{R}^N$ and suppose that for some $h > 0$,*

$$0 < \liminf_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_2(C(\lambda,h)) \le \limsup_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_2(C(\lambda,h)) < \infty. \tag{2.12}$$

Then for any constant $T > 0$ with $ThN < \log 2$, there exist positive and finite constants $c_{2,7}$ and $c_{2,8}$ such that

$$\begin{aligned} c_{2,7}\int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda &\le \int_{\mathbb{R}^N} |g(\lambda)|^2\Delta_2(d\lambda) \\ &\le c_{2,8}\int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda \end{aligned} \tag{2.13}$$

for all $g(\lambda)$ of the form (2.11).

Proof First we claim that there is a positive constant $c \le 1$ such that

$$\begin{aligned} c\int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda &\le \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(1+\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda \\ &\le \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda \end{aligned} \tag{2.14}$$

for all functions g of the form (2.11).

Clearly only the first inequality in (2.14) needs a proof. For this purpose, we split the first integral in (2.14) over $\{\lambda : \|\lambda\| \le c_{2,5}\}$ and $\{\lambda : \|\lambda\| > c_{2,5}\}$ and apply Lemma 2.2 with

$$\Delta_1(d\lambda) = \frac{d\lambda}{(1+\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}$$

[which satisfies (2.10)] to derive

$$\begin{aligned} &\int_{\{\|\lambda\|\le c_{2,5}\}} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda \\ &\quad\le c_{2,6}^2\int_{\{\|\lambda\|\le c_{2,5}\}} \frac{\|\lambda\|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\,d\lambda\cdot\int_{\mathbb{R}^N} |g(\xi)|^2\,\Delta_1(d\xi) \\ &\quad= c_{2,9}\int_{\mathbb{R}^N} |g(\xi)|^2\,\Delta_1(d\xi), \end{aligned}$$

because the first integral in the second line is convergent. It follows from the above that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda &\le c_{2,9} \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(1+\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda \\ &\quad + \int_{\{\lambda:\|\lambda\|>c_{2,5}\}} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda \\ &\le c_{2,10} \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(1+\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda.\end{aligned}$$

This verifies the first inequality in (2.14).

Next we take a constant $s>0$ such that $(T+s)hN<\log 2$ and denote $T_1 = T+s$. Let $\varphi \in L^2(C(0,s))$ be a function with the following property

$$c_{2,11} \le |\widehat{\varphi}(\lambda)|^2 \cdot (1+\rho(0,\lambda))^{Q+2} \le c_{2,12} \tag{2.15}$$

for all $\lambda \in \mathbb{R}^N$, where $c_{2,11}$ and $c_{2,12}$ are positive and finite constants. Such a function φ can be constructed as follows. Observe that the function $\lambda \mapsto (1+\rho(0,\lambda))^{-(Q+2)/2}$ is in $L^2(\mathbb{R}^N)$. Hence it is the Fourier transform of a function $\kappa \in L^2(\mathbb{R}^N)$. For the constant $s>0$ chosen above we consider the function

$$P_s(t) = \prod_{j=1}^N \left(1 - \frac{|t_j|}{s}\right)^+ \quad \text{for all } t \in \mathbb{R}^N,$$

where $a^+ := \max(a,0)$ for all real numbers a. Then the support of P_s is $C(0,s)$. Recall that the Fourier transform of P_s is

$$\widehat{P_s}(\xi) := 2^N \prod_{j=1}^N \frac{1-\cos(s\xi_j)}{s\xi_j^2} \quad \text{for all } \xi \in \mathbb{R}^N.$$

Define $\varphi(t) = \kappa(t)P_s(t)$. Then $\varphi \in L^1(C(0,s)) \cap L^2(C(0,s))$ and its Fourier transform is given by

$$\begin{aligned}\widehat{\varphi}(\lambda) &= \widehat{\kappa} \star \widehat{P_s}(\lambda) \\ &= \int_{\mathbb{R}^N} \frac{2^N}{(1+\rho(0,\lambda-\xi))^{(Q+2)/2}} \prod_{j=1}^N \frac{1-\cos(s\xi_j)}{s\xi_j^2}\, d\xi.\end{aligned}$$

It is clear that $\widehat{\varphi}(\lambda) > 0$ for all $\lambda \in \mathbb{R}^N$. Writing

$$\widehat{\varphi}(\lambda) \cdot (1+\rho(0,\lambda))^{(Q+2)/2} = \int_{\mathbb{R}^N} \frac{2^N (1+\rho(0,\lambda))^{(Q+2)/2}}{(1+\rho(0,\lambda-\xi))^{(Q+2)/2}} \prod_{j=1}^N \frac{1-\cos(s\xi_j)}{s\xi_j^2}\, d\xi$$

and using the dominated convergence theorem, we see that

$$\lim_{\|\lambda\|\to\infty} \widehat{\varphi}(\lambda)\cdot(1+\rho(0,\lambda))^{(Q+2)/2} = 2^N \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{1-\cos(s\xi_j)}{s\xi_j^2}\, d\xi.$$

Hence (2.15) follows.

Now we continue with the proof of (2.13). Let

$$\widehat{\psi}(\lambda) := g(\lambda)\widehat{\varphi}(\lambda) = \sum_{j=1}^n a_j(e^{i\langle s^j,\lambda\rangle}-1)\widehat{\varphi}(\lambda),$$

where $s^j \in C(0,T)$ for $j=1,\dots,n$. Since $\varphi \in L^1(C(0,s)) \cap L^2(C(0,s))$, we use the Fourier inversion formula to verify that $\psi \in L^2(C(0,T_1))$. Moreover, by (2.14) and (2.15), there is a constant $c \ge 1$ such that

$$c^{-1}\int_{\mathbb{R}^N} |g(\lambda)\widehat{\varphi}(\lambda)|^2 d\lambda \le \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}} d\lambda \le c\int_{\mathbb{R}^N} |g(\lambda)\widehat{\varphi}(\lambda)|^2 d\lambda \tag{2.16}$$

for all functions g of the form (2.11).

Consider the new positive measure $\widetilde{\Delta}(d\lambda)$ on $\mathbb{R}^N$ defined by $\widetilde{\Delta}(d\lambda) = |\widehat{\varphi}(\lambda)|^{-2} \times \Delta_2(d\lambda)$. It follows from (2.12) and (2.15) that

$$\liminf_{\|\lambda\|\to\infty} \widetilde{\Delta}(C(\lambda,h)) \ge c \liminf_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_2(C(\lambda,h)) > 0$$

and

$$\limsup_{\|\lambda\|\to\infty} \widetilde{\Delta}(C(\lambda,h)) \le c \limsup_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_2(C(\lambda,h)) < \infty.$$

Hence the measure $\widetilde{\Delta}(d\lambda)$ satisfies (2.8). We apply Lemma 2.1 to derive that

$$\begin{aligned} c_{2,2}\int_{\mathbb{R}^N} |g(\lambda)\widehat{\varphi}(\lambda)|^2 d\lambda &\le \int_{\mathbb{R}^N} |g(\lambda)\widehat{\varphi}(\lambda)|^2 \widetilde{\Delta}(d\lambda) \\ &= \int_{\mathbb{R}^N} |g(\lambda)|^2 \Delta_2(\lambda) \le c_{2,3}\int_{\mathbb{R}^N} |g(\lambda)\widehat{\varphi}(\lambda)|^2 d\lambda. \end{aligned}$$

for all functions g of the form (2.11) provided $s^j \in C(0,T)$ for $j=1,\dots,n$. This and (2.16) yield (2.13). □

We are ready to prove the main result of this section.

Theorem 2.4 *Let $\{X_0(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and $X_0(0)=0$. If for some constant $h>0$ the spectral measure F of X_0 satisfies*

$$0 < \liminf_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}F(C(\lambda,h)) \le \limsup_{\|\lambda\|\to\infty} \rho(0,\lambda)^{Q+2}F(C(\lambda,h)) < \infty, \tag{2.17}$$

then for any $T > 0$ *such that* $ThN < \log 2$, X_0 *satisfies Conditions* (C1) *and* (C2) *on* $C(0, T)$.

Proof First we verify X_0 satisfies Condition (C1). For any $s, t \in C(0, T)$, we apply the stochastic representation of X_0 and Lemma 2.3 to write

$$\begin{aligned}\mathbb{E}(|X_0(s) - X_0(t)|^2) &= \int_{\mathbb{R}^N} |e^{i\langle s,\lambda\rangle} - e^{i\langle t,\lambda\rangle}|^2\, F(d\lambda)\\ &\asymp \int_{\mathbb{R}^N} \frac{|e^{i\langle s,\lambda\rangle} - e^{i\langle t,\lambda\rangle}|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda. \end{aligned} \tag{2.18}$$

Since it has been proved in Xiao [32] that

$$\int_{\mathbb{R}^N} \frac{|e^{i\langle s,\lambda\rangle} - e^{i\langle t,\lambda\rangle}|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}\, d\lambda \asymp \rho(s,t)^2, \quad \forall s, t \in C(0, T),$$

we conclude that X_0 satisfies (C1) on $C(0, T)$.

Now we prove that X_0 satisfies Condition (C2) on $C(0, T)$. Denote $r = \min_{0\le j\le n} \rho(u, t^j)$. It is sufficient to prove that for all $a_j \in \mathbb{R}$ $(1 \le j \le n)$ we have

$$\mathbb{E}(|X_0(u) - \sum_{j=1}^n a_j X_0(t^j)|^2) \ge c_{2,10} r^2 \tag{2.19}$$

and $c_{2,10}$ is a positive constant which is independent of n, a_j and the choice of $\{t^j\}$ and u. Again by using the stochastic representation of X_0, the left hand side of (2.19) can be written as

$$\begin{aligned}&\mathbb{E}\bigg(\bigg|X_0(u) - \sum_{j=1}^n a_j X_0(t^j)\bigg|^2\bigg)\\ &\quad= \int_{\mathbb{R}^N} \bigg|e^{i\langle u,\lambda\rangle} - 1 - \sum_{j=1}^n a_j (e^{i\langle t^j,\lambda\rangle} - 1)\bigg|^2 F(d\lambda).\end{aligned}$$

Note that the function inside the integral is of the form (2.11). We apply Lemma 2.3 to get

$$\begin{aligned}&\int_{\mathbb{R}^N} \bigg|e^{i\langle u,\lambda\rangle} - 1 - \sum_{j=1}^n a_j (e^{i\langle t^j,\lambda\rangle} - 1)\bigg|^2 F(d\lambda)\\ &\quad\ge c_{2,7} \int_{\mathbb{R}^N} \bigg|e^{i\langle u,\lambda\rangle} - 1 - \sum_{j=1}^n a_j (e^{i\langle t^j,\lambda\rangle} - 1)\bigg|^2 \frac{d\lambda}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}}.\end{aligned}$$

However, it has been proved in Theorem 3.2 of [32] that the last integral is bounded from below by $c_{2,11} r^2$, and $c_{2,11}$ is a positive constant which is independent of n, a_j and the choice of $\{t^j\}$ and u. This proves (2.19) and Theorem 2.4. □

Theorem 2.4 can be applied directly to Gaussian random fields with stationary increments and with discrete spectral measure (or of mixed form $F = F_{ac} + F_{dis}$). It is useful for analyzing many space-time Gaussian random fields in the literature; see [33] and the references therein for some examples. In the following we give an example of Gaussian random field with discrete spectral measure F.

Let $\{\xi_n, n \in \mathbb{Z}^N\}$ and $\{\eta_n, n \in \mathbb{Z}^N\}$ be two independent sequences of i.i.d. $N(0,1)$ random variables, where $\mathbb{Z}$ is the set of integers. Let $\{a_n, n \in \mathbb{Z}^N\}$ be a sequence of real numbers such that

$$\sum_{n\in\mathbb{Z}^N} a_n^2 < \infty.$$

Then

$$Y(t) = \sum_{n\in\mathbb{Z}^N} a_n(\xi_n \cos\langle n,t\rangle + \eta_n \sin\langle n,t\rangle), \quad t \in \mathbb{R}^N$$

is a centered stationary Gaussian random field with covariance function

$$\mathbb{E}(Y(t)Y(s)) = \sum_{n\in\mathbb{Z}^N} a_n^2 \cos\langle n, t-s\rangle.$$

Hence the spectral measure F of Y is supported on $\mathbb{Z}^N$ with $F(\{n\}) = a_n^2$. If we choose $\{a_n\}$ such that as $\|n\| \to \infty$,

$$a_n^2 \asymp \frac{1}{(\sum_{j=1}^N n_j^{H_j})^{Q+2}},$$

then for any fixed constant $h > 1$, F satisfies (2.17). Consider the Gaussian random field $\{X_0(t), t \in \mathbb{R}^N\}$ defined by $X_0(t) = Y(t) - Y(0)$. Theorem 2.4 implies that, for any constant $T > 0$ with $ThN < \log 2$, $\{X_0(t), t \in \mathbb{R}^N\}$ satisfies Conditions (C1) and (C2) on $C(0,T)$.

Consequently, many sample path properties of Y such as uniform and local moduli of continuity, Chung's law of the iterated logarithm, existence and joint continuity of the local times can be derived from the results in [15, 32], and [16].

Finally we mention that, as special cases of this example, we can show that the solution of a fractional stochastic heat equation on the circle $\mathbb{S}_1$ (see [18, 25]) has the property of strong local nondeterminism in the space variable (at fixed time t). Hence we can apply the results in [16, 17, 32] to obtain fine properties of the solution which improve significantly those in [18, 25]. Similarly, we can prove that the spherical fractional Brownian motion on $\mathbb{S}_1$ introduced by Istas [12] is also strongly locally nondeterministic. Details of these results will be given elsewhere.

3 Exact Hausdorff Measure Function for the Range $X([0,1]^N)$

In this section, we determine the exact Hausdorff measure function for the range of an (N,d)-Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ defined in (1.1), where

$X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field X_0 with stationary increments, which satisfies Conditions (C1) and (C2).

First we recall briefly the definition of Hausdorff measure, an upper density theorem due to [21] and two useful inequalities for large and small tails of the supremum of Gaussian processes. Then we extend a result of [23] to anisotropic Gaussian random fields, which is applied to derive an upper bound for the φ_1-Hausdorff measure of $X([0,1]^N)$. Finally we prove a law of the iterated logarithm for the sojourn time of X and derive a lower bound for the φ_1-Hausdorff measure of $X([0,1]^N)$.

3.1 Hausdorff Measure

Let Φ be the class of functions $\phi : (0,\delta) \to (0,1)$ which are right continuous, monotone increasing with $\phi(0_+) = 0$ and such that there exists a finite constant $c_{3,1} > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \le c_{3,1}, \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbb{R}^d$ is defined by

$$\phi\text{-}m(E) = \lim_{\epsilon \to 0} \inf\left\{\sum_i \phi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \epsilon\right\},$$

where $B(x,r)$ denotes the Euclidean open ball of radius r centered at x. It is known that ϕ-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is ϕ-m measurable. We say that a function ϕ is an exact Hausdorff measure function for E if $0 < \phi\text{-}m(E) < \infty$. The Hausdorff dimension of E is defined by

$$\begin{aligned}\dim E &= \inf\{\alpha > 0; s^\alpha\text{-}m(E) = 0\}\\ &= \sup\{\alpha > 0; s^\alpha\text{-}m(E) = \infty\}.\end{aligned}$$

We refer to Falconer [11] for more properties of Hausdorff measure and Hausdorff dimension.

The following lemma can be easily derived from the results in Rogers and Taylor [21], which gives a way to get a lower bound for ϕ-$m(E)$. For any Borel measure μ on $\mathbb{R}^d$ and $\phi \in \Phi$, the upper ϕ-density of μ at $x \in \mathbb{R}^d$ is defined by

$$\overline{D}^{\phi}_{\mu}(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{\phi(2r)}.$$

Lemma 3.1 *For a given $\phi \in \Phi$ there exists a positive constant $c_{3,2}$ such that for any Borel measure μ on $\mathbb{R}^d$ and every Borel set $E \subseteq \mathbb{R}^d$, we have*

$$\phi\text{-}m(E) \ge c_{3,2}\, \mu(E) \inf_{x \in E} \{\overline{D}^{\phi}_{\mu}(x)\}^{-1}.$$

Now we recall some basic facts about Gaussian processes. Consider a set S and a centered Gaussian process $\{Y(t), t \in S\}$. We provide S with the following canonical pseudo-metric

$$d(s,t) = \|Y(s) - Y(t)\|_2,$$

where $\|Y\|_2 = (\mathbb{E}(Y^2))^{1/2}$. Denote by $N_d(S,\epsilon)$ the smallest number of open d-balls of radius ϵ needed to cover S and let $D = \sup\{d(s,t) : s,t \in S\}$ be the d-diameter of S.

The following lemma is well known. It is a consequence of the Gaussian isoperimetric inequality and Dudley's entropy bound (see [23]).

Lemma 3.2 *There exists a positive constant $c_{3,3}$ such that for all $u > 0$, we have*

$$\mathbb{P}\left\{\sup_{s,t\in S} |Y(s) - Y(t)| \geq c_{3,3}\left(u + \int_0^D \sqrt{\log N_d(S,\epsilon)}d\epsilon\right)\right\} \leq \exp\left(-\frac{u^2}{D^2}\right).$$

Lemma 3.3 *Consider a function Ψ such that $N_d(S,\epsilon) \leq \Psi(\epsilon)$ for all $\epsilon > 0$. Assume that for some constant $c_{3,4} \geq 1$ and all $\epsilon > 0$ we have*

$$\Psi(\epsilon)/c_{3,4} \leq \Psi\left(\frac{\epsilon}{2}\right) \leq c_{3,4}\Psi(\epsilon).$$

Then

$$\mathbb{P}\left\{\sup_{s,t\in S} |Y(s) - Y(t)| \leq u\right\} \geq \exp\left(-c_{3,5}\Psi(u)\right),$$

where $c_{3,5} > 0$ is a constant depending only on $c_{3,4}$.

This was proved in Talagrand [22]. It gives a general lower bound for the small ball probability of Gaussian processes.

3.2 Some Basic Estimates

Let $X_0 = \{X_0(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and satisfying Conditions (C1) and (C2). Without loss of generality, we assume that $H_1, \ldots, H_N$ are ordered as

$$0 < H_1 \leq H_2 \leq \cdots \leq H_N < 1. \tag{3.1}$$

In order to solve some dependence problems that are a major obstacle, we consider for any given $0 < a < b < \infty$ the random field

$$X_0(a,b,t) = \int_{a<\rho(0,\lambda)\leq b} (e^{i\langle t,\lambda\rangle} - 1)\, W(d\lambda), \quad t \in \mathbb{R}^N.$$

An essential fact is that if $0 < a < b < a' < b' < \infty$, then the Gaussian random fields $\{X_0(a,b,t), t \in \mathbb{R}^N\}$ and $\{X_0(a',b',t), t \in \mathbb{R}^N\}$ are independent.

Let $X_1(a,b,t), \dots, X_d(a,b,t)$ be independent copies of $X_0(a,b,t)$ and let

$$X(a,b,t) = (X_1(a,b,t), \dots, X_d(a,b,t)), \quad t \in \mathbb{R}^N.$$

Then we have the following lemma. For convenience, we write $I = [0,1]^N$.

Lemma 3.4 *Given any* $0 < a < b$ *and* $0 < \epsilon < r$, *we have*

$$\mathbb{P}\left\{ \sup_{t \in I: \rho(0,t) \le r} \|X(a,b,t)\| \le \epsilon \right\} \ge \exp\left(-c \left(\frac{r}{\epsilon}\right)^Q \right), \tag{3.2}$$

where $0 < c < \infty$ *is an absolute constant.*

Proof It is sufficient to prove (3.2) for $X_0(a,b,t)$. Let $S = \{t \in I : \rho(0,t) \le r\}$ and define a distance d on S by

$$d(s,t) = \|X_0(a,b,s) - X_0(a,b,t)\|_2.$$

Then (C1) implies $d(s,t) \le c_{1,1} \sum_{i=1}^N |s_i - t_i|^{H_i}$ for all $s,t \in I$, independent of the choices of $0 < a < b$. It follows that

$$N_d(S,\epsilon) \le c \left(\frac{r}{\epsilon}\right)^Q.$$

By Lemma 3.3 we have

$$\mathbb{P}\left\{ \sup_{t \in I: \rho(0,t) \le r} |X_0(a,b,t)| \le \epsilon \right\} \ge \exp\left(-c \left(\frac{r}{\epsilon}\right)^Q \right).$$

This proves Lemma 3.4. □

The following truncation inequalities are extensions of those in [14, p. 209] for $N = 1$ and (3.4) and (3.5) in [28] for $N > 1$ and ρ being replaced by the Euclidean metric.

Lemma 3.5 *There exist positive finite constants* $c_{3,6}$ *and* $c_{3,7}$ *such that the following hold.*

(i) *For any* $a > 0$ *and any* $t \in \mathbb{R}^N$ *with* $\rho(0,t)a \le 1/N$ *we have*

$$\int_{\{\lambda: \rho(0,\lambda) \le a\}} \langle t, \lambda \rangle^2 F(d\lambda) \le c_{3,6} \int_{\mathbb{R}^N} (1 - \cos\langle t, \lambda \rangle) F(d\lambda). \tag{3.3}$$

(ii) *For all* $a > 0$

$$\int_{\{\lambda: \rho(0,\lambda) > a\}} F(d\lambda) \le c_{3,7} a^{-2}. \tag{3.4}$$

Proof Notice that when $\rho(0,\lambda) \le a$, the condition $\rho(0,t)a \le 1/N$ implies that $|\langle t,\lambda\rangle| < 1$. It follows that

$$1-\cos\langle t,\lambda\rangle \ge \frac{\langle t,\lambda\rangle^2}{2}\left(1-\frac{\langle t,\lambda\rangle^2}{12}\right) \ge \frac{11}{24}\langle t,\lambda\rangle^2.$$

Then for any $t \in \mathbb{R}^N$ with $\rho(0,t)a \le 1/N$ we have

$$\begin{aligned}\int_{\mathbb{R}^N}(1-\cos\langle t,\lambda\rangle)F(d\lambda) &\ge \frac{11}{24}\int_{\{\lambda:|\langle t,\lambda\rangle|\le 1\}}\langle t,\lambda\rangle^2 F(d\lambda)\\ &\ge \frac{11}{24}\int_{\{\lambda:\rho(0,\lambda)\le a\}}\langle t,\lambda\rangle^2 F(d\lambda).\end{aligned}$$

That is

$$\int_{\{\lambda:\rho(0,\lambda)\le a\}}\langle t,\lambda\rangle^2 F(d\lambda) \le \frac{24}{11}\int_{\mathbb{R}^N}(1-\cos\langle t,\lambda\rangle)F(d\lambda).$$

To prove (3.4), we make the following two claims:

(a) For any $u>0$, if $\lambda_i \ne 0$ for $i=1,\dots,N$, then

$$\frac{1}{2^N u^Q}\int_{\prod_{i=1}^N[-u^{\frac{1}{H_i}},u^{\frac{1}{H_i}}]}\cos\langle t,\lambda\rangle dt = \prod_{i=1}^N \frac{\sin(u^{\frac{1}{H_i}}\lambda_i)}{u^{\frac{1}{H_i}}\lambda_i}.$$

(b) For any $u>0$,

$$\int_{\{\lambda:\rho(0,\lambda)>\frac{1}{u}\}}F(d\lambda) \le \frac{c}{2^N u^Q}\int_{\prod_{i=1}^N[-u^{\frac{1}{H_i}},u^{\frac{1}{H_i}}]}dt\int_{\mathbb{R}^N}(1-\cos\langle t,\lambda\rangle)F(d\lambda).$$

Claim (a) is obviously true when $N=1$. Suppose it is true for $N=k$, then for $N=k+1$, we have

$$\begin{aligned}&\frac{1}{2^{k+1}u^{\frac{1}{H_1}+\cdots+\frac{1}{H_{k+1}}}}\int_{\prod_{i=1}^k[-u^{\frac{1}{H_i}},u^{\frac{1}{H_i}}]}dt_1\cdots dt_k\\ &\quad\times\int_{[-u^{\frac{1}{H_{k+1}}},u^{\frac{1}{H_{k+1}}}]}\cos(t_1\lambda_1+\cdots+t_{k+1}\lambda_{k+1})dt_{k+1}\\ &=\frac{1}{2^k u^{\frac{1}{H_1}+\cdots+\frac{1}{H_k}}}\int_{\prod_{i=1}^k[-u^{\frac{1}{H_i}},u^{\frac{1}{H_i}}]}dt_1\cdots dt_k\\ &\quad\times\frac{\sin(t_1\lambda_1+\cdots+t_k\lambda_k+u^{\frac{1}{H_{k+1}}}\lambda_{k+1})-\sin(t_1\lambda_1+\cdots+t_k\lambda_k-u^{\frac{1}{H_{k+1}}}\lambda_{k+1})}{2u^{\frac{1}{H_{k+1}}}\lambda_{k+1}}\end{aligned}$$

$$= \frac{1}{2^k u^{\frac{1}{H_1}+\cdots+\frac{1}{H_k}}} \int_{\prod_{i=1}^k [-u^{\frac{1}{H_i}}, u^{\frac{1}{H_i}}]} \cos(t_1\lambda_1 + \cdots + t_k\lambda_k) dt_1 \cdots dt_k \frac{\sin u^{\frac{1}{H_{k+1}}} \lambda_{k+1}}{u^{\frac{1}{H_{k+1}}} \lambda_{k+1}}$$

$$= \frac{\sin u^{\frac{1}{H_1}} \lambda_1}{u^{\frac{1}{H_1}} \lambda_1} \cdots \frac{\sin u^{\frac{1}{H_{k+1}}} \lambda_{k+1}}{u^{\frac{1}{H_{k+1}}} \lambda_{k+1}}.$$

Hence claim (a) is true for all $N \geq 1$.

By Fubini's theorem and claim (a), we have

$$\begin{aligned}
&\frac{1}{2^N u^Q} \int_{\prod_{i=1}^N [-u^{\frac{1}{H_i}}, u^{\frac{1}{H_i}}]} dt \int_{\mathbb{R}^N} (1 - \cos\langle t, \lambda\rangle) F(d\lambda) \\
&\quad = \int_{\mathbb{R}^N} \left[\frac{1}{2^N u^Q} \int_{\prod_{i=1}^N [-u^{\frac{1}{H_i}}, u^{\frac{1}{H_i}}]} (1 - \cos\langle t, \lambda\rangle) dt \right] F(d\lambda) \\
&\quad = \int_{\mathbb{R}^N} \left(1 - \prod_{i=1}^N \frac{\sin u^{\frac{1}{H_i}} \lambda_i}{u^{\frac{1}{H_i}} \lambda_i} \right) F(d\lambda) \\
&\quad \geq \int_{\mathbb{R}^N \setminus \{\lambda : |\lambda_i| \leq (Nu)^{-\frac{1}{H_i}}, \forall i\}} \left(1 - \prod_{i=1}^N \frac{\sin u^{\frac{1}{H_i}} \lambda_i}{u^{\frac{1}{H_i}} \lambda_i} \right) F(d\lambda) \\
&\quad \geq c \int_{\mathbb{R}^N \setminus \{\lambda : |\lambda_i| \leq (Nu)^{-\frac{1}{H_i}}, \forall i\}} F(d\lambda) \\
&\quad \geq c \int_{\{\lambda : \rho(0,\lambda) > \frac{1}{u}\}} F(d\lambda).
\end{aligned}$$

Hence claim (b) is verified.

Now we turn to the proof of (3.4). With claim (b), (2.4) and Condition (C1) in hand, we have for $a > 0$,

$$\begin{aligned}
\int_{\{\lambda : \rho(0,\lambda) > a\}} F(d\lambda) &\leq \frac{c a^Q}{2^N} \int_{\prod_{i=1}^N [-a^{-\frac{1}{H_i}}, a^{-\frac{1}{H_i}}]} dt \int_{\mathbb{R}^N} (1 - \cos\langle t, \lambda\rangle) F(d\lambda) \\
&\leq \frac{c a^Q}{2^N} \int_{\prod_{i=1}^N [-a^{-\frac{1}{H_i}}, a^{-\frac{1}{H_i}}]} \sum_{i=1}^N |t_i|^{2H_i} dt \\
&\leq c a^{-2}.
\end{aligned}$$

This finishes the proof of Lemma 3.5. □

Lemma 3.6 gives estimates on the small ball probability of the (N, d)-Gaussian random field X in (1.1).

Lemma 3.6 *There exist constants $c_{3,8}$ and $c_{3,9}$ such that for all* $0 < \epsilon < r$,

$$\exp\left(-c_{3,8}\left(\frac{r}{\epsilon}\right)^Q\right) \le \mathbb{P}\left\{\sup_{t\in I:\rho(0,t)\le r} \|X(t)\| \le \epsilon\right\} \le \exp\left(-c_{3,9}\left(\frac{r}{\epsilon}\right)^Q\right). \tag{3.5}$$

Proof Let $S = \{t \in I : \rho(0,t) \le r\}$. It follows from (C1) that for all $\epsilon \in (0,r)$,

$$N_\rho(S,\epsilon) \le c \prod_{i=1}^N \left(\frac{r}{\epsilon}\right)^{\frac{1}{H_i}} = c\left(\frac{r}{\epsilon}\right)^Q := \psi(\epsilon).$$

Clearly $\psi(\epsilon)$ satisfies the condition in Lemma 3.3. Hence the lower bound in (3.5) follows from Lemma 3.3.

The proof of the upper bound in (3.5) is based on Condition (C2) and a conditioning argument and is similar to the proof of Theorem 5.1 in [32] (see also [17]). We include it for the sake of completeness. Let $T = \prod_{i=1}^N [0, (\frac{r}{N})^{\frac{1}{H_i}}]$. Then $T \subseteq S$. We divide T into

$$\ell := \prod_{i=1}^N \left(\left\lfloor \left(\frac{r}{N\epsilon}\right)^{\frac{1}{H_i}} \right\rfloor + 1\right) \ge \left(\frac{r}{N\epsilon}\right)^Q$$

sub-rectangles of side-lengths ϵ^{1/H_i} $(i = 1,\dots,N)$, where $\lfloor x \rfloor$ is the largest integer no more than x. And denote the lower-left vertices of these rectangles (in any order) by t_k $(k = 1,\dots,\ell)$. Then

$$\mathbb{P}\left\{\sup_{t\in S}\|X(t)\| \le \epsilon\right\} \le \mathbb{P}\left\{\sup_{1\le k\le \ell}\|X(t_k)\| \le \epsilon\right\}. \tag{3.6}$$

It follows from Condition (C2) that for every $1 \le k \le \ell$

$$\mathrm{Var}(X_0(t_k)|X_0(t_i) : 1 \le i \le k-1) \ge c_{1,2}\,\epsilon^2.$$

By this and Anderson's inequality for Gaussian measures (see [2]), we have the following upper bound for the conditional probabilities

$$\mathbb{P}\{\|X(t_k)\| \le \epsilon | X(t_i) : 1 \le i \le k-1\} \le \Phi\left(\frac{1}{\sqrt{c_{1,2}}}\right)^d, \tag{3.7}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. It follows from (3.6) and (3.7) that

$$\mathbb{P}\left\{\sup_{t\in S}\|X(t)\| \le \epsilon\right\} \le \Phi\left(\frac{1}{\sqrt{c_{1,2}}}\right)^{\ell d} \le \exp\left(-c_{3,9}\left(\frac{r}{\epsilon}\right)^Q\right).$$

Thus we obtain the upper bound in (3.5). □

The main estimate is given in the following proposition.

Proposition 3.7 *There exist positive constants δ_1 and $c_{3,10}$ such that for any $0 < r_0 \le \delta_1$, we have*

$$\mathbb{P}\left\{\exists\, r \in [r_0^2, r_0], \sup_{t\in I:\rho(0,t)\le r} \|X(t)\| \le c_{3,10} r \left(\log\log \frac{1}{r}\right)^{-1/Q}\right\} \ge 1 - \exp\left(-\left(\log \frac{1}{r_0}\right)^{1/2}\right). \tag{3.8}$$

Proof Though the main idea of the proof is similar to the proof of Proposition 4.1 in Talagrand [23], some modifications are needed to characterize the anisotropic nature of X. Let $U > 1$ be a number whose value will be determined later. For $k \ge 0$, let $r_k = r_0 U^{-2k}$. Consider the largest integer k_0 such that

$$k_0 \le \frac{\log(1/r_0)}{2\log U}. \tag{3.9}$$

Thus, for $k \le k_0$ we have $r_0^2 \le r_k \le r_0$. It thereby suffices to prove that

$$\mathbb{P}\left\{\exists k \le k_0, \sup_{t\in I:\rho(0,t)\le r_k} \|X(t)\| \le c\, r_k \left(\log\log \frac{1}{r_k}\right)^{-1/Q}\right\} \ge 1 - \exp\left(-\left(\log \frac{1}{r_0}\right)^{1/2}\right).$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \ldots$

$$X_{0,k}(t) = X_0(a_k, a_{k+1}, t)$$

and

$$\widehat{X}_k(t) = (X_{1,k}(t), \ldots, X_{d,k}(t)),$$

where $X_{1,k}(t), \ldots, X_{d,k}(t)$ are independent copies of $X_{0,k}(t)$. Furthermore, we assume $X_1 - X_{1,k}, \ldots, X_d - X_{d,k}$ are independent copies of $X_0 - X_{0,k}$. We note that the Gaussian random fields $\widehat{X}_0, \widehat{X}_1, \ldots$ are independent. By Lemma 3.4 we can find a constant $c_{3,11} > 0$ such that, if r_0 is small enough, then for each $k \ge 0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in I:\rho(0,t)\le r_k} \|\widehat{X}_k(t)\| \le c_{3,11}\, r_k \left(\log\log \frac{1}{r_k}\right)^{-1/Q}\right\} &\ge \exp\left(-\frac{1}{4}\log\log\frac{1}{r_k}\right) = \frac{1}{(\log 1/r_k)^{\frac{1}{4}}} \\ &\ge \frac{1}{(2\log 1/r_0)^{\frac{1}{4}}}.\end{aligned} \tag{3.10}$$

By independence,

$$\begin{aligned}
&\mathbb{P}\left\{\exists k \le k_0, \sup_{t\in I:\rho(0,t)\le r_k} \|\widehat{X}_k(t)\| \le c_{3,11} r_k\left(\log\log\frac{1}{r_k}\right)^{-1/Q}\right\}\\
&\quad\ge 1-\left(1-\frac{1}{(2\log 1/r_0)^{1/4}}\right)^{k_0}\\
&\quad\ge 1-\exp\left(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\right),
\end{aligned}\tag{3.11}$$

where the last inequality follows from the elementary inequality $1-x\le e^{-x}$ for all $x\ge 0$.

Let $\beta=\min\{\frac{1}{H_N}-1,2\}$. We claim that for any $u\ge cr_kU^{-\frac{\beta}{2}}\sqrt{\log U}$,

$$\mathbb{P}\left\{\sup_{t\in I:\rho(0,t)\le r_k}\|X(t)-\widehat{X}_k(t)\|\ge u\right\}\le \exp\left(-\frac{u^2}{cr_k^2U^{-\beta}}\right).\tag{3.12}$$

To see this, it's enough to prove that (3.12) holds for $X_0-X_{0,k}$. Consider $S=\{t\in I:\rho(0,t)\le r_k\}$ and on S the distance

$$d(s,t)=\|(X_0(s)-X_{0,k}(s))-(X_0(t)-X_{0,k}(t))\|_2.$$

Then $d(s,t)\le c\sum_{i=1}^N|s_i-t_i|^{H_i}$ and $N_d(S,\epsilon)\le c(\frac{r_k}{\epsilon})^Q$. Now we estimate the diameter D of S. For any $t\in S$,

$$\begin{aligned}
\mathbb{E}[(X_0(t)-X_{0,k}(t))^2] &= 2\int_{\{\lambda:\rho(0,\lambda)\le a_k\}\cup\{\lambda:\rho(0,\lambda)>a_{k+1}\}}(1-\cos\langle t,\lambda\rangle)F(d\lambda)\\
&\le 2\int_{\{\lambda:\rho(0,\lambda)\le a_k\}}(1-\cos\langle t,\lambda\rangle)F(d\lambda)\\
&\quad+4\int_{\{\lambda:\rho(0,\lambda)>a_{k+1}\}}F(d\lambda)\\
&=: I_1+I_2.
\end{aligned}\tag{3.13}$$

The second term is easy to estimate: By Lemma 3.5,

$$I_2\le ca_{k+1}^{-2}.\tag{3.14}$$

For the first term I_1, we use the elementary inequality $1-\cos\langle t,\lambda\rangle \le \frac{1}{2}\langle t,\lambda\rangle^2$ to derive that for all $t\in S$

$$\begin{aligned}
I_1 &\le \int_{\{\lambda:\rho(0,\lambda)\le a_k\}} \langle t,\lambda\rangle^2 F(d\lambda)\\
&= N^{\frac{2}{H_1}} U^{-\frac{1}{H_N}} \int_{\{\lambda:\rho(0,\lambda)\le a_k\}} \left\langle \frac{U^{\frac{1}{2H_N}}}{N^{\frac{1}{H_1}}} t,\lambda\right\rangle^2 F(d\lambda)\\
&= cU^{-\frac{1}{H_N}} \int_{\{\lambda:\rho(0,\lambda)\le a_k\}} \langle t',\lambda\rangle^2 F(d\lambda),
\end{aligned}$$

where $t' = U^{\frac{1}{2H_N}} N^{-\frac{1}{H_1}} t$. Since $\rho(0,t') \le \frac{1}{N} U^{\frac{1}{2}} \rho(0,t) \le \frac{1}{N} U^{\frac{1}{2}} r_k < \frac{1}{Na_k}$, it follows from Lemma 3.5 and (C1) that

$$I_1 \le cU^{-\frac{1}{H_N}} \rho(0,t')^2 \le cU^{1-\frac{1}{H_N}} \rho(0,t)^2 \le cr_k^2 U^{-(\frac{1}{H_N}-1)}. \tag{3.15}$$

With (3.13), (3.14) and (3.15) in hand, the diameter of S satisfies

$$\begin{aligned}
D^2 &\le c[r_k^2 U^{-(\frac{1}{H_N}-1)} + a_{k+1}^{-2}]\\
&\le cr_k^2[U^{-(\frac{1}{H_N}-1)} + U^{-2}]\\
&\le cr_k^2 U^{-\beta},
\end{aligned}\tag{3.16}$$

where $\beta = \min\{\frac{1}{H_N}-1, 2\}$. Some simple calculations yield

$$\begin{aligned}
\int_0^D \sqrt{\log N_d(S,\epsilon)}\,d\epsilon &\le c\int_0^{cr_k U^{-\frac{\beta}{2}}} \sqrt{\log\frac{r_k}{\epsilon}}\,d\epsilon\\
&\le cr_k U^{-\frac{\beta}{2}} \sqrt{\log U}.
\end{aligned}\tag{3.17}$$

Hence we use Lemma 3.2 and (3.17) to derive that for any $u \ge cr_k U^{-\frac{\beta}{2}}\sqrt{\log U}$,

$$\mathbb{P}\left\{ \sup_{\rho(0,t)\le r_k} |X_0(t) - X_{0,k}(t)| \ge u \right\} \le \exp\left(-\frac{u^2}{cr_k^2 U^{-\beta}}\right). \tag{3.18}$$

Thus we have proved (3.12).

Now we continue our proof of (3.8). Let $U = (\log 1/r_0)^{1/\beta}$. We see that for $r_0 > 0$ small

$$U^{\beta/2}(\log U)^{-1/2} \ge \left(\log\log\frac{1}{r_0}\right)^{1/Q}.$$

Take $u = c_{3,11} r_k (\log\log 1/r_0)^{-1/Q}$. It follows from (3.12) that

$$\mathbb{P}\left\{\sup_{t\in I:\rho(0,t)\le r_k} \|X(t) - \widehat{X}_k(t)\| \ge c_{3,11} r_k \left(\log\log \frac{1}{r_0}\right)^{-1/Q}\right\}$$
$$\le \exp\left(-\frac{U^\beta}{c_{3,12}(\log\log 1/r_0)^{2/Q}}\right).$$

Combining this with (3.11), we get

$$\begin{aligned}\mathbb{P}&\left\{\exists k \le k_0, \sup_{\rho(0,t)\le r_k} \|X(t)\| \le 2c_{3,11} r_k \left(\log\log \frac{1}{r_k}\right)^{-1/Q}\right\}\\ &\ge 1 - \exp\left(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\right)\\ &\quad - k_0 \exp\left(-\frac{U^\beta}{c_{3,12}(\log\log 1/r_0)^{2/Q}}\right).\end{aligned} \tag{3.19}$$

We recall that

$$\frac{\log(1/r_0)}{4\log U} \le k_0 \le \log\frac{1}{r_0}.$$

Then the right-hand side of (3.19) is at least $1 - \exp(-(\log 1/r_0)^{1/2})$ when $r_0 > 0$ is small enough. This completes the proof. □

3.3 Upper Bound for the Hausdorff Measure of the Range

We start with the following result on the uniform modulus of continuity of X_0. See, e.g., Xiao [32]. More precise result can be found in Meerschaert et al. [16].

Lemma 3.8 *Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}$. If Condition (C1) is satisfied, then there exists a positive and finite constant $c_{3,13}$ such that*

$$\limsup_{\|\varepsilon\|\to 0} \frac{\sup_{t\in[0,1]^N,\, s\in[0,\varepsilon]} |X_0(t+s) - X_0(t)|}{\rho(0,\varepsilon)\sqrt{\log(1+\rho(0,\varepsilon)^{-1})}} \le c_{3,13}, \quad a.s. \tag{3.20}$$

Now we derive an upper bound for the Hausdorff measure of $X([0,1]^N)$.

Theorem 3.9 *If $d > Q$, then there exists a constant $c_{3,14} > 0$ such that*

$$\varphi_1\text{-}m(X([0,1]^N)) \le c_{3,14} \quad a.s., \tag{3.21}$$

where $\varphi_1(r) = r^Q \log\log 1/r$.

Proof For $k \geq 1$, consider the set

$$R_k = \left\{ t \in [0,1]^N : \exists\, r \in [2^{-2k}, 2^{-k}] \text{ such that} \right.$$
$$\left. \sup_{s\in I:\rho(s,t)\le r} \|X(s) - X(t)\| \le c_{3,10}\, r \left(\log\log \frac{1}{r}\right)^{-1/Q} \right\}. \tag{3.22}$$

By Proposition 3.7 we have

$$\mathbb{P}\{t \in R_k\} \ge 1 - \exp(-\sqrt{k/2}).$$

Denote by L_N the Lebesgue measure on $\mathbb{R}^N$. It follows from Fubini's theorem that $\mathbb{P}(\Omega_0) = 1$, where

$$\Omega_0 = \{\omega :\ L_N(R_k) \ge 1 - \exp(-\sqrt{k}/4) \text{ infinitely often}\}.$$

On the other hand, by Lemma 3.8, there exists an event Ω_1 such that $\mathbb{P}(\Omega_1) = 1$ and for all $\omega \in \Omega_1$, there exists $n_1 = n_1(\omega)$ large enough such that for all $n \ge n_1$ and any rectangle I_n of side-lengths $2^{-n/H_i}$ $(i = 1, \dots, N)$ that meets $[0,1]^N$, we have

$$\sup_{s,t\in I_n} \|X(t) - X(s)\| \le c2^{-n}\sqrt{\log[1 + (N2^{-n})^{-1}]} \le c2^{-n}\sqrt{n}. \tag{3.23}$$

Now for a fixed $\omega \in \Omega_0 \cap \Omega_1$, we show that φ_1-$m(X([0,1]^N)) \le c_{3,14} < \infty$. Consider $k \ge 1$ such that

$$L_N(R_k) \ge 1 - \exp(-\sqrt{k}/4).$$

For any $n \ge 1$, we divide $[0,1]^N$ into 2^{nQ} disjoint (half-open and half closed) rectangles of side-lengths $2^{-n/H_i}$ $(i = 1, \dots, N)$. Denote by $I_n(x)$ the rectangle of side-lengths $2^{-n/H_i}$ $(i = 1, \dots, N)$ containing x. For any $x \in R_k$ we can find the smallest integer n with $k \le n \le 2k + \ell_0$ (where ℓ_0 depends on N only) such that

$$\sup_{s,t\in I_n(x)} \|X(t) - X(s)\| \le c2^{-n}(\log\log 2^n)^{-1/Q}. \tag{3.24}$$

Thus we have

$$R_k \subseteq V = \bigcup_{n=k}^{2k+\ell_0} V_n$$

and each V_n is a union of rectangles $I_n(x)$ satisfying (3.24). Clearly $X(I_n(x))$ can be covered by a ball of radius

$$\rho_n = c2^{-n}(\log\log 2^n)^{-1/Q}.$$

Since $\varphi_1(2\rho_n) \le c2^{-nQ} = cL_N(I_n)$, we obtain

$$\sum_{n=k}^{k+\ell_0} \sum_{I_n\in V_n} \varphi_1(2\rho_n) \le \sum_{n} \sum_{I_n\in V_n} cL_N(I_n) = cL_N(V) \le c. \tag{3.25}$$

Thus $X(V)$ is contained in the union of a family of balls B_n of radius ρ_n with $\sum_n \varphi_1(2\rho_n) \le c$.

On the other hand, $[0,1]^N \backslash V$ is contained in a union of rectangles of side-lengths $2^{-q/H_i}$ $(i = 1, \dots, N)$ where $q = 2k + \ell_0$, none of which meets R_k. There can be at most

$$2^{Qq} L_N([0,1]^N \backslash V) \le c 2^{Qq} \exp(-\sqrt{k}/4)$$

such rectangles. Since $\omega \in \Omega_1$, (3.23) implies that, for each of these rectangles I_q, $X(I_q)$ is contained in a ball of radius $c2^{-q}\sqrt{q}$. Thus $X([0,1]^N \backslash V)$ can be covered by a family B_n of balls of radius $\rho_n = c2^{-q}\sqrt{q}$ such that

$$\sum_n \varphi_1(2\rho_n) \le (c 2^{Qq} \exp(-\sqrt{k}/4)) \cdot (c 2^{-qQ} q^{Q/2} \log\log(c 2^q q^{-1/2})) \le 1 \qquad (3.26)$$

for k large enough. Since k can be arbitrarily large, Theorem 3.9 follows from (3.25) and (3.26). $\square$

3.4 Lower Bound for the Hausdorff Measure of the Range

Theorem 3.10 *If $d > Q$, then there exists a constant $c_{3,15} > 0$ such that*

$$\varphi_1\text{-}m(X([0,1]^N)) \ge c_{3,15} \quad a.s., \qquad (3.27)$$

where $\varphi_1(r) = r^Q \log\log 1/r$.

In order to prove Theorem 3.10, we first study the asymptotic behavior of the sojourn measure of X. For any $r > 0$ and $y \in \mathbb{R}^d$, define

$$T_y(r) = \int_I \mathbf{1}_{\{\|X(t) - y\| \le r\}}\, dt,$$

the sojourn time of X in the ball $B(y,r)$. If $y = 0$, we write $T(r)$ for $T_0(r)$.

Lemma 3.11 *If $d > Q$, then there is a finite constant $c_{3,16}$ such that*

$$\mathbb{E}(T(r)^n) \le c_{3,16}^n n! r^{Qn} \qquad (3.28)$$

for all for all integers $n \ge 1$ and $0 < r < 1$.

Proof For $n=1$, by Fubini's theorem and (C1) we have

$$\begin{aligned}
\mathbb{E}(T(r)) &= \int_I \mathbb{P}\{\|X(t)\| < r\}dt \\
&\le \int_I \min\{1, c\Big(\frac{r}{\rho(0,t)}\Big)^d\}dt \\
&= \int_{\{t\in I:\rho(0,t)\le cr\}} dt + c\int_{\{t\in I:\rho(0,t)>cr\}} \Big(\frac{r}{\rho(0,t)}\Big)^d dt \\
&=: J_1 + J_2.
\end{aligned}$$

The first term is easy to estimate:

$$J_1 \le c\prod_{i=1}^N r^{\frac{1}{H_i}} = cr^Q. \tag{3.29}$$

For the second term, we use the following elementary fact: Given positive constants β and γ, there exists a finite constant $c_{3,17}$ such that for all $a>0$,

$$\int_0^\infty \frac{dx}{(a+x^\beta)^\gamma} = \begin{cases} c_{3,17}a^{-(\gamma-\frac{1}{\beta})} & \text{if } \beta\gamma > 1, \\ +\infty & \text{if } \beta\gamma \le 1. \end{cases} \tag{3.30}$$

Since $\rho(0,t) > cr$ implies that $t_{j_0} \ge cr^{1/H_{j_0}}$ for some $j_0 \in \{1,\dots,N\}$, without loss of generality we assume $j_0 = 1$. Then using (3.30) $(N-1)$ times, we obtain

$$\begin{aligned}
J_2 &\le cr^d \int_{cr^{\frac{1}{H_1}}}^1 dt_1 \int_{[0,1]^{N-1}} \frac{dt_2,\dots,dt_N}{(\sum_{i=1}^N t_i^{H_i})^d} \\
&\le cr^d \int_{cr^{\frac{1}{H_1}}}^1 dt_1 \int_{[0,1]^{N-2}} \frac{dt_2,\dots,dt_{N-1}}{(\sum_{i=1}^{N-1} t_i^{H_i})^{d-\frac{1}{H_N}}} \\
&\le cr^d \int_{cr^{\frac{1}{H_1}}}^1 \frac{dt_1}{(t_1^{H_1})^{d-\sum_{i=2}^N \frac{1}{H_i}}} \\
&\le c\,r^Q,
\end{aligned} \tag{3.31}$$

where the last step follows from the assumption that $d > Q$. It follows from (3.29) and (3.31) that

$$\mathbb{E}(T(r)) \le cr^Q. \tag{3.32}$$

For $n \ge 2$,

$$\mathbb{E}(T(r)^n) = \int_{I^n} \mathbb{P}\{\|X(t^j)\| < r, 1 \le j \le n\}dt^1\cdots dt^n. \tag{3.33}$$

Consider $t^1, \dots, t^n \in I$ satisfying

$$t^j \neq 0, \quad \text{for } j = 1, \dots, n \quad \text{and} \quad t^j \neq t^k \quad \text{for } j \neq k.$$

By Condition (C2), we have

$$\mathrm{Var}(X_0(t^n)|X_0(t^1), \dots, X_0(t^{n-1})) \geq c_{1,2} \min_{0 \leq k \leq n-1} \rho(t^n, t^k)^2, \tag{3.34}$$

where $t^0 = 0$. Since conditional distributions in Gaussian processes are still Gaussian, (3.34) and Anderson's inequality yield that for all $x^1, \dots, x^{n-1} \in \mathbb{R}^d$,

$$\begin{aligned} &\mathbb{P}\{\|X(t^n)\| < r | X(t^1) = x^1, \dots, X(t^{n-1}) = x^{n-1}\} \\ &\quad \leq c \min\left\{1, \left(\frac{r}{\min\limits_{0 \leq k \leq n-1} \rho(t^n, t^k)}\right)^d\right\}. \end{aligned} \tag{3.35}$$

It follows from (3.35) and an argument similar to the proof of (3.32) that

$$\begin{aligned} &\int_I \mathbb{P}\{\|X(t^n)\| < r | X(t^1) = x^1, \dots, X(t^{n-1}) = x^{n-1}\} dt^n \\ &\quad \leq c \int_I \sum_{k=0}^{n-1} \min\left\{1, c\left(\frac{r}{\rho(t^n, t^k)}\right)^d\right\} dt^n \\ &\quad \leq c\, n \int_I \min\left\{1, c\left(\frac{r}{\rho(0, t^n)}\right)^d\right\} dt^n \\ &\quad \leq c\, n r^Q. \end{aligned} \tag{3.36}$$

Combining (3.33) and (3.36), we obtain

$$\begin{aligned} \mathbb{E}(T(r)^n) &\leq cnr^Q \int_{I^{n-1}} \mathbb{P}\{\|X(t^1)\| < r, \dots, \|X(t^{n-1})\| < r\} dt^1 \cdots dt^{n-1} \\ &= cnr^Q \mathbb{E}(T(r)^{n-1}). \end{aligned}$$

Hence the inequality (3.28) follows from this and induction. □

Let $0 < b < 1/c_{3,16}$. Then by (3.28) we have

$$\mathbb{E}(\exp(br^{-Q} T(r))) \leq \sum_{n=0}^{\infty} (c_{3,16} b)^n < \infty. \tag{3.37}$$

This and the exponential Chebychev's inequality imply that for any constant $0 < b < 1/c_{3,16}$,

$$\mathbb{P}\{T(r) \geq r^Q u\} \leq \frac{e^{-bu}}{1 - c_{3,16} b} \tag{3.38}$$

for all $u > 0$.

The following is a law of the iterated logarithm for the sojourn measure of X.

Proposition 3.12 *For every $\tau \in I$, we have*

$$\limsup_{r \to 0} \frac{T_{X(\tau)}(r)}{\varphi_1(r)} \le c_{3,16}, \quad a.s. \tag{3.39}$$

Proof Since $\{X(t), t \in \mathbb{R}^N\}$ has stationary increments, it is sufficient to consider $\tau = 0$. Then (3.39) follows from (3.38), the Borel-Cantelli lemma and a monotonicity argument in a standard way. □

Proof of Theorem 3.10 We can prove this theorem by using Lemma 3.1 and Proposition 3.12, in the same way as that of Theorem 4.1 in [28]. □

Proof of Theorem 1.1 It follows immediately from Theorems 3.9 and 3.10. □

Acknowledgements This paper was written while Nana Luan was visiting Department of Statistics and Probability, Michigan State University (MSU) with the support of a grant from China Scholarship Council (CSC). She thanks MSU for the good working condition and CSC for the financial support.

The authors thank the referees for their careful reading of the manuscript and their helpful comments.

References

1. Adler, R.J.: The Geometry of Random Fields. Wiley, New York (1981)
2. Anderson, T.W.: The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Am. Math. Soc. **6**, 170–176 (1955)
3. Ayache, A., Xiao, Y.: Asymptotic growth properties and Hausdorff dimension of fractional Brownian sheets. J. Fourier Anal. Appl. **11**, 407–439 (2005)
4. Baraka, D., Mountford, T.: A law of iterated logarithm for fractional Brownian motions. In: Séminaire de Probabilités XLI. Lecture Notes in Math. vol. 1934, pp. 161–179. Springer, Berlin (2008)
5. Baraka, D., Mountford, T.: The exact Hausdorff measure of the zero set of fractional Brownian motion. J. Theor. Probab. **24**, 271–293 (2011)
6. Benassi, A., Jaffard, S., Roux, D.: Elliptic Gaussian random processes. Rev. Mat. Iberoam. **13**, 19–90 (1997)
7. Berg, C., Forst, G.: Potential Theory on Locally Compact Abelian Groups. Springer, New York (1975)
8. Berman, S.M.: Local nondeterminism and local times of Gaussian processes. Indiana Univ. Math. J. **23**, 69–94 (1973)
9. Berman, S.M.: Spectral conditions for local nondeterminism. Stoch. Process. Appl. **27**, 73–84 (1988)
10. Biermé, H., Lacaux, C., Xiao, Y.: Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. Bull. Lond. Math. Soc. **41**, 253–273 (2009)
11. Falconer, K.J.: Fractal Geometry—Mathematical Foundations and Applications. Wiley, New York (1990)
12. Istas, J.: Spherical and hyperbolic fractional Brownian motion. Electron. Commun. Probab. **10**, 254–262 (2005)
13. Kahane, J.-P.: Some Random Series of Functions, 2nd edn. Cambridge University Press, Cambridge (1985)
14. Loéve, L.: Probability Theory I. Springer, New York (1977)
15. Luan, N., Xiao, Y.: Chung's law of the iterated logarithm for anisotropic Gaussian random fields. Stat. Probab. Lett. **80**, 1886–1895 (2010)
16. Meerschaert, M.M., Wang, W., Xiao, Y.: Fernique-type inequalities and moduli of continuity of anisotropic Gaussian random fields. Trans. Amer. Math. Soc. (2011, to appear)
17. Monrad, D., Rootzén, H.: Small values of Gaussian processes and functional laws of the iterated logarithm. Probab. Theory Relat. Fields **101**, 173–192 (1995)

18. Nualart, E., Viens, F.: The fractional stochastic heat equation on the circle: time regularity and potential theory. Stoch. Process. Appl. **119**, 1505–1540 (2009)
19. Pitt, L.D.: Stationary Gaussian Markov fields on $\mathbb{R}^d$ with a deterministic component. J. Multivar. Anal. **5**, 300–311 (1975)
20. Pitt, L.D.: Local times for Gaussian vector fields. Indiana Univ. Math. J. **27**, 309–330 (1978)
21. Rogers, C.A., Taylor, S.J.: Functions continuous and singular with respect to a Hausdorff measure. Mathematika **8**, 1–31 (1961)
22. Talagrand, M.: New Gaussian estimates for enlarged balls. Geom. Funct. Anal. **3**, 502–526 (1993)
23. Talagrand, M.: Hausdorff measure of trajectories of multiparameter fractional Brownian motion. Ann. Probab. **23**, 767–775 (1995)
24. Talagrand, M.: Multiple points of trajectories of multiparameter fractional Brownian motion. Probab. Theory Relat. Fields **112**, 545–563 (1998)
25. Tindel, S., Tudor, C.A., Viens, F.: Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation. J. Funct. Anal. **217**, 280–313 (2004)
26. Wu, D., Xiao, Y.: Uniform Hausdorff dimension results for Gaussian random fields. Sci. China Ser. A **52**, 1478–1496 (2009)
27. Wu, D., Xiao, Y.: On local times of anisotropic Gaussian random fields. Commun. Stoch. Anal. **5**, 15–39 (2011)
28. Xiao, Y.: Hausdorff measure of the sample paths of Gaussian random fields. Osaka J. Math. **33**, 895–913 (1996)
29. Xiao, Y.: Hausdorff dimension of the graph of fractional Brownian motion. Math. Proc. Camb. Philos. Soc. **122**, 565–576 (1997a)
30. Xiao, Y.: Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. Probab. Theory Relat. Fields **109**, 129–157 (1997b)
31. Xiao, Y.: Strong local nondeterminism of Gaussian random fields and its applications. In: Lai, T.-L., Shao, Q.-M., Qian, L. (eds.) Asymptotic Theory in Probability and Statistics with Applications, pp. 136–176. Higher Education Press, Beijing (2007)
32. Xiao, Y.: Sample path properties of anisotropic Gaussian random fields. In: Khoshnevisan, D., Rassoul-Agha, F. (eds.) A Minicourse on Stochastic Partial Differential Equations. Lecture Notes in Math., vol. 1962, pp. 145–212. Springer, New York (2009)
33. Xue, Y., Xiao, Y.: Fractal and smoothness properties of anisotropic Gaussian models. Frontiers Math. China (2011, to appear)
34. Yaglom, A.M.: Some classes of random fields in n-dimensional space, related to stationary random processes. Theory Probab. Appl. **2**, 273–320 (1957)