

PROPAGATION OF SINGULARITIES FOR THE STOCHASTIC WAVE EQUATION

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ABSTRACT. We study the existence and propagation of singularities of the solution to a one-dimensional linear stochastic wave equation driven by an additive Gaussian noise that is white in time and colored in space. Our approach is based on a simultaneous law of the iterated logarithm and general methods for Gaussian processes.

1. INTRODUCTION

Consider the linear stochastic wave equation in one spatial dimension:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = \dot{W}(t, x), & t \geq 0, x \in \mathbb{R}, \\ u(0, x) = 0, \quad \frac{\partial}{\partial t} u(0, x) = 0, \end{cases} \quad (1.1)$$

where \dot{W} is a Gaussian noise that is white in time and colored in space with spatial covariance given by the Riesz kernel of exponent $0 < \beta < 1$, i.e.

$$\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta_0(t - s)|x - y|^{-\beta}. \quad (1.2)$$

The existence of the mild solution $u(t, x)$ of (1.1) was studied by Dalang [4], see (2.3) below. The purpose of this article is to study the existence and propagation of singularities of $\{u(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$.

In this article, singularities are associated with the law of the iterated logarithm (LIL), or the local modulus of continuity. They refer to the random points at which the process exhibits local oscillations that are much larger than those given by the LIL. For Brownian motion, this phenomenon was first studied by Orey and Taylor [15]. It is well known that at a fixed time point, the local oscillation of a Brownian sample path satisfies the LIL almost surely. However, it is not true that the LIL holds simultaneously for all points with probability 1. Indeed, Orey and Taylor [15] show that, according to Lévy's uniform modulus of continuity, one can find random points at which the LIL fails and the oscillation is exceptionally large. Therefore these exceptional points may be defined as singularities. Similarly, singularities can be defined for other stochastic processes and, more generally, random fields.

The singularities of the Brownian sheet and the one-dimensional stochastic wave equation driven by the space-time white noise were studied by Walsh [21, 22], and the case of semi-fractional Brownian sheet was studied by Blath and Martin [2]. Based on a simultaneous law of the iterated logarithm, Walsh [21] showed that the singularities of the Brownian sheet propagate parallel to the coordinate axis. Moreover, Walsh [22] established an interesting relation between the Brownian sheet and the solution $u(t, x)$ to (1.1) driven by the space-time

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white noise. Indeed, it follows from Theorem 3.1 in [22] that the solution can be decomposed into three components:

$$u(t, x) = \frac{1}{2} \left[B\left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\right) + \hat{W}\left(\frac{t-x}{\sqrt{2}}, 0\right) + \hat{W}\left(0, \frac{t+x}{\sqrt{2}}\right) \right], \quad (1.3)$$

where the main component B is a Brownian sheet and \hat{W} is the modified Brownian sheet defined in Chapter 1 of Walsh [22], and the processes $\{B(s, t), s, t \geq 0\}$, $\{\hat{W}(s, 0), s \geq 0\}$ and $\{\hat{W}(0, t), t \geq 0\}$ are independent. This relation implies that the singularities of $u(t, x)$ propagate along the characteristic lines $t - x = c$ and $t + x = c$.

The study of singularities of the solution to (1.1) driven by the space-time white noise in [21, 22] was later extended by Carmona and Nualart [3] to one-dimensional nonlinear stochastic wave equations driven by the space-time white noise on the whole line \mathbb{R} and on a bounded interval, respectively. Their method is based on the general theory of semimartingales and two-parameter strong martingales. They showed that, in the white noise case, their solution $X(t, x)$ has the following important properties:

- (i). For any $x \in \mathbb{R}$, $\{X(\frac{h}{\sqrt{2}}, x + \frac{h}{\sqrt{2}}), h \geq 0\}$ is a continuous semimartingale (cf, [3, p.741]).
- (ii). The increments of $X(t, x)$ over a certain class of rectangles form a two-parameter strong martingale (cf, [3, p.740]).

In [3], Carmona and Nualart proved the law of the iterated logarithm for a semimartingale by the LIL of Brownian motion and a time change. They also proved that, for a class of two-parameter strong martingales, the law of the iterated logarithm in one variable holds simultaneously for all values of the other variable. By applying these results and properties (i) and (ii), Carmona and Nualart proved the existence and propagation of singularities of the solution in [3, Theorem 3.1].

The main objective of this article is to study the existence and propagation of singularities of the solution to the linear stochastic wave equation (1.1) driven by a Gaussian noise that is white in time and colored in space with spatial covariance given by (1.2) with $0 < \beta < 1$. In this case, the solution is a mean zero Gaussian random field that shares some similarity with the fractional Brownian sheet, but it seems to us that there is not a natural relation like (1.3) between the solution and the fractional Brownian sheet. Also, by applying the computation in the proofs of Lemmas 2.3 and 2.4 below, one can verify that, when $0 < \beta < 1$, the solution $u(t, x)$ does not have the aforementioned properties (i) and (ii). Consequently, the method in Carmona and Nualart [3] can not be applied. Our approach for studying the existence and propagation of singularities of the solution $u(t, x)$ is based on a simultaneous LIL that is established by applying general methods for Gaussian processes. We believe that this approach can be applied in more general settings. For example, one may study the singularities of the solutions of linear stochastic wave equations driven by more general Gaussian noises or in two spatial dimensions. One may also consider non-vanishing initial data and study the effects of both initial data and the Gaussian noise on the existence and propagation of singularities. These extensions are non-trivial and go beyond the scope of the present paper.

The rest of this article is organized as follows. First, we establish a simultaneous LIL in Propositions 2.1 and 3.1 for the solution of the linear stochastic wave equation (1.1). We prove that after a rotation, the LIL in one variable holds simultaneously for all values of the other variable. The proof consists of two parts. The upper bound for the LIL is proved in Section 2 and the corresponding lower bound is proved in Section 3. In Section 4, we define singularity for the solution and apply the simultaneous LIL to study the propagation

of singularities. The main result Theorem 4.3 shows that singularities propagate along the characteristic lines.

It would be interesting to study the existence and propagation of singularities of the solutions of the nonlinear stochastic wave equations with a Gaussian noise which is white in time and colored in space ([4]) or stochastic wave equations with a multiplicative space-time homogeneous Gaussian noise ([1]). For solving these problems, the martingale based methods of Carmona and Nualart [3] are not applicable. We believe that the Gaussian method in this article, together with an appropriate approximation argument, would be useful. However, we have not been able to find a quantitative bound on the approximation that is precise enough for our purpose. Some new argument seems to be needed.

2. SIMULTANEOUS LAW OF ITERATED LOGARITHM: UPPER BOUND

The noise in (1.1) is defined as the mean zero Gaussian process $W(\varphi)$ indexed by compactly supported smooth functions $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with covariance function

$$\begin{aligned} \mathbb{E}[W(\varphi)W(\psi)] &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \varphi(s, y) |y - y'|^{-\beta} \psi(s, y') \\ &= \frac{1}{2\pi} \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} \mu(d\xi) \mathcal{F}(\varphi(s, \cdot))(\xi) \overline{\mathcal{F}(\psi(s, \cdot))(\xi)} \end{aligned} \quad (2.1)$$

for all $\varphi, \psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, where μ is the measure whose Fourier transform is $|\cdot|^{-\beta}$ and $\mathcal{F}(\varphi(s, \cdot))(\xi)$ is the Fourier transform of the function $y \mapsto \varphi(s, y)$ in the following convention:

$$\mathcal{F}(\varphi(s, \cdot))(\xi) = \int_{\mathbb{R}} e^{-i\xi y} \varphi(s, y) dy.$$

Note that $\mu(d\xi) = C_\beta |\xi|^{-1+\beta} d\xi$, where

$$C_\beta = \frac{\pi^{1/2} 2^{1-\beta} \Gamma(\frac{1}{2} - \frac{\beta}{2})}{\Gamma(\frac{\beta}{2})},$$

see [19, p.117]. We assume that W is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Following [4, 5], for any bounded Borel set A in $\mathbb{R}_+ \times \mathbb{R}$, we can define

$$W(A) = \lim_{n \rightarrow \infty} W(\varphi_n)$$

in the sense of $L^2(\mathbb{P})$ -limit, where (φ_n) is a sequence in $C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with a compact set K such that $\text{supp } \varphi_n \subset K$ for all n and $\varphi_n \rightarrow \mathbf{1}_A$. From (2.1), it follows that for any bounded Borel sets A, B in $\mathbb{R}_+ \times \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}[W(A)W(B)] &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \mathbf{1}_A(s, y) |y - y'|^{-\beta} \mathbf{1}_B(s, y') \\ &= \frac{1}{2\pi} \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathcal{F}(\mathbf{1}_A(s, \cdot))(\xi) \overline{\mathcal{F}(\mathbf{1}_B(s, \cdot))(\xi)}. \end{aligned} \quad (2.2)$$

In dimension one, the fundamental solution of the wave equation is $\frac{1}{2} \mathbf{1}_{\{|x| \leq t\}}$, so the mild solution of (1.1) is

$$u(t, x) = \frac{1}{2} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-y| \leq t-s\}}(s, y) W(ds dy) = \frac{1}{2} W(\Delta(t, x)), \quad (2.3)$$

where $\Delta(t, x) = \{(s, y) \in \mathbb{R}_+ \times \mathbb{R} : 0 \leq s \leq t, |x - y| \leq t - s\}$. The set $\Delta(t, x)$ is indicated by the shaded region in Figure 1.

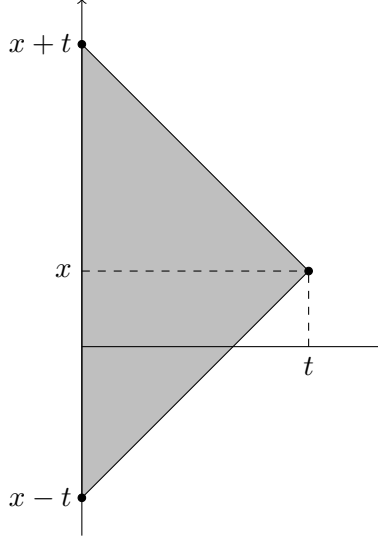


FIGURE 1

Consider a new coordinate system (τ, λ) obtained by rotating the (t, x) -coordinates by -45° . In other words,

$$(\tau, \lambda) = \left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}} \right) \quad \text{and} \quad (t, x) = \left(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}} \right).$$

For $\tau \geq 0, \lambda \geq 0$, let us denote

$$\tilde{u}(\tau, \lambda) = u\left(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\right).$$

We are going to prove a simultaneous LIL for the Gaussian random field $\{\tilde{u}(\tau, \lambda), \tau \geq 0, \lambda \geq 0\}$. This is divided into Propositions 2.1 and 3.1, where an upper bound and the corresponding lower bound for the LIL in λ , which holds simultaneously for all values of τ , are proved. By using a symmetry argument, we can also prove that the LIL in τ holds simultaneously for all λ .

Proposition 2.1. *For any $\lambda > 0$, we have*

$$\mathbb{P}\left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta} \log \log(1/h)}} \leq K_\beta \text{ for all } \tau \in [0, \infty)\right) = 1, \quad (2.4)$$

where

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)} \right)^{1/2}. \quad (2.5)$$

In order to prove (2.4), we will need several lemmas.

Lemma 2.2. *For any $0 < \beta < 1$, $a < b$ and $c < d$, we have*

$$C_\beta \int_{-\infty}^{\infty} |\mathcal{F}\mathbf{1}_{[a,b]}(\xi)|^2 \frac{d\xi}{|\xi|^{1-\beta}} = \frac{4\pi}{(2-\beta)(1-\beta)} (b-a)^{2-\beta} \quad (2.6)$$

and

$$C_\beta \int_{-\infty}^{\infty} \mathcal{F}\mathbf{1}_{[a,b]}(\xi) \overline{\mathcal{F}\mathbf{1}_{[c,d]}(\xi)} \frac{d\xi}{|\xi|^{1-\beta}} = \frac{2\pi}{(2-\beta)(1-\beta)} \left(|c-b|^{2-\beta} + |d-a|^{2-\beta} - |c-a|^{2-\beta} - |d-b|^{2-\beta} \right). \quad (2.7)$$

Proof. The Fourier transform of the function $\mathbf{1}_{[a,b]}$ is

$$\mathcal{F}\mathbf{1}_{[a,b]}(\xi) = \frac{e^{-i\xi a} - e^{-i\xi b}}{i\xi}.$$

It follows that

$$\begin{aligned} C_\beta \int_{-\infty}^{\infty} |\mathcal{F}\mathbf{1}_{[a,b]}(\xi)|^2 \frac{d\xi}{|\xi|^{1-\beta}} &= C_\beta \int_{-\infty}^{\infty} |e^{i\xi(b-a)} - 1|^2 \frac{d\xi}{|\xi|^{3-\beta}} \\ &= C_\beta (b-a)^{2-\beta} \int_{-\infty}^{\infty} |e^{i\xi} - 1|^2 \frac{d\xi}{|\xi|^{3-\beta}}. \end{aligned}$$

The last equality follows by scaling. The proof of Proposition 7.2.8 of [17] shows that

$$\int_{-\infty}^{\infty} |e^{i\xi} - 1|^2 \frac{d\xi}{|\xi|^{3-\beta}} = \frac{2\pi}{(2-\beta)\Gamma(2-\beta)\sin(\frac{\pi\beta}{2})}.$$

Also, using the relations $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1)$, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and $z\Gamma(z) = \Gamma(z+1)$ (cf. [8, p.895–896]), we can show that

$$C_\beta = \frac{2\Gamma(2-\beta)\sin(\frac{\pi\beta}{2})}{1-\beta}.$$

Hence (2.6) follows.

For the second part,

$$\int_{-\infty}^{\infty} \mathcal{F}\mathbf{1}_{[a,b]}(\xi) \overline{\mathcal{F}\mathbf{1}_{[c,d]}(\xi)} \frac{d\xi}{|\xi|^{1-\beta}} = \int_{-\infty}^{\infty} (e^{i\xi(c-a)} + e^{i\xi(d-b)} - e^{i\xi(c-b)} - e^{i\xi(d-a)}) \frac{d\xi}{|\xi|^{3-\beta}}.$$

Note that this integral is real, so we have

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{F}\mathbf{1}_{[a,b]}(\xi) \overline{\mathcal{F}\mathbf{1}_{[c,d]}(\xi)} \frac{d\xi}{|\xi|^{1-\beta}} &= \frac{1}{2} \int_{-\infty}^{\infty} (e^{i\xi(c-a)} + e^{-i\xi(c-a)} + e^{i\xi(d-b)} + e^{-i\xi(d-b)} \\ &\quad - e^{i\xi(c-b)} - e^{-i\xi(c-b)} - e^{i\xi(d-a)} - e^{-i\xi(d-a)}) \frac{d\xi}{|\xi|^{3-\beta}}. \end{aligned}$$

Since $|e^{i\xi(x-y)} - 1|^2 = 2 - e^{i\xi(x-y)} - e^{-i\xi(x-y)}$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathcal{F}\mathbf{1}_{[a,b]}(\xi) \overline{\mathcal{F}\mathbf{1}_{[c,d]}(\xi)} \frac{d\xi}{|\xi|^{1-\beta}} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (-|e^{i\xi(c-a)} - 1|^2 - |e^{i\xi(d-b)} - 1|^2 + |e^{i\xi(c-b)} - 1|^2 + |e^{i\xi(d-a)} - 1|^2) \frac{d\xi}{|\xi|^{3-\beta}}. \end{aligned}$$

Hence the result (2.7) follows from the first part of the proof. \square

The next two lemmas are concerned with the variances of the increments of \tilde{u} between two points and over an arbitrary rectangle, respectively. They will be used in the proofs of Propositions 2.1 and 3.1 below.

Lemma 2.3. For any $\tau, \lambda, h > 0$,

$$\mathbb{E}[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda))^2] = \frac{1}{2} K_\beta^2 \left[(\tau + \lambda) h^{2-\beta} + (3 - \beta)^{-1} h^{3-\beta} \right],$$

where K_β is the constant defined in (2.5).

Proof. Note that

$$\begin{aligned} \mathbb{E}[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda))^2] &= \mathbb{E} \left[\left(u \left(\frac{\tau + \lambda + h}{\sqrt{2}}, \frac{-\tau + \lambda + h}{\sqrt{2}} \right) - u \left(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}} \right) \right)^2 \right] \\ &= \frac{1}{4} \mathbb{E} \left[\left(W \left(\Delta \left(\frac{\tau + \lambda + h}{\sqrt{2}}, \frac{-\tau + \lambda + h}{\sqrt{2}} \right) \setminus \Delta \left(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}} \right) \right) \right)^2 \right]. \end{aligned}$$

Then by (2.2) and Lemma 2.2,

$$\begin{aligned} &\mathbb{E}[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda))^2] \\ &= \frac{1}{8\pi} \left\{ \int_0^{\frac{\tau+\lambda}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} |\mathcal{F} \mathbf{1}_{[\sqrt{2}\lambda-s, \sqrt{2}(\lambda+h)-s]}(\xi)|^2 \right. \\ &\quad \left. + \int_{\frac{\tau+\lambda}{\sqrt{2}}}^{\frac{\tau+\lambda+h}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} |\mathcal{F} \mathbf{1}_{[-\sqrt{2}\tau+s, \sqrt{2}(\lambda+h)-s]}(\xi)|^2 \right\} \\ &= \frac{1}{2(2-\beta)(1-\beta)} \left\{ \int_0^{\frac{\tau+\lambda}{\sqrt{2}}} (\sqrt{2}h)^{2-\beta} ds + \int_{\frac{\tau+\lambda}{\sqrt{2}}}^{\frac{\tau+\lambda+h}{\sqrt{2}}} (\sqrt{2}(\tau + \lambda + h) - 2s)^{2-\beta} ds \right\} \\ &= \frac{1}{2} K_\beta^2 \left[(\tau + \lambda) h^{2-\beta} + (3 - \beta)^{-1} h^{3-\beta} \right]. \end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4. Fix $\lambda \geq 0$. Then, for any $0 \leq \tau \leq \tau'$ and $0 \leq h \leq h'$,

$$\begin{aligned} &\mathbb{E}[(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda + h) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda + h))^2] \\ &= \begin{cases} \frac{1}{2} K_\beta^2 (h' - h)^{2-\beta} \left[(\tau' - \tau) - \frac{1-\beta}{3-\beta} (h' - h) \right] & \text{if } h' - h \leq \tau' - \tau, \\ \frac{1}{2} K_\beta^2 (\tau' - \tau)^{2-\beta} \left[(h' - h) - \frac{1-\beta}{3-\beta} (\tau' - \tau) \right] & \text{if } h' - h > \tau' - \tau. \end{cases} \end{aligned} \quad (2.8)$$

Proof. Note that

$$\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda + h) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda + h) = \frac{1}{2} W(Q),$$

where Q is the image of the rectangle $(\tau, \tau'] \times (\lambda + h, \lambda + h']$ under the rotation $(\tau, \lambda) \mapsto (\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}})$. Suppose $h' - h \leq \tau' - \tau$. Then $Q = Q_1 \cup Q_2 \cup Q_3$, where

$$\begin{aligned} Q_1 &= \left\{ (t, x) : \frac{\tau + \lambda + h}{\sqrt{2}} < t \leq \frac{\tau + \lambda + h'}{\sqrt{2}}, \sqrt{2}(\lambda + h) - s < x < -\sqrt{2}\tau + s \right\}, \\ Q_2 &= \left\{ (t, x) : \frac{\tau + \lambda + h'}{\sqrt{2}} < t \leq \frac{\tau' + \lambda + h}{\sqrt{2}}, \sqrt{2}(\lambda + h) - s < x \leq \sqrt{2}(\lambda + h') - s \right\}, \\ Q_3 &= \left\{ (t, x) : \frac{\tau' + \lambda + h}{\sqrt{2}} < t \leq \frac{\tau' + \lambda + h'}{\sqrt{2}}, -\sqrt{2}\tau' + s \leq x \leq \sqrt{2}(\lambda + h') - s \right\}. \end{aligned}$$

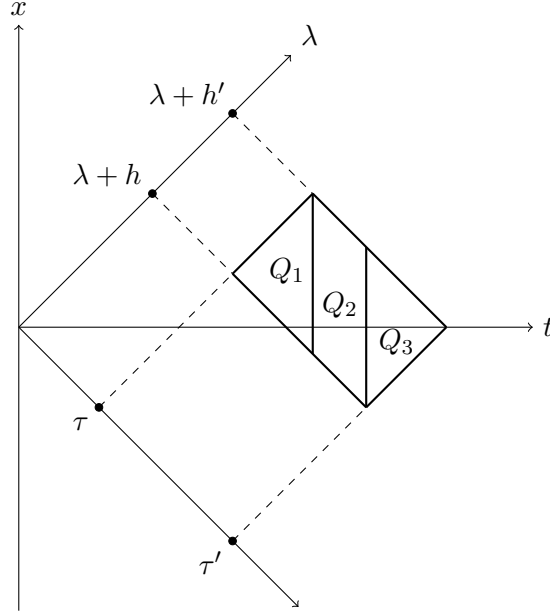


FIGURE 2

The sets Q_1 , Q_2 and Q_3 are shown in Figure 2. By (2.2),

$$\begin{aligned}
& \mathbb{E}[(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda + h) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda + h))^2] = \frac{1}{4} \mathbb{E}[W(Q)^2] \\
&= \frac{1}{8\pi} \left\{ \int_{\frac{\tau+\lambda+h}{\sqrt{2}}}^{\frac{\tau+\lambda+h'}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \left| \mathcal{F} \mathbf{1}_{[\sqrt{2}(\lambda+h)-s, -\sqrt{2}\tau+s]}(\xi) \right|^2 \right. \\
&\quad + \int_{\frac{\tau+\lambda+h'}{\sqrt{2}}}^{\frac{\tau'+\lambda+h}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \left| \mathcal{F} \mathbf{1}_{[\sqrt{2}(\lambda+h)-s, \sqrt{2}(\lambda+h')-s]}(\xi) \right|^2 \\
&\quad \left. + \int_{\frac{\tau'+\lambda+h}{\sqrt{2}}}^{\frac{\tau'+\lambda+h'}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \left| \mathcal{F} \mathbf{1}_{[-\sqrt{2}\tau'+s, \sqrt{2}(\lambda+h')-s]}(\xi) \right|^2 \right\}.
\end{aligned}$$

Then, by Lemma 2.2,

$$\begin{aligned}
& \mathbb{E}[(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda + h) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda + h))^2] \\
&= \frac{1}{2(2-\beta)(1-\beta)} \left\{ \int_{\frac{\tau+\lambda+h}{\sqrt{2}}}^{\frac{\tau+\lambda+h'}{\sqrt{2}}} (2s - \sqrt{2}(\tau + \lambda + h))^{2-\beta} ds \right. \\
&\quad \left. + \int_{\frac{\tau+\lambda+h'}{\sqrt{2}}}^{\frac{\tau'+\lambda+h}{\sqrt{2}}} (\sqrt{2}(h' - h))^{2-\beta} ds + \int_{\frac{\tau'+\lambda+h}{\sqrt{2}}}^{\frac{\tau'+\lambda+h'}{\sqrt{2}}} (\sqrt{2}(\tau' + \lambda + h') - 2s)^{2-\beta} ds \right\} \\
&= \frac{1}{2(2-\beta)(1-\beta)} \left\{ 2 \times \frac{2^{\frac{1-\beta}{2}}}{3-\beta} (h' - h)^{3-\beta} + 2^{\frac{1-\beta}{2}} (h' - h)^{2-\beta} [(\tau' - \tau) - (h' - h)] \right\} \\
&= \frac{1}{2} K_\beta^2 (h' - h)^{2-\beta} \left\{ (\tau' - \tau) - \frac{1-\beta}{3-\beta} (h' - h) \right\}.
\end{aligned}$$

This proves (2.8) in the case of $h' - h \leq \tau' - \tau$. The case of $h' - h > \tau' - \tau$ can be dealt with in a similar way. We omit the details. \square

It is known that for establishing the law of the iterated logarithm for a Gaussian process at a fix point, the following large deviation result (cf. [10, 12]) is useful: If $\{Z(t), t \in T\}$ is a continuous centered Gaussian random field which is a.s. bounded, then

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma^2} \log \mathbb{P} \left(\sup_{t \in T} Z(t) > \gamma \right) = -\frac{1}{2 \sup_{t \in T} \mathbb{E}(Z(t)^2)}. \quad (2.9)$$

The result (2.9), together with the scaling property of fractional Brownian sheets, also allows Blath and Martin [2] to prove a simultaneous LIL and study the propagation of singularities of the semi-fractional Brownian sheet. However, it turns out that (2.9) is not precise enough for proving Proposition 2.1 due to the facts that the index set in the event A_n in (2.12) depends on n and \tilde{u} does not have scaling property. To overcome this difficulty, we will make use of the following two results of Talagrand [20, Theorem 2.4, Proposition 2.7] regarding tail probability estimates for general Gaussian processes in terms of metric entropy.

Lemma 2.5. *Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process and $\sigma_T^2 = \sup_{t \in T} \mathbb{E}[Z(t)^2]$. Consider the canonical metric d_Z on T defined by $d_Z(s, t) = \mathbb{E}[(Z(s) - Z(t))^2]^{1/2}$. Assume that for some constant $M > \sigma_T$, some $\alpha > 0$ and some $0 < \varepsilon_0 \leq \sigma_T$,*

$$N(T, d_Z, \varepsilon) \leq \left(\frac{M}{\varepsilon} \right)^\alpha \quad \text{for all } \varepsilon < \varepsilon_0,$$

where $N(T, d_Z, \varepsilon)$ is the smallest number of d_Z -balls of radius ε needed to cover T . Then for any $\gamma > \sigma_T^2[(1 + \sqrt{\alpha})/\varepsilon_0]$, we have

$$\mathbb{P} \left\{ \sup_{t \in T} Z(t) \geq \gamma \right\} \leq \left(\frac{KM\gamma}{\sqrt{\alpha} \sigma_T^2} \right)^\alpha \Phi \left(\frac{\gamma}{\sigma_T} \right), \quad (2.10)$$

where $\Phi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-z^2/2) dz$ and K is a universal constant.

The conditions of Lemma 2.5 is easier to verify and the upper bound in (2.10) is more precise than that is given by (2.9) if M/σ_T is not too large. However, the upper bound in (2.10) may not be useful when M/σ_T becomes very large [this is the case for the even A_n in (2.12) below.] On the other hand, Lemma 2.6 is more efficient if the variance of $Z(t)$ attains its maximum at a unique point because the size of the set T_ρ can be very small.

Lemma 2.6. *Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process as in Lemma 2.5. For $\rho > 0$, set*

$$T_\rho = \{t \in T : \mathbb{E}[Z(t)^2] \geq \sigma_T^2 - \rho^2\}.$$

Assume that there exist constants $v \geq w \geq 1$ such that for all $\rho > 0$, and $0 < \varepsilon \leq \rho(1 + \sqrt{v})/\sqrt{w}$, we have

$$N(T_\rho, d_Z, \varepsilon) \leq A\rho^w \varepsilon^{-v}.$$

Then for any $\gamma > 2\sigma_T\sqrt{w}$, we have

$$\mathbb{P} \left\{ \sup_{t \in T} Z(t) \geq \gamma \right\} \leq \frac{Aw^{w/2}}{v^{v/2}} K^{v+w} \left(\frac{\gamma}{\sigma_T^2} \right)^{v-w} \Phi \left(\frac{\gamma}{\sigma_T} \right).$$

Now, we prove Proposition 2.1.

Proof of Proposition 2.1. Fix $\lambda > 0$. It suffices to show that for any $0 \leq a < b < \infty$ and any $0 < \varepsilon < 1$,

$$\mathbb{P} \left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau + \lambda)h^{2-\beta} \log \log(1/h)}} \leq (1 + \varepsilon)K_\beta \text{ for all } \tau \in [a, b] \right) = 1. \quad (2.11)$$

Let $c \in [a, b]$, $\delta = (c + \lambda)\varepsilon/2$ and $d = c + \delta$. Choose a real number q such that $1 < q < (1 + \varepsilon)^{1/(2-\beta)}$. For every integer $n \geq 1$, consider the event

$$A_n = \left\{ \sup_{\tau \in [c, d]} \sup_{h \in [0, q^{-n}]} |\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)| > \gamma_n \right\}, \quad (2.12)$$

where

$$\gamma_n = (1 + \varepsilon)K_\beta \sqrt{(c + \lambda)(q^{-n-1})^{2-\beta} \log \log q^n}.$$

To estimate $\mathbb{P}(A_n)$, we will apply Lemma 2.6. Define $T = [c, d] \times [0, q^{-n}]$ and $Z(\tau, h) = \tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)$ for $(\tau, h) \in T$. It follows from Lemma 2.3 that $\mathbb{E}[Z(\tau, h)^2]$ attains its unique maximum σ_T^2 at (d, q^{-n}) , where

$$\sigma_T^2 = \frac{1}{2}K_\beta^2 [(d + \lambda)q^{-n(2-\beta)} + (3 - \beta)^{-1}q^{-n(3-\beta)}].$$

For any $(\tau, h), (\tau', h') \in T$, without loss of generality, we may assume that $\tau \leq \tau'$. By the triangle inequality,

$$\begin{aligned} d_Z((\tau, h), (\tau', h')) &\leq \mathbb{E}[(Z(\tau, h) - Z(\tau, h'))^2]^{1/2} + \mathbb{E}[(Z(\tau', h') - Z(\tau, h'))^2]^{1/2} \\ &= \mathbb{E}[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda + h'))^2]^{1/2} \\ &\quad + \mathbb{E}[(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda))^2]^{1/2} \\ &=: S_1 + S_2. \end{aligned}$$

By Lemma 2.3, $S_1 \leq C|h - h'|^{(2-\beta)/2}$. The term S_2 can be evaluated by using Lemma 2.4. In particular, if $|\tau - \tau'| \geq h'$, we have

$$S_2 \leq \frac{1}{\sqrt{2}}K_\beta q^{-n(2-\beta)/2} |\tau - \tau'|^{1/2}.$$

If $|\tau - \tau'| < h'$, since $h' \leq q^{-n}$ and $(2 - \beta)/2 > 1/2$, we have

$$\begin{aligned} S_2 &\leq \frac{1}{\sqrt{2}}K_\beta |\tau - \tau'|^{(2-\beta)/2} (h')^{1/2} \\ &\leq \frac{1}{\sqrt{2}}K_\beta q^{-n(2-\beta)/2} |\tau - \tau'|^{1/2}. \end{aligned}$$

Hence,

$$d_Z((\tau, h), (\tau', h')) \leq C(q^{-n(2-\beta)/2} |\tau - \tau'|^{1/2} + |h - h'|^{(2-\beta)/2}). \quad (2.13)$$

Next, we estimate $N(T_\rho, d_Z, \varepsilon)$, where

$$T_\rho = \{(\tau, h) \in T : \sigma_T^2 - \mathbb{E}[Z(\tau, h)^2] \leq \rho^2\}.$$

We can write $\sigma_T^2 - \mathbb{E}[Z(\tau, h)^2]$ as the sum of three terms:

$$\frac{1}{2}K_\beta^2 \left[(d - \tau)q^{-n(2-\beta)} + (\tau + \lambda)(q^{-n(2-\beta)} - h^{2-\beta}) + (3 - \beta)^{-1}(q^{-n(3-\beta)} - h^{3-\beta}) \right].$$

It follows that $(\tau, h) \in T_\rho$ implies that each of the three terms is at most ρ^2 . Then, together with the elementary inequality $x^{1/(2-\beta)} - y^{1/(2-\beta)} \leq (x - y)^{1/(2-\beta)}$ for $x \geq y > 0$ and $0 < 1/(2 - \beta) < 1$, we can deduce that

$$T_\rho \subset [d - C_1 q^{n(2-\beta)} \rho^2, d] \times [q^{-n} - C_2 \rho^{2/(2-\beta)}, q^{-n}]$$

for some constants C_1 and C_2 . This and (2.13) imply that

$$N(T_\rho, d_Z, \varepsilon) \leq C_0(\rho/\varepsilon)^{2+\frac{2}{2-\beta}}.$$

Hence, we apply Lemma 2.6 with $v = w = 2 + \frac{2}{2-\beta}$ and the standard estimate $\Phi(x) \leq (2\pi)^{-1/2} e^{-x^2/2}$ for $x \geq 1$ to get that

$$\mathbb{P}(A_n) \leq C \exp\left(-\frac{\gamma_n^2}{2\sigma_T^2}\right) = (n \log q)^{-p_n},$$

where

$$p_n = \frac{(1 + \varepsilon)^2}{q^{2-\beta} \left[\frac{d+\lambda}{c+\lambda} + (3-\beta)^{-1}(c+\lambda)^{-1} q^{-n} \right]}.$$

Recall that $d = c + \delta$ and $\delta = (c + \lambda)\varepsilon/2$. When n is large enough, $(3-\beta)^{-1}(c+\lambda)^{-1} q^{-n} \leq \varepsilon/2$, which implies that

$$p_n \geq \frac{1 + \varepsilon}{q^{2-\beta}} > 1.$$

Hence $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ and by the Borel-Cantelli lemma, we have $\mathbb{P}(A_n \text{ i.o.}) = 0$. It follows that with probability 1,

$$\sup_{\tau \in [c, d]} \sup_{h \in [q^{-n-1}, q^{-n}]} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(c + \lambda)(q^{-n-1})^{2-\beta} \log \log q^n}} \leq (1 + \varepsilon)K_\beta$$

eventually for all large n . Hence

$$\mathbb{P}\left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau + \lambda)h^{2-\beta} \log \log(1/h)}} \leq (1 + \varepsilon)K_\beta \text{ for all } \tau \in [c, d]\right) = 1.$$

From this, we can deduce (2.11) by covering the interval $[a, b]$ by finitely many intervals $[c, d]$ of length δ . \square

3. SIMULTANEOUS LAW OF ITERATED LOGARITHM: LOWER BOUND

Next, we prove the lower bound for the simultaneous LIL:

Proposition 3.1. *For any $\lambda > 0$,*

$$\mathbb{P}\left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau + \lambda)h^{2-\beta} \log \log(1/h)}} \geq K_\beta \text{ for all } \tau \in [0, \infty)\right) = 1, \quad (3.1)$$

where K_β is the constant in (2.5) of Proposition 2.1.

Recall the following version of Borel-Cantelli lemma [16, p.391].

Lemma 3.2. *Let $\{A_n : n \geq 1\}$ be a sequence of events. If*

- (i) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and
- (ii) $\liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \sum_{k=1}^n \mathbb{P}(A_j \cap A_k)}{[\sum_{j=1}^n \mathbb{P}(A_j)]^2} = 1,$

then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

We will also use the following lemma, which is essentially proved in [18]. For the sake of completeness, we provide a proof for this result.

Lemma 3.3. *Let Z_1 and Z_2 be jointly Gaussian random variables with $\mathbb{E}(Z_i) = 0$, $\mathbb{E}(Z_i^2) = 1$ and $\mathbb{E}(Z_1 Z_2) = r$. Then for any $\gamma_1, \gamma_2 > 0$, there exists a number r^* between 0 and r such that*

$$\mathbb{P}(Z_1 > \gamma_1, Z_2 > \gamma_2) - \mathbb{P}(Z_1 > \gamma_1)\mathbb{P}(Z_2 > \gamma_2) = rg(\gamma_1, \gamma_2; r^*),$$

where $g(x, y; r)$ is the standard bivariate Gaussian density with correlation r , i.e.

$$g(x, y; r) = \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left(-\frac{x^2 + y^2 - 2rxy}{2(1-r^2)}\right).$$

Proof. Let $\gamma_1, \gamma_2 > 0$ and $p(r) = \int_{\gamma_1}^{\infty} \int_{\gamma_2}^{\infty} g(x, y; r) dx dy$. Define the Fourier transform of a function $f(x, y)$ as $\mathcal{F}f(\xi, \zeta) = \iint_{\mathbb{R}^2} e^{-i(x\xi + y\zeta)} f(x, y) dx dy$. Note that

$$g(x, y; r) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi + y\zeta)} [\mathcal{F}g(*; r)](\xi, \zeta) d\xi d\zeta$$

and

$$[\mathcal{F}g(*; r)](\xi, \zeta) = e^{-\frac{1}{2}(\xi^2 + 2r\xi\zeta + \zeta^2)}.$$

By the dominated convergence theorem,

$$\partial_r g(x, y; r) = \frac{-1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi + y\zeta)} \xi\zeta [\mathcal{F}g(*; r)](\xi, \zeta) d\xi d\zeta.$$

Since $(i\xi)(i\zeta) \cdot \mathcal{F}f(\xi, \zeta) = [\mathcal{F}\partial_x \partial_y f](\xi, \zeta)$, we have

$$\partial_r g(x, y; r) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi + y\zeta)} [\mathcal{F}\partial_x \partial_y g(*; r)](\xi, \zeta) d\xi d\zeta = \partial_x \partial_y g(x, y; r).$$

Therefore,

$$\partial_r p = \int_{\gamma_1}^{\infty} \int_{\gamma_2}^{\infty} \partial_x \partial_y g(x, y; r) dx dy = g(\gamma_1, \gamma_2; r).$$

The mean value theorem implies that $p(r) - p(0) = rg(\gamma_1, \gamma_2; r^*)$ for some r^* between 0 and r , and hence the result. \square

Let σ and $\tilde{\sigma}$ be the canonical metric on $\mathbb{R}_+ \times \mathbb{R}$ for u and \tilde{u} respectively, i.e.,

$$\sigma[(t, x), (t', x')] = \mathbb{E}[(u(t, x) - u(t', x'))^2]^{1/2},$$

$$\tilde{\sigma}[(\tau, \lambda), (\tau', \lambda')] = \mathbb{E}[(\tilde{u}(\tau, \lambda) - \tilde{u}(\tau', \lambda'))^2]^{1/2}.$$

For a rectangle $I = [a, a'] \times [-b, b]$, where $0 < a < a' < \infty$ and $0 < b < \infty$, recall from [6, Proposition 4.1] that there exist positive finite constants C_1 and C_2 such that

$$C_1(|t - t'| + |x - x'|)^{(2-\beta)/2} \leq \sigma[(t, x), (t', x')] \leq C_2(|t - t'| + |x - x'|)^{(2-\beta)/2} \quad (3.2)$$

for all $(t, x), (t', x') \in I$.

The proof of the following lemma is based on the method in [13, 14].

Lemma 3.4. *Let $\tau > 0$, $\lambda > 0$ and $q > 1$. Then for all $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(\frac{\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda + q^{-n-1})}{\tilde{\sigma}[(\tau, \lambda + q^{-n}), (\tau, \lambda + q^{-n-1})]} \geq (1 - \varepsilon)\sqrt{2 \log \log q^n} \text{ infinitely often in } n\right) = 1. \quad (3.3)$$

Proof. For $n \geq 1$, let $A_n = \{Z_n > \gamma_n\}$, where

$$Z_n = \frac{\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda + q^{-n-1})}{\tilde{\sigma}[(\tau, \lambda + q^{-n}), (\tau, \lambda + q^{-n-1})]}$$

and

$$\gamma_n = (1 - \varepsilon)\sqrt{2 \log \log q^n}.$$

We will complete the proof by showing that (i) and (ii) of Lemma 3.2 are satisfied. For (i), by using the standard estimate

$$\mathbb{P}(Z > x) \geq (2\sqrt{2\pi})^{-1}x^{-1} \exp(-x^2/2), \quad x > 1, \quad (3.4)$$

for a standard Gaussian random variable Z , we derive that for large n ,

$$\mathbb{P}(Z_n > \gamma_n) \geq \frac{C}{n^{(1-\varepsilon)^2} \sqrt{\log n}}$$

and hence $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$.

Next, we show that (ii) is satisfied. Since

$$\sum_{j=1}^n \sum_{k=1}^n [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)] = \mathbb{E} \left[\left(\sum_{j=1}^n (\mathbf{1}_{A_j} - \mathbb{P}(A_j)) \right)^2 \right] \geq 0$$

and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, it is enough to prove that

$$\liminf_{n \rightarrow \infty} \frac{\sum_{1 \leq j < k \leq n} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)]}{[\sum_{j=1}^n \mathbb{P}(A_j)]^2} \leq 0. \quad (3.5)$$

We are going to use Lemma 3.3 to estimate the difference in the numerator. First, we estimate the correlation r_{jk} between Z_j and Z_k for $j < k$. Let $(t_n, x_n) = (\frac{\tau + \lambda + q^{-n}}{\sqrt{2}}, \frac{-\tau + \lambda + q^{-n}}{\sqrt{2}})$. By (2.2), we have

$$\begin{aligned} & \mathbb{E}[(\tilde{u}(\tau, \lambda + q^{-j}) - \tilde{u}(\tau, \lambda + q^{-j-1}))(\tilde{u}(\tau, \lambda + q^{-k}) - \tilde{u}(\tau, \lambda + q^{-k-1}))] \\ &= \frac{1}{4} \mathbb{E}[W(\Delta(t_j, x_j) \setminus \Delta(t_{j+1}, x_{j+1}))W(\Delta(t_k, x_k) \setminus \Delta(t_{k+1}, x_{k+1}))] \\ &= \frac{1}{8\pi} \int_0^\infty ds \int_{-\infty}^\infty \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathcal{F}(\mathbf{1}_{\Delta(t_j, x_j) \setminus \Delta(t_{j+1}, x_{j+1})}(s, \cdot))(\xi) \overline{\mathcal{F}(\mathbf{1}_{\Delta(t_k, x_k) \setminus \Delta(t_{k+1}, x_{k+1})}(s, \cdot)}(\xi))} \\ &= \frac{1}{8\pi} \int_0^{t_{k+1}} ds \int_{-\infty}^\infty \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathcal{F}(\mathbf{1}_{[x_{k+1} + t_{k+1} - s, x_k + t_k - s]}(s, \cdot))(\xi) \overline{\mathcal{F}(\mathbf{1}_{[x_{j+1} + t_{j+1} - s, x_j + t_j - s]}(s, \cdot)}(\xi))} \\ &+ \frac{1}{8\pi} \int_{t_{k+1}}^{t_k} ds \int_{-\infty}^\infty \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathcal{F}(\mathbf{1}_{[x_k - t_k + s, x_k + t_k - s]}(s, \cdot))(\xi) \overline{\mathcal{F}(\mathbf{1}_{[x_{j+1} + t_{j+1} - s, x_j + t_j - s]}(s, \cdot)}(\xi))}. \end{aligned}$$

Note that (2.2) also implies that this covariance is nonnegative. Then by Lemma 2.2,

$$\begin{aligned} & \mathbb{E}[(\tilde{u}(\tau, \lambda + q^{-j}) - \tilde{u}(\tau, \lambda + q^{-j-1}))(\tilde{u}(\tau, \lambda + q^{-k}) - \tilde{u}(\tau, \lambda + q^{-k-1}))] \\ &= Ct_{k+1} \left[(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - q^{-k-1})^{2-\beta} + (q^{-j} - q^{-k-1})^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \right] \\ &+ C \int_{q^{-k-1}}^{q^{-k}} \left[(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - s)^{2-\beta} + (q^{-j} - s)^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \right] ds \\ &=: J_1 + J_2. \end{aligned} \quad (3.6)$$

Let us consider the first term J_1 . By the mean value theorem, we can find some a and b such that

$$q^{-j-1} - q^{-k} \leq a \leq q^{-j-1} - q^{-k-1} < q^{-j} - q^{-k} \leq b \leq q^{-j} - q^{-k-1}.$$

and

$$\begin{aligned} & (q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - q^{-k-1})^{2-\beta} + (q^{-j} - q^{-k-1})^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \\ &= (2-\beta)(b^{1-\beta} - a^{1-\beta})(q^{-k} - q^{-k-1}) \\ &\leq (2-\beta)[(q^{-j} - q^{-k-1})^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta}]q^{-k}. \end{aligned}$$

Suppose $j \leq k-2$. By the mean value theorem again, we can find some ξ between $q^{-j} - q^{-k-1}$ and $q^{-j-1} - q^{-k}$ such that

$$\begin{aligned} (q^{-j} - q^{-k-1})^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta} &= (1-\beta)\xi^{-\beta}[(q^{-j} - q^{-k-1}) - (q^{-j-1} - q^{-k})] \\ &\leq (1-\beta)(q^{-j-1} - q^{-k})^{-\beta}q^{-j} \\ &\leq (1-\beta)(q^{-j-1} - q^{-j-2})^{-\beta}q^{-j} \\ &\leq (1-\beta)(q^{-1} - q^{-2})(q^{-j})^{1-\beta}. \end{aligned}$$

It follows that

$$J_1 \leq C(q^{-j})^{1-\beta}q^{-k}.$$

Next, we consider the term J_2 in (3.6). For every $s \in [q^{-k-1}, q^{-k}]$, we can find some \tilde{a} and \tilde{b} (depending on s) such that

$$q^{-j-1} - q^{-k} \leq \tilde{a} \leq q^{-j-1} - s < q^{-j} - q^{-k} \leq \tilde{b} \leq q^{-j} - s$$

and

$$\begin{aligned} & (q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - s)^{2-\beta} + (q^{-j} - s)^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \\ &= (2-\beta)(\tilde{b}^{1-\beta} - \tilde{a}^{1-\beta})(q^{-k} - s) \\ &\leq (2-\beta)[(q^{-j} - s)^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta}]q^{-k}. \end{aligned}$$

For $j \leq k-2$, by the mean value theorem again, there exists η between $q^{-j} - s$ and $q^{-j-1} - q^{-k}$ such that

$$\begin{aligned} (q^{-j} - s)^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta} &= (1-\beta)\eta^{-\beta}[(q^{-j} - s) - (q^{-j-1} - q^{-k})] \\ &\leq (1-\beta)(q^{-j-1} - q^{-j-2})^{-\beta}q^j \\ &\leq (1-\beta)(q^{-1} - q^{-2})(q^{-j})^{1-\beta}. \end{aligned}$$

Then we have

$$\begin{aligned} J_2 &= C \int_{q^{-k-1}}^{q^{-k}} \left[(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - s)^{2-\beta} + (q^{-j} - s)^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \right] ds \\ &\leq C(q^{-j})^{1-\beta}(q^{-k})^2 \leq C(q^{-j})^{1-\beta}q^{-k}. \end{aligned}$$

Therefore, by combining (3.6), the upper bounds for J_1 , J_2 , and recalling (3.2), we see that for $j \leq k-2$, the correlation r_{jk} between Z_j and Z_k satisfies

$$0 \leq r_{jk} = \mathbb{E}(Z_j Z_k) \leq \frac{C(q^{-j})^{1-\beta}q^{-k}}{(q^{-j})^{1-\beta/2}(q^{-k})^{1-\beta/2}} = C_0(q^{-(k-j)})^{\beta/2} =: \xi_{jk}. \quad (3.7)$$

By (3.7), we can choose a fixed $l \geq 2$ such that $r := \sup\{r_{jk} : j \leq k-l\} < 1$. Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, in order to prove (3.5), it suffices to prove that for any $\delta > 0$, there exists m such that

$$\liminf_{n \rightarrow \infty} \frac{\sum_{k=m}^n \sum_{j=1}^{k-l} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)]}{[\sum_{j=1}^n \mathbb{P}(A_j)]^2} \leq \delta. \quad (3.8)$$

Let $\delta > 0$ be given and let m be a large integer that will be chosen appropriately depending on δ . Let $\rho_k = \frac{4}{(\beta/2)\log q} \log \gamma_k$, so that for $1 \leq j \leq k - \rho_k$,

$$\xi_{jk} \leq C_0 \gamma_k^{-4}. \quad (3.9)$$

Provided m is large, $1 < k - \rho_k < k - l$ for all $k \geq m$. By Lemma 3.3, we have

$$\sum_{k=m}^n \sum_{j=1}^{k-l} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)] \leq \left(\sum_{k=m}^n \sum_{j=1}^{\lfloor k-\rho_k \rfloor} + \sum_{k=m}^n \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \right) r_{jk} g(\gamma_j, \gamma_k; r_{jk}^*), \quad (3.10)$$

where r_{jk}^* is a number such that $0 \leq r_{jk}^* \leq r_{jk}$ for each j, k . Let us consider the two sums on the right-hand side of (3.10) separately. By (3.7), the first sum is

$$\begin{aligned} & \sum_{k=m}^n \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{r_{jk}}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{\gamma_j^2 + \gamma_k^2 - 2r_{jk}^* \gamma_j \gamma_k}{2(1-r_{jk}^{*2})}\right) \\ & \leq \sum_{k=m}^n \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{\xi_{jk} \gamma_j \gamma_k}{2\pi(1-\xi_{jk}^2)^{1/2}} \exp\left(\frac{-r_{jk}^{*2}(\gamma_j^2 + \gamma_k^2) + 2r_{jk}^* \gamma_j \gamma_k}{2(1-r_{jk}^{*2})}\right) \gamma_j^{-1} e^{-\gamma_j^2/2} \gamma_k^{-1} e^{-\gamma_k^2/2}. \end{aligned}$$

Note that $\gamma_j < \gamma_k$ for $j < k$. Then by (3.4), (3.7) and (3.9), the sum is

$$\leq 4 \sum_{k=m}^n \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{C_0 \gamma_k^{-2}}{(1-C_0^2 \gamma_k^{-8})^{1/2}} \exp\left(\frac{C_0 \gamma_k^{-2}}{1-C_0^2 \gamma_k^{-8}}\right) \mathbb{P}(A_j)\mathbb{P}(A_k).$$

Since $\gamma_k \rightarrow \infty$, we may choose m to be large enough such that this sum is $\leq \delta [\sum_{j=1}^n \mathbb{P}(A_j)]^2$.

By (3.4), the second sum on the right-hand side of (3.10) is

$$\begin{aligned} & \sum_{k=m}^n \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{r_{jk}}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{\gamma_j^2 + \gamma_k^2 - 2r_{jk}^* \gamma_j \gamma_k}{2(1-r_{jk}^{*2})}\right) \\ & \leq \sum_{k=m}^n \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{r_{jk} \gamma_j}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{(\gamma_k - r_{jk}^* \gamma_j)^2}{2(1-r_{jk}^{*2})}\right) \gamma_j^{-1} e^{-\gamma_j^2/2} \\ & \leq \frac{2}{\sqrt{2\pi}} \sum_{k=m}^n \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{\gamma_k}{(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{(1-r_{jk}^*)^2 \gamma_k^2}{2(1-r_{jk}^{*2})}\right) \mathbb{P}(A_j). \end{aligned}$$

Recall that $r = \sup\{r_{jk} : j \leq k-l\} < 1$. Moreover, if m is large enough, then

$$\frac{\gamma_k \log \gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \leq \delta$$

and $k - \rho_k > k/2$ for all $k \geq m$, so that the last sum above is

$$\begin{aligned} &\leq \frac{2}{\sqrt{2\pi}} \sum_{k=m}^n \frac{\rho_k \gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \mathbb{P}(A_{[k-\rho_k]}) \\ &\leq C \sum_{k=m}^n \frac{\gamma_k \log \gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \mathbb{P}(A_{[k/2]}) \\ &\leq 2C\delta \sum_{k=1}^n \mathbb{P}(A_k). \end{aligned}$$

We get that

$$\sum_{k=m}^n \sum_{j=1}^{k-1} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)] \leq \delta \left(\sum_{j=1}^n \mathbb{P}(A_j) \right)^2 + 2C\delta \sum_{j=1}^n \mathbb{P}(A_j).$$

Hence (3.8) follows and the proof of Lemma 3.4 is complete. \square

We now come to the proof of Proposition 3.1.

Proof of Proposition 3.1. Fix $\lambda > 0$. It suffices to show that for any $0 \leq a < b < \infty$ and $0 < \varepsilon < 1$,

$$\mathbb{P} \left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau + \lambda)h^{2-\beta} \log \log(1/h)}} \geq (1 - \varepsilon)K_\beta \text{ for all } \tau \in [a, b] \right) = 1. \quad (3.11)$$

To this end, let us fix a, b and ε for the rest of the proof.

Note that when q is large, $q^{-\frac{2-\beta}{2}} \left(1 + \frac{q^{-n-1}}{\tau + \lambda}\right)^{1/2} < \varepsilon/4$ uniformly for all $\tau \in [a, b]$. So we can choose and fix a large $q > 1$ such that

$$(1 - \varepsilon/4) \left(\frac{q-1}{q} \right)^{\frac{2-\beta}{2}} - q^{-\frac{2-\beta}{2}} \left(1 + \frac{q^{-n-1}}{\tau + \lambda} \right)^{1/2} - (1 - \varepsilon) > \varepsilon/4 \quad (3.12)$$

for all $\tau \in [a, b]$. We also choose $\delta > 0$ small such that

$$\frac{\lambda(\varepsilon/4)^2}{\delta} > 1. \quad (3.13)$$

Since we can cover $[a, b]$ by finitely many intervals $[c, d]$ of length δ , we only need to show that (3.11) holds for all $\tau \in [c, d]$, where $[c, d] \subset [a, b]$ and $d = c + \delta$.

Let us define the increment of \tilde{u} over a rectangle $(\tau, \tau'] \times (\lambda, \lambda']$ by

$$\Delta \tilde{u}((\tau, \tau'] \times (\lambda, \lambda']) = \tilde{u}(\tau', \lambda') - \tilde{u}(\tau, \lambda') - \tilde{u}(\tau', \lambda) + \tilde{u}(\tau, \lambda).$$

Then for all $\tau \in [c, d]$ we can write

$$\begin{aligned} \tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda) &= \tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1}) \\ &\quad + \tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda) \\ &\quad - \Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n})). \end{aligned} \quad (3.14)$$

By Lemma 3.4, we have

$$\frac{|\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})|}{\tilde{\sigma}[(d, \lambda + q^{-n}), (d, \lambda + q^{-n-1})]} \geq (1 - \varepsilon/4) \sqrt{2 \log \log q^n}$$

infinitely often in n with probability 1. By Lemma 2.3,

$$\begin{aligned} & \tilde{\sigma}[(d, \lambda + q^{-n}), (d, \lambda + q^{-n-1})] \\ &= \frac{K_\beta}{\sqrt{2}} \sqrt{(d + \lambda + q^{-n-1})(q^{-n} - q^{-n-1})^{2-\beta} + (3 - \beta)^{-1}(q^{-n} - q^{-n-1})^{3-\beta}}, \end{aligned}$$

so we have

$$|\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})| \geq (1 - \varepsilon/4)K_\beta \sqrt{(d + \lambda)(q^{-n} - q^{-n-1})^{2-\beta} \log \log q^n} \quad (3.15)$$

infinitely often in n with probability 1. Also, by Proposition 2.1, with probability 1, for all $\tau \in [c, d]$ simultaneously,

$$|\tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda)| \leq K_\beta \sqrt{(\tau + \lambda + q^{-n-1})(q^{-n-1})^{2-\beta} \log \log q^n} \quad (3.16)$$

eventually for all large n .

Next, we derive a bound for the term $\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])$. For $\tau \in [c, d]$, let

$$\phi_n(\tau) = (1 - \varepsilon/4) \left(\frac{q-1}{q} \right)^{\frac{2-\beta}{2}} (d + \lambda)^{1/2} - q^{-\frac{2-\beta}{2}} (\tau + \lambda + q^{-n-1})^{1/2} - (1 - \varepsilon)(\tau + \lambda)^{1/2}.$$

Consider the event

$$A_n = \left\{ \sup_{\tau \in [c, d]} |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\},$$

where

$$\gamma_n = K_\beta \phi_n(d) \sqrt{(q^{-n})^{2-\beta} \log \log q^n}.$$

Consider n large enough such that $q^{-n} - q^{-n-1} \leq d - c$. Then

$$\mathbb{P}(A_n) \leq \mathbb{P}(A_n^1) + \mathbb{P}(A_n^2),$$

where

$$\begin{aligned} A_n^1 &= \left\{ \sup_{\tau \in [c, d - (q^{-n} - q^{-n-1})]} |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\}, \\ A_n^2 &= \left\{ \sup_{\tau \in [d - (q^{-n} - q^{-n-1}), d]} |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\}. \end{aligned}$$

We first estimate $\mathbb{P}(A_n^1)$ using Lemma 2.6. Let $Z(\tau) = \Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])$, $T = [c, d - (q^{-n} - q^{-n-1})]$ and

$$\begin{aligned} d_Z(\tau, \tau') &:= \mathbb{E}[(Z(\tau) - Z(\tau'))^2]^{1/2} \\ &= \mathbb{E}[(\tilde{u}(\tau', \lambda + q^{-n}) - \tilde{u}(\tau', \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda + q^{-n}) + \tilde{u}(\tau, \lambda + q^{-n-1}))^2]^{1/2}. \end{aligned}$$

By Lemma 2.4,

$$\begin{aligned} d_Z(\tau, \tau') &\leq \begin{cases} C(q^{-n} - q^{-n-1})^{(2-\beta)/2} |\tau - \tau'|^{1/2} & \text{if } q^{-n} - q^{-n-1} \leq |\tau - \tau'| \\ C|\tau - \tau'|^{(2-\beta)/2} (q^{-n} - q^{-n-1})^{1/2} & \text{if } q^{-n} - q^{-n-1} > |\tau - \tau'| \end{cases} \\ &\leq Cq^{-n(2-\beta)/2} |\tau - \tau'|^{1/2}. \end{aligned} \quad (3.17)$$

Let $\sigma_T^2 = \sup_{\tau \in T} \mathbb{E}[Z(\tau)^2]$ and $T_\rho = \{\tau \in T : \sigma_T^2 - \mathbb{E}[Z(\tau)^2] \leq \rho^2\}$. For all $\tau \in T$, $d - \tau \geq q^{-n} - q^{-n-1}$, so by Lemma 2.4,

$$\sigma_T^2 = \frac{1}{2} K_\beta^2 (q^{-n} - q^{-n-1})^{2-\beta} \left[(d - c) - \frac{1 - \beta}{3 - \beta} (q^{-n} - q^{-n-1}) \right]$$

and

$$\sigma_T^2 - \mathbb{E}[Z(\tau)^2] = \frac{1}{2} K_\beta^2 (q^{-n} - q^{-n-1})^{2-\beta} (d - \tau).$$

Hence, $T_\rho \subset [d - Cq^{n(2-\beta)}\rho^2, d]$. This and (3.17) imply that $N(T_\rho, d_Z, \varepsilon) \leq C(\rho/\varepsilon)^2$. Now, we can apply Lemma 2.6 with $v = w = 2$ to get that for n large,

$$\mathbb{P}(A_n^1) \leq C \exp\left(-\frac{\gamma_n^2}{2\sigma_T^2}\right) \leq (n \log q)^{-p_n},$$

where

$$p_n = \frac{1}{d - c} \left(\frac{q}{q - 1} \right)^{2-\beta} \phi_n(d)^2.$$

Recall that $d = c + \delta$. By (3.12), $\phi_n(d)^2 > (d + \lambda)(\varepsilon/4)^2$ for all n , so $p_n > \frac{\lambda(\varepsilon/4)^2}{\delta} > 1$ by (3.13). Hence, $\sum_{n=1}^{\infty} \mathbb{P}(A_n^1) < \infty$.

To get an upper bound for $\mathbb{P}(A_n^2)$, we take $T' = [d - (q^{-n} - q^{-n-1}), d]$. Since the size of T' is quite small, we can apply Lemma 2.5. Notice that $\sigma_{T'}^2 \leq Cq^{-n(3-\beta)}$, and by (3.17),

$$N(T', d_Z, \varepsilon) \leq C \frac{q^{-n} - q^{-n-1}}{q^{n(2-\beta)} \varepsilon^2} \leq Cq^{-n(3-\beta)} \varepsilon^{-2}.$$

Therefore, by Lemma 2.5, for n large, we get that

$$\mathbb{P}(A_n^2) \leq C\phi_n(d)^2 (q^n \log n) \exp(-C' \phi_n(d)^2 q^n \log n).$$

Since $(d + \lambda)(\varepsilon/4)^2 \leq \phi_n(d)^2 \leq d + \lambda$ for all n , we have $\sum_{n=1}^{\infty} \mathbb{P}(A_n^2) < \infty$.

Hence, $\mathbb{P}(A_n \text{ i.o.}) = 0$ by the Borel–Cantelli lemma. Then, by the monotonic decreasing property of ϕ , this implies that, with probability 1, simultaneously for all $\tau \in [c, d]$,

$$|\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| \leq K_\beta \phi_n(\tau) \sqrt{(q^{-n})^{2-\beta} \log \log q^n} \quad (3.18)$$

eventually for all large n . By (3.14) and the triangle inequality,

$$\begin{aligned} |\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda)| &\geq |\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})| \\ &\quad - |\tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda)| \\ &\quad - |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])|. \end{aligned}$$

Then (3.15), (3.16) and (3.18) together imply that with probability 1, for all $\tau \in [c, d]$ simultaneously,

$$\begin{aligned} &|\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda)| \\ &\geq \left[(1 - \varepsilon/4) \left(\frac{q - 1}{q} \right)^{\frac{2-\beta}{2}} (d + \lambda)^{1/2} - q^{-\frac{2-\beta}{2}} (\tau + \lambda + q^{-n-1})^{1/2} - \phi_n(\tau) \right] \\ &\quad \times K_\beta \sqrt{(q^{-n})^{2-\beta} \log \log q^n} \\ &\geq (1 - \varepsilon) K_\beta \sqrt{(\tau + \lambda)(q^{-n})^{2-\beta} \log \log q^n} \end{aligned}$$

infinitely often in n . This shows that (3.11) holds for all $\tau \in [c, d]$ and concludes the proof of Proposition 3.1. \square

4. SINGULARITIES AND THEIR PROPAGATION

In this section, we study the existence and propagation of singularities of the stochastic wave equation (1.1). The main result is Theorem 4.3.

Let us first discuss the interpretation of singularities and how they may arise. Propositions 2.1 and 3.1 imply that LIL holds at any fixed point (t, x) :

$$\limsup_{h \rightarrow 0^+} \frac{|u(t + \frac{h}{\sqrt{2}}, x + \frac{h}{\sqrt{2}}) - u(t, x)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = K_\beta (\sqrt{2}t)^{1/2} \quad \text{a.s.}$$

It describes the rate of local oscillation of u when (t, x) is fixed. On the other hand, when (t, x) is not fixed, the behavior will be different. Indeed, from the uniform modulus of continuity in Theorem 3.1 of [11], we know that for $I = [a, a'] \times [-b, b]$, where $0 < a < a'$ and $b > 0$, there exists a positive finite constant K such that

$$\lim_{h \rightarrow 0^+} \sup_{\substack{(t,x), (t',x') \in I: \\ 0 < \sigma[(t,x), (t',x')] \leq h}} \frac{|u(t', x') - u(t, x)|}{\sigma[(t, x), (t', x')] \sqrt{\log(1 + \sigma[(t, x), (t', x')]^{-1})}} = K \quad \text{a.s.}$$

Recalling (3.2), this result shows that the largest oscillation in I is of order $\sqrt{h^{2-\beta} \log(1/h)}$, which is larger than $\sqrt{h^{2-\beta} \log \log(1/h)}$ as specified by the LIL. It suggests that the LIL does not hold simultaneously for all $(t, x) \in I$ and there may be (random) exceptional points with much larger oscillation. Therefore, we can define singularities as such points where the LIL fails. More precisely, we say that (τ, λ) is a *singular point* of \tilde{u} in the λ -direction if

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty$$

and a *singular point* in the τ -direction if

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau + h, \lambda) - \tilde{u}(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty.$$

Our goal is to justify the existence of random singular points and study their propagation.

Fix $\tau_0 > 0$. Let us decompose \tilde{u} into $\tilde{u}_1 + \tilde{u}_2$, where

$$\tilde{u}_i(\tau, \lambda) = u_i\left(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\right), \quad i = 1, 2,$$

and

$$\begin{aligned} u_1(t, x) &= \frac{1}{2} W\left(\Delta(t, x) \cap \Pi(\tau_0)\right), \\ u_2(t, x) &= \frac{1}{2} W\left(\Delta(t, x) \cap \Pi(\tau_0)^c\right), \end{aligned}$$

where $\Pi(\tau_0)$ is the vertical stripe $\Pi(\tau_0) = \{(s, y) : 0 \leq s < \tau_0/\sqrt{2}, y \in \mathbb{R}\}$ and $\Pi(\tau_0)^c$ is the complement of $\Pi(\tau_0)$.

Let \mathcal{F}_{τ_0} be the σ -field generated by $\{W(B \cap \Pi(\tau_0)) : B \in \mathcal{B}_b(\mathbb{R}^2)\}$ and the \mathbb{P} -null sets. Note that \mathcal{F}_{τ_0} is independent of the process \tilde{u}_2 .

Following the approach of Walsh [21] and Blath and Martin [2], we will use Meyer's section theorem to prove the existence of a random singularity. Let us recall Meyer's section theorem ([7], Theorem 37, p.18):

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space and S be a $\mathcal{B}(\mathbb{R}_+) \times \mathcal{G}$ -measurable subset of $\mathbb{R}_+ \times \Omega$. Then there exists a \mathcal{G} -measurable random variable T with values in $(0, \infty]$ such that

- (a) the graph of T , denoted by $[T] := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : T(\omega) = t\}$, is contained in S ;
- (b) $\{T < \infty\}$ is equal to the projection $\pi(S)$ of S onto Ω .

Lemma 4.1. *Let $\tau_0 > 0$. Then there exists a positive, finite, \mathcal{F}_{τ_0} -measurable random variable Λ such that*

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}_1(\tau_0, \Lambda + h) - \tilde{u}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty \quad a.s.$$

Proof. Note that

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}_1(\tau_0, \Lambda + h) - \tilde{u}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \limsup_{h \rightarrow 0^+} \frac{|\tilde{v}_1(\tau_0, \Lambda + h) - \tilde{v}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}},$$

where $\tilde{v}_1(\tau_0, \lambda) = \tilde{u}_1(\tau_0, \lambda) - \tilde{u}_1(\tau_0, 0)$. The covariance for the process $\{\tilde{v}_1(\tau_0, \lambda), \lambda \geq 0\}$ is

$$\mathbb{E}[\tilde{v}_1(\tau_0, \lambda)\tilde{v}_1(\tau_0, \lambda')] = \frac{1}{4} \mathbb{E}[W(A_\lambda)W(A_{\lambda'})]$$

for $\lambda, \lambda' \geq 0$, where $A_\lambda = \{(t, x) : 0 \leq t < \tau_0/\sqrt{2}, -t < x \leq \sqrt{2}\lambda - t\}$. By (2.2) and Lemma 2.2,

$$\begin{aligned} \mathbb{E}[\tilde{v}_1(\tau_0, \lambda)\tilde{v}_1(\tau_0, \lambda')] &= \frac{1}{8\pi} \int_0^{\tau_0/\sqrt{2}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathcal{F} \mathbf{1}_{[-s, \sqrt{2}\lambda-s]}(\xi) \overline{\mathcal{F} \mathbf{1}_{[-s, \sqrt{2}\lambda'-s]}(\xi)} \\ &= \frac{1}{4(2-\beta)(1-\beta)} \int_0^{\tau_0/\sqrt{2}} \left(|\sqrt{2}\lambda|^{2-\beta} + |\sqrt{2}\lambda'|^{2-\beta} - |\sqrt{2}\lambda - \sqrt{2}\lambda'|^{2-\beta} \right) ds \\ &= \frac{2^{-(3+\beta)/2} \tau_0}{(2-\beta)(1-\beta)} \left(|\lambda|^{2-\beta} + |\lambda'|^{2-\beta} - |\lambda - \lambda'|^{2-\beta} \right). \end{aligned}$$

It follows that $\{C_0 \tilde{v}_1(\tau_0, \lambda), \lambda \geq 0\}$ is a fractional Brownian motion of Hurst parameter $(2-\beta)/2$ for some constant C_0 depending on τ_0 and β .

Let

$$S = \left\{ (\lambda, \omega) \in \mathbb{R}_+ \times \Omega : \limsup_{h \rightarrow 0^+} \frac{|\tilde{v}_1(\tau_0, \lambda + h)(\omega) - \tilde{v}_1(\tau_0, \lambda)(\omega)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty \right\}.$$

Then S is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}_{\tau_0}$ -measurable. Using Meyer's section theorem, we can find a positive \mathcal{F}_{τ_0} -measurable random variable Λ such that (a) $[\Lambda] \subset S$, and (b) $\pi(S) = \{\Lambda < \infty\}$.

We claim that $\Lambda < \infty$ a.s. Indeed, by the uniform modulus of continuity for fractional Brownian motion (cf. [9], Theorem 1.1), for any $0 \leq a < b$,

$$\limsup_{h \rightarrow 0^+} \sup_{\lambda \in [a, b]} \frac{|\tilde{v}_1(\tau_0, \lambda + h) - \tilde{v}_1(\tau_0, \lambda)|}{\sqrt{h^{2-\beta} \log(1/h)}} = C_0^{-1} \sqrt{2} \quad a.s. \quad (4.1)$$

We now use an argument with nested intervals (cf. [15], Theorem 1) to show the existence of a random λ^* such that

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{v}_1(\tau_0, \lambda^* + h) - \tilde{v}_1(\tau_0, \lambda^*)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty \quad (4.2)$$

with probability 1. First take an event Ω^* of probability 1 such that (4.1) holds for all intervals $[a, b]$, where a and b are rational numbers. Let

$$\varphi(h) = \frac{1}{2}C_0^{-1}\sqrt{2h^{2-\beta}\log(1/h)}.$$

Let $h_0 > 0$ be small such that φ is increasing on $[0, h_0]$. For an $\omega \in \Omega^*$, we define two sequences $(\lambda_n), (\lambda'_n)$ as follows. By (4.1), we can choose λ_1, λ'_1 , say in $[1, 2]$, with $\lambda_1 < \lambda'_1$ such that $\lambda'_1 - \lambda_1 < h_0$ and

$$|\tilde{v}_1(\tau_0, \lambda'_1) - \tilde{v}_1(\tau_0, \lambda_1)| > \varphi(\lambda'_1 - \lambda_1).$$

Suppose $n \geq 1$ and λ_n, λ'_n are chosen with $0 < \lambda'_n - \lambda_n \leq 2^{-n+1}$ and

$$|\tilde{v}_1(\tau_0, \lambda'_n) - \tilde{v}_1(\tau_0, \lambda_n)| > \varphi(\lambda'_n - \lambda_n).$$

Since \tilde{v}_1 is continuous in λ and $\varphi(h)$ is increasing for h small, we can find some $\tilde{\lambda}_n$ such that $\lambda_n < \tilde{\lambda}_n < \min\{\lambda'_n, \lambda_n + 2^{-n}\}$ and

$$|\tilde{v}_1(\tau_0, \lambda'_n) - \tilde{v}_1(\tau_0, \lambda)| > \varphi(\lambda'_n - \lambda) \quad \text{for all } \lambda \in [\lambda_n, \tilde{\lambda}_n]. \quad (4.3)$$

Then we can apply (4.1) for a rational interval $[a, b] \subseteq [\lambda_n, \tilde{\lambda}_n]$ to find λ_{n+1} and λ'_{n+1} such that $\lambda_n \leq \lambda_{n+1} < \lambda'_{n+1} \leq \tilde{\lambda}_n$ and

$$|\tilde{v}_1(\tau_0, \lambda'_{n+1}) - \tilde{v}_1(\tau_0, \lambda_{n+1})| > \varphi(\lambda'_{n+1} - \lambda_{n+1}).$$

We obtain a sequence of nested intervals $[\lambda_1, \lambda'_1] \supset [\lambda_2, \lambda'_2] \supset \dots$ with lengths $\lambda'_n - \lambda_n \leq 2^{-n+1}$. Therefore, the intervals contain a common point $\lambda^* \in [1, 2]$ such that $\lambda'_{n+1} \downarrow \lambda^*$. Since $\lambda^* \in [\lambda_n, \tilde{\lambda}_n]$ for all n , by (4.3) we have

$$|\tilde{v}_1(\tau_0, \lambda'_n) - \tilde{v}_1(\tau_0, \lambda^*)| > \varphi(\lambda'_n - \lambda^*).$$

Hence, for each $\omega \in \Omega^*$, there is at least one $\lambda^* > 0$ (depending on ω) such that (4.2) holds. It implies that $\Omega^* \subset \pi(S)$. Then from (b) we deduce that $\Lambda < \infty$ a.s., and from (a) we conclude that

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{v}_1(\tau_0, \Lambda + h) - \tilde{v}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty \quad \text{a.s.}$$

The proof of Lemma 4.1 is complete. \square

Lemma 4.2. *For any $\tau_0 > 0$ and $\lambda > 0$,*

$$\mathbb{P}\left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}_2(\tau, \lambda + h) - \tilde{u}_2(\tau, \lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\tau - \tau_0 + \lambda)^{1/2} \text{ for all } \tau \geq \tau_0\right) = 1.$$

Proof. By Propositions 2.1 and 3.1,

$$\mathbb{P}\left(\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\tau + \lambda)^{1/2} \text{ for all } \tau \geq 0\right) = 1.$$

Then the result can be obtained by the observation that $\{\tilde{u}_2(\tau_0 + \tau, \lambda), \tau, \lambda \geq 0\}$ has the same distribution as $\{\tilde{u}(\tau, \lambda), \tau, \lambda \geq 0\}$. Indeed, for any bounded Borel sets A, B in $\mathbb{R}_+ \times \mathbb{R}$ and

$c = (c_1, c_2) \in \mathbb{R}_+ \times \mathbb{R}$, by (2.2) and change of variables we have

$$\begin{aligned} \mathbb{E}[W(A+c)W(B+c)] &= \int_{c_1}^{\infty} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \mathbf{1}_A(s-c_1, y-c_2) |y-y'|^{-\beta} \mathbf{1}_B(s-c_1, y-c_2) \\ &= \int_0^{\infty} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \mathbf{1}_A(s, y) |y-y'|^{-\beta} \mathbf{1}_B(s, y) \\ &= \mathbb{E}[W(A)W(B)]. \end{aligned}$$

Since

$$\Delta\left(\frac{\tau_0 + \tau + \lambda}{\sqrt{2}}, \frac{-\tau_0 - \tau + \lambda}{\sqrt{2}}\right) \cap \Pi(\tau_0)^c = \Delta\left(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\right) + c,$$

where $c = (\frac{\tau_0}{\sqrt{2}}, -\frac{\tau_0}{\sqrt{2}})$, it follows that for any $\tau, \lambda, \tau', \lambda' \geq 0$,

$$\begin{aligned} \mathbb{E}[\tilde{u}_2(\tau_0 + \tau, \lambda)\tilde{u}_2(\tau_0 + \tau', \lambda')] &= \frac{1}{4} \mathbb{E}\left[W\left(\Delta\left(\frac{\tau_0 + \tau + \lambda}{\sqrt{2}}, \frac{-\tau_0 - \tau + \lambda}{\sqrt{2}}\right) \cap \Pi(\tau_0)^c\right)\right. \\ &\quad \left.\times W\left(\Delta\left(\frac{\tau_0 + \tau' + \lambda'}{\sqrt{2}}, \frac{-\tau_0 - \tau' + \lambda'}{\sqrt{2}}\right) \cap \Pi(\tau_0)^c\right)\right] \\ &= \frac{1}{4} \mathbb{E}\left[W\left(\Delta\left(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\right)\right)W\left(\Delta\left(\frac{\tau' + \lambda'}{\sqrt{2}}, \frac{-\tau' + \lambda'}{\sqrt{2}}\right)\right)\right] \\ &= \mathbb{E}[\tilde{u}(\tau, \lambda)\tilde{u}(\tau', \lambda')]. \end{aligned}$$

The result follows immediately. \square

We are now in a position to state and prove our main theorem below. The first part shows that if we fix $\tau_0 > 0$, then based on the information from the σ -field \mathcal{F}_{τ_0} , we are able to show the existence of a random singularity (τ_0, Λ) in the λ -direction. The second part says that if (τ_0, Λ) is a singular point in the λ -direction, then (τ, Λ) is also a singular point for all $\tau \geq \tau_0$. In other words, singularities in the λ -direction propagate orthogonally, along the line that is parallel to the τ -axis. By symmetry, it follows immediately that singularities in the τ -direction propagate along the line parallel to the λ -axis. These are the directions of the characteristic lines $t + x = c$ and $t - x = c$.

Theorem 4.3. *Let $\tau_0 > 0$. The following statements hold.*

- (i) *There exists a positive, finite, \mathcal{F}_{τ_0} -measurable random variable Λ such that*

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau_0, \Lambda + h) - \tilde{u}(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty \quad a.s.$$

- (ii) *If Λ is any positive, finite, \mathcal{F}_{τ_0} -measurable random variable, then on an event of probability 1, we have*

$$\limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau_0, \Lambda + h) - \tilde{u}(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty \Leftrightarrow \limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \Lambda + h) - \tilde{u}(\tau, \Lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}} = \infty$$

for all $\tau > \tau_0$ simultaneously.

Proof. To simplify notations, let us denote

$$L(\tau, \lambda) = \limsup_{h \rightarrow 0^+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}}$$

and for $i = 1, 2$,

$$L_i(\tau, \lambda) = \limsup_{h \rightarrow 0^+} \frac{|\tilde{u}_i(\tau, \lambda + h) - \tilde{u}_i(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log \log(1/h)}}.$$

As in Walsh [21] and Blath and Martin [2], we are going to use the fact that for two functions f and g , provided $\limsup_{h \rightarrow 0} |g(h)| < \infty$, we have

$$\limsup_{h \rightarrow 0} |f(h)| - \limsup_{h \rightarrow 0} |g(h)| \leq \limsup_{h \rightarrow 0} |f(h) + g(h)| \leq \limsup_{h \rightarrow 0} |f(h)| + \limsup_{h \rightarrow 0} |g(h)|. \quad (4.4)$$

(i). By Lemma 4.1, we can find a positive, finite, \mathcal{F}_{τ_0} -measurable random variable Λ such that

$$L_1(\tau_0, \Lambda) = \infty \quad \text{a.s.}$$

Since Λ and the process \tilde{u}_2 are independent, Lemma 4.2 implies that

$$L_2(\tau_0, \Lambda) = K_\beta \Lambda^{1/2} \quad \text{a.s.}$$

and this is finite a.s. Since $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$, it follows from the lower bound of (4.4) that

$$L(\tau_0, \Lambda) \geq L_1(\tau_0, \Lambda) - L_2(\tau_0, \Lambda) = \infty \quad \text{a.s.}$$

This completes the proof of (i).

(ii). Suppose Λ is a positive, finite, \mathcal{F}_{τ_0} -measurable random variable. By (4.4), we have

$$L_1(\tau, \Lambda) - L_2(\tau, \Lambda) \leq L(\tau, \Lambda) \leq L_1(\tau, \Lambda) + L_2(\tau, \Lambda) \quad (4.5)$$

for all $\tau \geq \tau_0$, provided that $L_2(\tau, \Lambda) < \infty$. Observe that for $\tau \geq \tau_0$,

$$\tilde{u}_1(\tau, \Lambda + h) - \tilde{u}_1(\tau, \Lambda) = \tilde{u}_1(\tau_0, \Lambda + h) - \tilde{u}_1(\tau_0, \Lambda),$$

hence $L_1(\tau, \Lambda) = L_1(\tau_0, \Lambda)$. Also, by Lemma 4.2 and independence between Λ and \tilde{u}_2 , we have

$$\mathbb{P}\left(L_2(\tau, \Lambda) = K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \text{ for all } \tau \geq \tau_0\right) = 1.$$

Since Λ is finite a.s., it follows from (4.5) that

$$\mathbb{P}\left(L_1(\tau_0, \Lambda) - K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \leq L(\tau, \Lambda) \leq L_1(\tau_0, \Lambda) + K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \text{ for all } \tau \geq \tau_0\right) = 1,$$

and it implies

$$\mathbb{P}\left(L(\tau_0, \Lambda) = \infty \Leftrightarrow L(\tau, \Lambda) = \infty \text{ for all } \tau \geq \tau_0\right) = 1. \quad \square$$

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