Ten Lectures on
"Gaussian Random Fields, SPDEs, Fractals, and Extremes"

Yimin Xiao
Michigan State University

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1 Lecture 1. Introduction to (Gaussian) random fields

Multivariate random fields (or spatial processes) have recently been the focus of much attention in probability and statistics, due to their extensive applications as spatial or spatio-temporal models in scientific areas where many problems involve data sets with multivariate measurements obtained at spatial locations.

In this section, we present an overview on random fields and provide concrete examples of random fields that are drawn from science and engineering.

We introduce important statistical characteristics such as self-similarity, operator-self- similarity, anisotropy, long range dependence of random fields.

A random field $X = \{X(t), t \in T\}$ is a family of random variables with values in state space S, where T is the parameter set. We consider $T \subseteq \mathbb{R}^N$ and $S = \mathbb{R}^d$ $(d \geq 1)$. Then X is called an (N, d) random field.

Random fields arise naturally in turbulence (A. N. Kolmogorov, 1941), oceanography (M.S. Longuet-Higgins, 1953, ...) spatial statistics, spatio-temporal geostatistics (G. Mathron, 1962), stochastic hydrology (Mandelbrot and Wallis, 1968–1969), ... image and signal processing (Krueger et al. 1996; Kaplan and Kuo,1996); Han and Denney 1999; Bonami and Estrade, 2003), just to mention a few.

In theories on random fields, the following questions are addressed:

1. How to construct random fields?
2. How to characterize and analyze random fields?
3. How to estimate parameters in random fields?

4. How to use random fields to make predictions?

In this lecture, we provide a brief introduction to answers to Questions (1) and (2).

The mathematical theory of random fields developed by Itô (1954), Yaglom (1957, 1987), Gihman and Skorohod (1974) provides an excellent framework for constructing and studying multivariate random fields. This lecture will introduce systematic methods for constructing univariate and multivariate Gaussian random fields, including characterization of cross-covariance matrices and the spectral method. Interesting examples of multivariate Gaussian random fields that can be constructed by using these methods include multivariate stationary Gaussian random fields with Matérn cross-covariance matrix and operator fractional Brownian motion. Another natural way to define multivariate random fields is through systems of stochastic partial differential equations.

1.1 Stationary random fields and their spectral representations

A real-valued random field $\{X(t), t \in \mathbb{R}^N\}$ is called second-order stationary if $\mathbb{E}(X(t)) \equiv m$, where m is a constant, and the covariance function depends on $s-t$ only:

$$\mathbb{E}\big[(X(s)-m)(X(t)-m)\big] = C(s-t), \quad \forall s,t \in \mathbb{R}^N.$$

The celebrated Bochner's Theorem (1932) says that a bounded and continuous function C is positive definite if and only if there is a finite Borel measure μ such that

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t,\, x\rangle}\, d\mu(x), \qquad \forall t \in \mathbb{R}^N.$$

Similar problems for second-order stationary processes were studied by N. Wiener (1930) and and by A. Y. Khinchin (1934). They proved the following representation theorem:

Theorem 1.1 *Every second-order stationary random field $\{X(t), t \in \mathbb{R}^N\}$ with continuous covariance can be represented as*

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t,\, x\rangle}\, d\widetilde{W}(x),$$

where $\widetilde{W}$ is certain random measure.

If $X = \{X(t), t \in \mathbb{R}^N\}$ is a centered, stationary Gaussian random field with values in $\mathbb{R}$ whose covariance function is the Fourier transform of μ, then there is a complex-valued Gaussian random measure $\widetilde{W}$ on $\mathcal{B}(\mathbb{R}^N)$ such that $\mathbb{E}\big(\widetilde{W}(A)\big) = 0$,

$$\mathbb{E}(\widetilde{W}(A)\overline{\widetilde{W}(B)}) = \mu(A \cap B) \quad \text{and} \quad \widetilde{W}(-A) = \overline{\widetilde{W}(A)}$$

and X has the following Wiener integral representation:

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t,\, x\rangle}\, d\widetilde{W}(x).$$

The finite measure μ is called the spectral measure of X.

An important class of isotropic stationary random fields are those with the Matérn covariance function

$$C(t) = \frac{1}{\Gamma(\nu)2^{\nu-1}}\Big(\sqrt{2\nu}\frac{|t|}{\rho}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{|t|}{\rho}\Big), \tag{1.1}$$

where Γ is the Gamma function, K_ν is the modified Bessel function of the second kind, and ρ and ν are non-negative parameters.

Since the covariance function $C(t)$ in (1.1) depends only on the Euclidean norm $|t|$, the corresponding Gaussian field X is called *isotropic*.

By the inverse Fourier transform, one can show that the spectral measure of X has the following density function:

$$f(\lambda) = \frac{1}{(2\pi)^N} \frac{1}{(|\lambda|^2 + \frac{\rho^2}{2\nu})^{\nu + \frac{N}{2}}}, \qquad \forall \lambda \in \mathbb{R}^N. \tag{1.2}$$

Whittle (1954) showed that the Gaussian random field X with Matérn covariance function (1.1) can be obtained as the solution to the following fractional SPDE

$$\big(\Delta + \frac{\rho^2}{2\nu}\big)^{\frac{\nu}{2} + \frac{N}{4}} X(t) = \dot{W}(t),$$

where $\Delta = \frac{\partial^2}{dt_1^2} + \cdots + \frac{\partial^2}{dt_N^2}$ is the N-dimensional Laplacian, and $\dot{W}(t)$ is the white noise.

1.2 Gaussian random fields with stationary increments

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Yaglom (1954) showed that, if $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \langle s, At\rangle + \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda),$$

where A is a nonnegative definite real $N \times N$ matrix and $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |\lambda|^2)\ \Delta(d\lambda) < \infty. \tag{1.3}$$

In analogy to the stationary case, the measure Δ is called the *spectral measure* of X.

We assume that $A = 0$. Then

$$\mathbb{E}\big[(X(s) - X(t))^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle s - t, \lambda\rangle\big)\Delta(d\lambda);$$

and X has the stochastic integral representation:

$$X(t) \overset{d}{=} \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\, \widetilde{W}(d\lambda), \tag{1.4}$$

where $\overset{d}{=}$ denotes equality of all finite-dimensional distributions, $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Gaussian fields with stationary increments can be constructed by choosing spectral measures Δ. In turn, the spectral measure characterizes the properties of the Gaussian random field.

We consider two examples.

Example 1.1 If Δ has a density function

$$f_H(\lambda) = c(H, N)|\lambda|^{-(2H+N)},$$

where $H \in (0,1)$ and $c(H,N) > 0$, then X is fractional Brownian motion with index H.

It can be verified that (for proper choice of $c(H,N)$),

$$\begin{aligned}\mathbb{E}\big[(X(s) - X(t))^2\big] &= 2c(H,N) \int_{\mathbb{R}^N} \frac{1 - \cos\langle s-t, \lambda\rangle}{|\lambda|^{2H+N}}\, d\lambda \\ &= |s-t|^{2H}.\end{aligned}$$

For the last identity, see, e.g., Schoenberg (1939). Moreover, one can verify the following properties:

- FBm X has stationary increments: for any $b \in \mathbb{R}^N$,

$$\big\{X(t+b) - X(b),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{X(t),\, t \in \mathbb{R}^N\big\}. \tag{1.5}$$

- FBm X is H-self-similar: for every constant $c > 0$,

$$\big\{X(ct),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{c^H X(t),\, t \in \mathbb{R}^N\big\}. \tag{1.6}$$

Example 1.2 A large class of Gaussian fields can be obtained by letting spectral density functions satisfy (1.3) and

$$f(\lambda) \asymp \frac{1}{\big(\sum_{j=1}^N |\lambda_j|^{\beta_j}\big)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \ge 1, \tag{1.7}$$

where $(\beta_1, \ldots, \beta_N) \in (0,\infty)^N$ and $\gamma > \sum_{j=1}^N \frac{1}{\beta_j}$.

This last condition is necessary for $f \in L^2(\mathbb{R}^N)$. More conveniently, we re-write (1.7) as

$$f(\lambda) \asymp \frac{1}{\big(\sum_{j=1}^N |\lambda_j|^{H_j}\big)^{Q+2}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \ge 1, \tag{1.8}$$

where $H_j = \frac{\beta_j}{2}\big(\gamma - \sum_{i=1}^N \frac{1}{\beta_i}\big)$ and $Q = \sum_{j=1}^N H_j^{-1}$.

1.3 More examples of non-stationary Gaussian random fields

The Brownian sheet and fractional Brownian sheets

The Brownian sheet $W = \{W(t),\, t \in \mathbb{R}_+^N\}$ is a centered (N,d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[W_i(s) W_j(t)\big] = \delta_{ij} \prod_{k=1}^N s_k \wedge t_k. \tag{1.9}$$

The Brownian sheet is a random field is a random field extension of the Wiener process (Brownian motion).

- When $N = 1$, W is Brownian motion in $\mathbb{R}^d$.
- W is $N/2$-self-similar, but it does not have stationary increments.
- It gives rise to the Gaussian white noise $\dot{W}$, which can be used as a stochastic integrator.

Fractional Brownian sheet

A fractional Brownian sheet $W^{\vec{H}} = \{W^{\vec{H}}(t), t \in \mathbb{R}^N\}$ is a mean zero Gaussian field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^{\vec{H}}(s)W^{\vec{H}}(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right),$$

where $\vec{H} = (H_1, \ldots, H_N) \in (0,1)^N$.

For all constants $c > 0$,

$$\left\{W^{\vec{H}}(c^E t),\, t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c\, W^{\vec{H}}(t),\ t \in \mathbb{R}^N\right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$.

This is referred to as an "operator-scaling" property.

Linear stochastic heat equation

Consider the linear stochastic heat equation with the Gaussian noise $\dot{W}$:

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma\, \dot{W}, \quad t \ge 0,\ x \in \mathbb{R}^k, \\ &u(0,x) \equiv 0, \end{aligned} \tag{1.10}$$

where Δ is the Laplacian operator in the spatial variables, σ is a constant or a deterministic function, and $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance [Dalang (1999)] given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise considered by Walsh (1986).

It follows from Walsh (1986) and Dalang (1999) that the mild solution of (1.10) is the mean zero Gaussian random field $u = \{u(t,x), t \ge 0, x \in \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \widetilde{G}_{t-r}(x-y)\, \sigma W(drdy), \quad t \ge 0, x \in \mathbb{R}, \tag{1.11}$$

where $\widetilde{G}_t(x)$ is the Green kernel given by

$$\widetilde{G}_t(x) = (2\pi t)^{-1/2} \exp\Big(-\frac{|x|^2}{2t}\Big), \qquad \forall\, t > 0, x \in \mathbb{R}^k.$$

Linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \ge 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0, \end{cases} \tag{1.12}$$

where $\dot{W}$ is a Gaussian noise as in the previous example with exponent $\beta \in (0, k \wedge 2)$.

The existence of real-valued process solution to (1.12) was studied by Walsh (1986) for the space-time white noise and by Dalang (1999) in the more general setting.

We recall briefly some known results.

Let G be the fundamental solution of the wave equation. Then

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}} \quad \text{if} \quad k=1;$$

$$G(t,x) = c_k\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(k-2)/2}(t^2-|x|^2)_+^{-1/2}, \quad \text{if } k \geq 2 \text{ is even;}$$

$$G(t,x) = c_k\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(k-3)/2}\frac{\sigma_t^k(dx)}{t}, \quad \text{if } k \geq 3 \text{ is odd,}$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$.

Note that for $k \geq 3$, G is not a function but a distribution.

For any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0,\, \xi \in \mathbb{R}^k. \tag{1.13}$$

Dalang (1999) extended Walsh's stochastic integration and proved that the real-valued process solution of equation (1.12) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{1.14}$$

where W is the martingale measure induced by the noise $\dot{W}$.

The range of β has been chosen so that the stochastic integral exists.

The solution $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$ is a centered Gaussian random field, which can be studied by using general Gaussian methods.

Recall from Theorem 2 of Dalang (1999) that

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^k} H(s,y) W(ds\, dy)\Big)^2\Big] = c\int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} |\mathscr{F}(H(s,\cdot))(\xi)|^2 \tag{1.15}$$

provided that $s \mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{k-\beta}} |\mathscr{F}(H(s,\cdot)(\xi)|^2 < \infty.$$

Eq. (1.15) is a basic tool for studying the Gaussian random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$.

Non-linear stochastic heat & wave equations

Many authors have studied the following nonlinear SPDE:

$$\begin{cases} \mathcal{L}u = b(u) + \sigma(u)\dot{W}, \quad t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, \end{cases} \tag{1.16}$$

where $\mathcal{L}$ is a partial differential operator, σ and b are non-random functions that satisfy some regularity conditions [e.g., σ and b are Lipschitz continuous.]

For example, $\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u$ and $\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \Delta u$ give the stochastic heat and wave equation, respectively.

The solutions, when they exist, are in general non-Gaussian random fields. We refer to Dalang (1999), Khoshnevisan (2014) for more information.

1.4 Multivariate Gaussian random fields

Consider a multivariate random field $\boldsymbol{X} = \{\boldsymbol{X}(t),\, t \in \mathbb{R}^N\}$ taking values in $\mathbb{R}^d$ defined by

$$\boldsymbol{X}(t) = (X_1(t), \cdots, X_d(t)), \qquad t \in \mathbb{R}^N. \tag{1.17}$$

Their key features are:

- the components $X_1, \ldots, X_d$ are dependent.
- $X_1, \ldots, X_d$ may have different smoothness properties.

For any $i,\, j = 1, \ldots, d$, define

$$C_{ij}(s,t) := \mathbb{E}[X_i(s)X_j(t)]. \tag{1.18}$$

They are called the cross-covariance functions of $\boldsymbol{X}$.

(i) The multivariate Matérn random fields

Gneiting, Kleiber and Schlather (2010) introduced a class of multivariate stationary Matérn models $\{\boldsymbol{X}(t),\, t \in \mathbb{R}^N\}$ in (1.17) with marginal and cross-covariance functions of the form

$$C_{ij}(s,t) = M(s-t|\nu_{ij}, a_{ij}),$$

where

$$M(h|\nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)}(a|h|)^{\nu} K_{\nu}(a|h|).$$

and provided conditions for such matrix-valued functions to form legitimate cross-covariance functions.

See also Apanansovich, Genton and Sun (2012), Kleiber and Nychka (2013).

The bivariate Matérn fields

Let $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$ be an $\mathbb{R}^2$-valued Gaussian field whose covariance matrix is determined by

$$C(h) = \begin{pmatrix} c_{11}(h) & c_{12}(h) \\ c_{21}(h) & c_{22}(h) \end{pmatrix}, \tag{1.19}$$

where $c_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} c_{11}(h) &= \sigma_1^2 M(h|\nu_1, a_1), \\ c_{22}(h) &= \sigma_2^2 M(h|\nu_2, a_2), \\ c_{12}(h) &= c_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}) \end{aligned} \tag{1.20}$$

with $a_1, a_2, a_{12}, \sigma_1, \sigma_2 > 0$ and $\rho \in (-1, 1)$.

Gneiting, et al. (2010) gave NSC for (1.19) to be valid. In particular, if $\rho \neq 0$, one must have

$$\frac{\nu_1 + \nu_2}{2} \le \nu_{12}.$$

The parameters ν_1 and ν_2 control the smoothness of the sample function $t \mapsto \boldsymbol{X}(t)$.

For example, if $\nu_1 > 1$, then a.s. the sample function $t \mapsto X_1(t)$ is continuously differentiable. This can be proved using the spectral density.

Zhou and X. (2017, 2018) studied extreme values and estimation problems for a class of bivariate random fields that includes the bivariate Matérn fields.

(ii). Multivariate random fields with stationary increments

An $\mathbb{R}^d$-valued Gaussian random field $\boldsymbol{X} = \{\boldsymbol{X}(t), t \in \mathbb{R}^N\}$ is said to have stationary increments if $\forall\, t_0 \in \mathbb{R}^N$,

$$\{\boldsymbol{X}(t + t_0) - \boldsymbol{X}(t_0), t \in \mathbb{R}^N\} \stackrel{d}{=} \{\boldsymbol{X}(t) - \boldsymbol{X}(0), t \in \mathbb{R}^N\}.$$

A general framework for multivariate random fields with stationary increments was provided by Yaglom (1957).

As an example, we consider a spacial class of *operator fractional Brownian motions*.

Let D be a linear operator on $\mathbb{R}^d$ (or a $d \times d$ real matrix). The operator norm of D is defined by

$$\|D\| = \max_{|x|=1} |Dx|.$$

It can be shown that if $D = (a_{ij})$ then

$$\max_{1 \le i,j \le d} |a_{ij}| \le \|D\| \le d^{3/2} \max_{1 \le i,j \le d} |a_{ij}|.$$

Denote the eigenvalues of D by

$$\lambda_k = \alpha_k + i\beta_k, \quad (k = 1, \ldots, d).$$

We assume that

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d < 1. \tag{1.21}$$

For any $c > 0$, we define the linear operator c^D by

$$c^D = \sum_{k=0}^{\infty} \frac{(\ln c)^k}{k!} D^k.$$

(a). Moving average representation

One can define *ofBm* $\boldsymbol{X} = \{\boldsymbol{X}(t), t \in \mathbb{R}\}$ in $\mathbb{R}^d$ by using the stochastic integration method:

$$\boldsymbol{X}(t) = \int_{\mathbb{R}} \big[(t - r)_+^{D - \frac{1}{2}I} - (-r)_+^{D - \frac{1}{2}I}\big] W(dr), \tag{1.22}$$

where W is d-dimensional Brownian motion, is an operator fractional Brownian motion with exponent D.

It has the following properties:

- stationary increments.

- (operator self-similarity) For every constant $c > 0$,

$$\{\boldsymbol{X}(c\,t), t \in \mathbb{R}\} \overset{d}{=} \{c^D \boldsymbol{X}(t), t \in \mathbb{R}\}.$$

(b). Harmonizable representation

The Gaussian random field $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$\boldsymbol{Y}(t) = \int_{\mathbb{R}^N} \frac{e^{i\langle t,r\rangle} - 1}{|r|^{D+\frac{N}{2}I}} \widetilde{\boldsymbol{W}}(dr), \tag{1.23}$$

where $\widetilde{\boldsymbol{W}}$ is a complex-valued Gaussian random measure on $\mathbb{R}^d$ with Lebesgue control measure and i.i.d. components, is also an operator fractional Brownian motion with exponent D.

In order to verify that the stochastic integrals in (1.22) and (1.23) are well defined, it is sufficient to verify respectively that

$$\int_{\mathbb{R}} \left\|(t-r)_+^{D-\frac{1}{2}I} - (-r)_+^{D-\frac{1}{2}I}\right\|^2 dr < \infty,$$

and

$$\int_{\mathbb{R}^N} (1 - \cos\langle t, r\rangle) \left\||r|^{-D-\frac{N}{2}I}\right\|^2 dr < \infty.$$

This is where condition (1.21) is needed.

(iii). Operator-scaling and operator-self-similar random fields

Li and X. (2011) constructed a large class of more general, namely, operator-scaling and operator-self-similar random fields with stationary increments.

Several authors have studied properties of these random fields. See, for example,

- Ercan Sönmez (2017, 2018, 2020).

- Kremer and Scheffler (2019) for further development and recent results.

- Shen, Stilian, and Hsing (2020).

(iv). Systems of stochastic partial differential equations

There has been a lot of recent research on this topic, which we do not discuss here. In the subsequent sections, we will consider the systems of stochastic heat and wave equations.

(v). Matrix-valued Gaussian random fields

Let $\xi = \{\xi(t) : t \in \mathbb{R}^N_+\}$ be a centered Gaussian random field and let $\{\xi_{i,j} : i, j \in \mathbb{N}\}$ be a family of independent copies of ξ.

Consider the symmetric $d \times d$ matrix-valued process $\boldsymbol{X} = \{X_{i,j}(t); t \in \mathbb{R}^N_+, 1 \le i, j \le d\}$ defined by

$$X_{i,j}(t) = \begin{cases} \xi_{i,j}(t), & i < j; \\ \sqrt{2}\xi_{i,i}(t), & i = j; \\ \xi_{j,i}(t), & i > j. \end{cases} \tag{1.24}$$

One may study statistical and sample path properties of the eigenvalues of $\boldsymbol{X}$.

2 Lecture 2. Gaussian random fields: general methods

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a random field. For each $\omega \in \Omega$, the function $X(\cdot, \omega) : \mathbb{R}^N \to \mathbb{R}^d$, $t \mapsto X(t, \omega)$, is called a sample function of X.

The following are natural questions:

(i) When are the sample functions of X bounded, or continuous?

(ii) When are the sample functions of X differentiable?

(iii) How to characterize the analytic and geometric properties of $X(\cdot)$ precisely?

We start with some general methods for Gaussian fields.

2.1 The entropy method

Let $X = \{X(t),\, t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$, where (T, τ) is a metric space; e.g., $T = [0, 1]^N$, or $T = \mathbb{S}^{N-1}$.

We define a *pseudo metric* $d_X(\cdot, \cdot) :\ T \times T \to [0, \infty)$ by

$$d_X(s, t) = \sqrt{\mathbb{E}\big[\big(X(t) - X(s)\big)^2\big]}\ .$$

(d_X is often called the canonical metric for X.)

Let $D = \sup_{t,s\in T} d_X(s, t)$ be the diameter of T, under d_X. For any $\varepsilon > 0$, let $N(T, d_X, \varepsilon)$ be the minimum number of d_X-balls of radius ε that cover T. $N(T, d_X, \varepsilon)$ is also called the metric entropy of T.

Theorem 2.1 *[Dudley, 1967] Assume $N(T,\, d_X, \varepsilon) < \infty$ for every $\varepsilon > 0$. If*

$$\int_0^D \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon < \infty.$$

Then $\exists$ a modification of X, still denoted by X, such that

$$\mathbb{E}\Big(\sup_{t\in T} X(t)\Big) \le 16\sqrt{2} \int_0^{\frac{D}{2}} \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon. \tag{2.1}$$

Fernique (1975) proved that (2.1) is also necessary if X is a Gaussian process which is stationary or has stationary increments.

The proof of Dudley's Theorem is based on a chaining argument. See Talagrand (2005), Marcus and Rosen (2007).

The proof of Dudley's Theorem gives an upper bound for the uniform modulus of continuity of X:

$$\omega_{X,\tau}(\delta) = \sup_{s,t\in T, \tau(s,t)\le\delta} \big|X(s) - X(t)\big|.$$

The following theorem is taken from Adler and Talor (2007).

Theorem 2.2 *Under the condition of Dudley's theorem, there is a random variable* $\eta \in (0, \infty)$ *such that for all* $0 < \delta < \eta$,

$$\omega_{X,d_X}(\delta) \le K \int_0^\delta \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon,$$

where $\omega_{X,d_X}(\delta)$ *is the modulus of continuity of* $X(t)$ *on* $(T,\, d_X)$ *and* K *is a universal constant.*

Theorem 2.2 can be applied easily to a wide class of Gaussian processes. For example, fractional Brownian motion, solutions of linear stochastic heat and wave equations, and a Gaussian random field $\{X(t), t \in T\}$ satisfying

$$d_X(s,t) \asymp \Big(\log \frac{1}{|s-t|}\Big)^{-\gamma},$$

its sample functions are continuous if $\gamma > 1/2$.

Corollary 2.3 *Let* $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ *be a fractional Brownian motion with index* $H \in (0,1)$. *Then* B^H *has a modification, still denoted by* B^H, *whose sample functions are almost surely continuous. Moreover,*

$$\limsup_{\varepsilon \to 0} \frac{\max_{t \in [0,1]^N, |s| \le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} \le K, \quad a.s.$$

Proof Recall that $d_{B^H}(s,t) = |s-t|^H$ and $\forall\ \varepsilon > 0$,

$$N\big([0,1]^N, d_{B^H},\, \varepsilon\big) \le\ K \Big(\frac{1}{\varepsilon^{1/H}}\Big)^N.$$

It follows from Theorem 2.2 that $\exists$ a random variable $\eta > 0$ and a constant $K > 0$ such that for all $0 < \delta < \eta$,

$$\omega_{B^H}(\delta)\ \le\ K \int_0^\delta \sqrt{\log\Big(\frac{1}{\varepsilon^{1/H}}\Big)}\ d\varepsilon \le K\ \delta \sqrt{\log \frac{1}{\delta}} \quad \text{a.s.}$$

Returning to the Euclidean metric and noticing

$$d_{B^H}(s,t) \le \delta \iff |s-t| \le \delta^{1/H},$$

yields the desired result. □

Later on, we will prove that there is a constant $K \in (0, \infty)$ such that

$$\limsup_{\varepsilon \to 0} \frac{\max_{t \in [0,1]^N, |s| \le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} = K, \quad a.s.$$

This is an analogue of Lévy's uniform modulus of continuity for Brownian motion.

2.2 Majorizing measure (generic chaining)

In general, (2.1) is not necessary for sample continuity.

Talagrand (1987) proved the following necessary and sufficient for the boundedness and continuity.

Theorem 2.4 [Talagrand, 1987] *Let $X = \{X(t),\, t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$. Suppose $D = \sup_{t,s\in T} d_X(s,t) < \infty$. Then*

(i) X has a modification which is bounded on T if and only if there exists a probability measure μ on T such that

$$\sup_{t\in T} \int_0^D \Big(\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}\Big)^{1/2} d\varepsilon < \infty, \tag{2.2}$$

where $B_{d_X}(t,\varepsilon) = \{s \in T : d_X(s,t) \le \varepsilon\}$. Moreover,

$$\mathbb{E}\Big(\sup_{t\in T} X(t)\Big) \le K \inf_{\mu} \sup_{t\in T} \int_0^\infty \Big(\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}\Big)^{1/2} d\varepsilon.$$

(ii) There exists a modification of X with bounded, uniformly continuous sample functions if and only if there exists a probability measure μ on T such that

$$\lim_{\varepsilon\to 0} \sup_{t\in T} \int_0^\varepsilon \Big(\log \frac{1}{\mu(B_{d_X}(t,u))}\Big)^{1/2} du = 0.$$

Kwapień and Rosiński (2004) provided an upper bound for the uniform modulus of continuity in terms of "weakly majorizing measure".

Theorem 2.5 *Under the condition of Theorem 2.4, there exists a random variable $\eta \in (0, \infty)$ and a constant $K > 0$ such that for all 0< $\delta < \eta$,*

$$\omega_{X,d_X}(\delta) \le K \int_0^\delta \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon, \quad a.s.$$

2.3 Differentiability

(i). *Mean-square differentiability*: the mean square partial derivative of X at t is defined as

$$\frac{\partial X(t)}{\partial t_j} = \mathrm{l.i.m}_{h\to 0} \frac{X(t + he_j) - X(t)}{h},$$

where e_j is the unit vector in the j-th direction.

For a Gaussian field, sufficient conditions can be given in terms of the differentiability of the covariance function (Adler, 1981).

(ii). *Sample path differentiability*: the sample function $t \mapsto X(t)$ is differentiable. This is much stronger and more useful than (i).

Sample path differentiability of $X(t)$ can be proved by using criteria for continuity.

Consider a centered Gaussian field with stationary increments whose spectral density function satisfies

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N, \quad |\lambda| \geq 1, \tag{2.3}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

$$\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}.$$

Theorem 2.6 [Xue and Xiao, 2011] *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and spectral density which satisfies (2.3).*

(i) If

$$\beta_j \Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) > 2, \tag{2.4}$$

then the partial derivative $\partial X(t)/\partial t_j$ is continuous almost surely. In particular, if (2.4) holds for all $1 \leq j \leq N$, then almost surely $X(t)$ is continuously differentiable.

(ii) If $\max\limits_{1\leq j\leq N} \beta_j\big(\gamma - \sum_{i=1}^{N} 1/\beta_i\big) \leq 2$, then $X(t)$ is not differentiable in any direction.

Proof of (i) : Under (2.4), we know that the mean square partial derivative $X_j'(t)$ exists. In order to show that $X_j'(t)$ has a continuous version, by Kolmorogov's continuity theorem, it is enough to show that for any compact interval $I \subset \mathbb{R}^N$, there exist constants $c > 0$ and $\eta > 0$ such that

$$\mathbb{E}\big[X_j'(s) - X_j'(t)\big]^2 \leq c\,|s - t|^{\eta} \quad \forall\, s, t \in I. \tag{2.5}$$

By the spectral representation of X, we have

$$\begin{aligned}\mathbb{E}\big(X_j'(s) - X_j'(t)\big)^2 &= \mathbb{E}\big[(X_j'(s))^2\big] + \mathbb{E}\big[(X_j'(t))^2\big] - 2\mathbb{E}\big[(X_j'(s)X_j'(t))\big] \\ &= 2\int_{\mathbb{R}^N} \lambda_j^2 \big(1 - \cos\langle s - t, \lambda\rangle\big) f(\lambda) d\lambda.\end{aligned}$$

From this, we can verify that (2.5) holds under (2.4).

It follows from (2.5) that the Gaussian field $X_j' = \{X_j'(t), t \in \mathbb{R}^N\}$ has a continuous version [still denoted by X_j'].

We define a new Gaussian field $\widetilde{X} = \{\widetilde{X}(t), t \in \mathbb{R}^N\}$ by

$$\begin{aligned}\widetilde{X}(t) &= X(t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_N) \\ &\qquad + \int_0^{t_j} X_j'(t_1, \cdots, t_{j-1}, s_j, t_{j+1}, \cdots, t_N)\, ds_j.\end{aligned} \tag{2.6}$$

Then we can verify that $\widetilde{X}$ is a continuous version of X and, for every $t \in \mathbb{R}^N$, $\widetilde{X}_j'(t) = X_j'(t)$ almost surely. This amounts to verify that for every $t \in \mathbb{R}^N$,

$$\mathbb{E}\big[\big(\widetilde{X}(t) - X(t)\big)^2\big] = 0,$$

which can be proved by using (2.6) and the representations for $X(t)$ and $X_j'(t)$. We omit the details. □

3 Lecture 3. Exact Results on Regularity of Gaussian Random Fields

For a Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$, we study:

(i) Local modulus of continuity: law of the iterated logarithm (LIL)

(ii) Chung's law of the iterated logarithm

(iii) Uniform modulus of continuity

(iv) Modulus of non-diffenerability

3.1 Local modulus of continuity

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered, real-valued Gaussian random field. For local oscillation of $X(t)$ near a fixed point $t^0 \in \mathbb{R}^N$, we may study the following question: Are there functions φ_1, $\varphi_2 : \mathbb{R}^N \to \mathbb{R}_+$ and constants $c_1, c_2 \in (0, \infty)$ such that

$$\limsup_{r\to 0} \max_{|h|\le r} \frac{|X(t^0+h) - X(t^0)|}{\varphi_1(h)} = \kappa_1, \qquad \text{a.s.}$$

and

$$\liminf_{r\to 0} \max_{|h|\le r} \frac{|X(t^0+h) - X(t^0)|}{\varphi_2(h)} = \kappa_2, \qquad \text{a.s.?}$$

The answers to these questions are referred to as the LIL and Chung's LIL, respectively. They describe different aspects of X near t^0 and rely on different methods.

Many authors have studied these questions for Gaussian random fields, usually under the extra condition of stationarity, or stationarity of increments. See, e.g., the book by Marcus and Rosen (2006) for Gaussian processes, Li and Shao (2001), Meerschaert, Wang and Xiao (2013) for Gaussian random fields.

We will use the following setting from Dalang, Mueller and Xiao (2017), which does not require stationarity of the Gaussian random field nor its increments, and can handle anisotropy. It is more convenient for applications to the solutions of linear SPDEs.

Condition (A1) Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\, t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X(\mathbf{t})$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

(b) There are constants $a_0 > 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \le a \le b \le \infty$ and $s = (s_1, \ldots, s_N), t = (t_1, \ldots, t_N) \in T$,

$$\begin{aligned} &\big\| v([a,b), s) - X(s) - v([a,b), t) + X(t) \big\|_{L^2} \\ &\le C\Big(\sum_{j=1}^N a^{\gamma_j} |s_j - t_j| + b^{-1} \Big), \end{aligned} \tag{3.1}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\big\|v([0,a_0),s) - v([0,a_0),t)\big\|_{L^2} \le C\sum_{j=1}^{N}|s_j - t_j|. \tag{3.2}$$

The parameters γ_j $(j = 1,\ldots,N)$ in Condition (A1) are important for characterizing sample path properties of $X(t)$.

Let

$$H_j = (\gamma_j + 1)^{-1} \quad \text{and} \quad Q = \sum_{j=1}^{N} H_j^{-1}.$$

Define the metric $\rho(s,t)$ on $\mathbb{R}^N$ by

$$\rho(s,t) = \sum_{j=1}^{N}|s_j - t_j|^{H_j}.$$

In order to see that (A1) is satisfied by the solution of an SPED, one needs to construct the random field $v(A,x)$. As an example, consider the solution of the linear one-dimensional heat equation driven by space-time white noise. In this case, $\mathbb{R}^N$ is replaced by $\mathbb{R}_+ \times \mathbb{R}$, and $X(t)$ is $u(t,x)$. Dalang, Mueller and X. (2017) defined

$$v(A,t,x) = \iint_{\max(|\tau|^{\frac{1}{4}},\,|\xi|^{\frac{1}{2}})\in A} e^{-i\xi x}\frac{e^{-i\tau t} - e^{-t\xi^2}}{|\xi|^2 - i\tau}W(d\tau,d\xi),$$

and verified that (A1) is satisfied with $\gamma_1 = 3$, $\gamma_2 = 1$. Thus, $H_1 = 1/4$ and $H_2 = 1/2$.

The following lemmas are needed for applying general Gaussian methods. For example, Lemma 2.1 can be applied to derive an upper bound for the uniform modulus of continuity for $\{X(t), t\in T\}$.

Lemma 3.1 [Dalang, Mueller, Xiao (2017)] *Under (A1), there is a constant $c\in(0,\infty)$ such that $\rho(s,t)$*

$$d_X(s,t) \le c\,\rho(s,t), \quad \forall\, s,t\in T, \tag{3.3}$$

where $d_X(s,t) = \|X(s) - X(t)\|_{L^2}$ is the canonical metric.

Proof For any $a > a_0$,

$$\begin{aligned} d_X(s,t) \le{}& \|X(s) - v([a_0,a[,s) - X(t) + v([a_0,a[,t)\|_{L^2} \\ &+ \|v([a_0,a[,s) - v([a_0,a[,\,t)\|_{L^2}. \end{aligned}$$

By (7.9) in (A1)(b), we have

$$\begin{aligned} &\|v([a_0,a[,\,s) - v([a_0,a[,\,t)\|_{L^2} \\ &\le \|X(s) - v([a,\infty[,\,s) - X(\mathbf{t}) + v([a,\infty[,\,t)\|_{L^2} \\ &\quad + \|-v([0,a_0[,\,s) + v([0,a_0[,\,t)\|_{L^2}. \end{aligned}$$

Applying (7.10) in (A1)(b), we see that

$$d_X(s,\,t) \le C\left[\sum_{j=1}^{N}(a_0^{H_j^{-1}-1} + a^{H_j^{-1}-1})|s_j - t_j| + a^{-1} + \sum_{j=1}^{N}|s_j - t_j|\right].$$

By hypothesis, $\max_{j=1,\ldots,N}|s_j - t_j|^{H_j} \le \rho(\mathbf{s},\mathbf{t}) \le C\,a_0^{-1}$, so we choose $a \ge a_0$ such that $\max_{j=1,\ldots,N}|s_j - t_j|^{H_j} = a^{-1}$.

Notice that

$$
\begin{aligned}
&(a_0^{H_j^{-1}-1} + a^{H_j^{-1}-1})|s_j - t_j| \\
&= \Big[\big(a_0\,|s_j - t_j|^{H_j}\big)^{\frac{1-H_j}{H_j}} + \big(a\,|s_j - t_j|^{H_j}\big)^{\frac{1-H_j}{H_j}}\Big]|s_j - t_j|^{H_j} \\
&\le 2\,\big(a\,|s_j - t_j|^{H_j}\big)^{\frac{1-H_j}{H_j}}\,|s_j - t_j|^{H_j} \\
&\le 2|s_j - t_j|^{H_j}
\end{aligned}
$$

by the choice of a. This proves (3.3). □

Condition (A1) indicates that $X(\mathbf{t})$ can be approximated by $v([a,b],\mathbf{t})$. The following lemma quantifies the approximation error.

Lemma 3.2 [Dalang, Mueller, Xiao (2017)] *Assume that (A1) holds. Consider $b > a > 1$ and $r > 0$ small. Set*

$$A = \sum_{j=1}^{N} a^{H_j^{-1}-1}\, r^{H_j^{-1}} + b^{-1}.$$

There are constants A_0, K and c such that for $A \le A_0 r$ and

$$u \ge KA \log^{1/2}\left(\frac{r}{A}\right), \tag{3.4}$$

we have for all $t^0 \in T$,

$$
\begin{aligned}
&\mathbb{P}\Big\{ \sup_{t\in S(t^0,r)} \big|X(t) - X(t^0) - (v([a,b],\, t) - v([a,b],\, t^0))\big| \ge u \Big\} \\
&\le \exp\Big(-\frac{u^2}{cA^2}\Big),
\end{aligned}
\tag{3.5}
$$

where $S(t^0, r) = \{t \in T : \rho(t,\, t^0) \le r\}$.

The proof of Lemma 3.2 makes use of the following important inequality from Lemma 2.1 in ' Talagrand (1995).

Lemma 3.3 *Let D be the d_X-diameter of a subset $S \subset \mathbb{R}^N$. There is a universal constant K such that, for all $u > 0$, we have*

$$
\begin{aligned}
&\mathbb{P}\left\{ \sup_{s,\,t\in S} \big|X(s) - X(t)\big| \ge K\Big(u + \int_0^D \sqrt{\log N(S, d_X, \varepsilon)}\, d\varepsilon\Big)\right\} \\
&\qquad \le \exp\Big(-\frac{u^2}{D^2}\Big).
\end{aligned}
\tag{3.6}
$$

Proof of Lemma 3.2 Set

$$\tilde{d}(s,t) = \big\|X(s) - X(t) - (v([a,b[,\, s) - v([a,b[,\, t)\big\|_{L^2}.$$

Then

$$\tilde{d}(s,\, t) \le \big\|X(s) - X(t)\|_{L^2} + \|v([a,b[,\, s) - v([a,b[,\, t)\big\|_{L^2}.$$

Since

$$
\begin{aligned}
X(s) - X(t) = &\big(v([a,b[,\, s) - v([a,b[,\, t)\big) \\
&+ \big(v(\mathbb{R}_+ \setminus [a,b[,\, s) - v(\mathbb{R}_+ \setminus [a,b[,\, t)\big),
\end{aligned}
$$

and the two terms on the right-hand side are independent by (A1)(a), we see that

$$\|v([a,b[,\, s) - v([a,b[,\, t)\|_{L^2} \le \|X(s) - X(t)\|_{L^2}.$$

By Lemma 3.3, we obtain

$$\tilde{d}(s,\, t) \le 2\big\|X(s) - X(t)\big\|_{L^2} \le K\rho(s,\, t) \tag{3.7}$$

Therefore, for small $\varepsilon > 0$, the number of ε-balls (in metric $\tilde{d}$) needed to cover $S(t^0, r)$ is

$$N(S(t^0, r), \tilde{d}, \varepsilon) \le c\frac{r^Q}{\varepsilon^Q}.$$

For $t \in S(t^0, r)$, $|t_j - t_j^0| \le r^{H_j^{-1}}$, so by (7.9), we have

$$\tilde{d}(t,\, t^0) \le CA.$$

Therefore the diameter D of $S(t^0, r)$ satisfies $D \le CA$.

Assuming that we have chosen the constant A_0 and that $A \le A_0 r$, then such that

$$\begin{aligned}&\int_0^D \sqrt{\log N(S(t^0,\, r), \tilde{d}, \varepsilon)}\, d\varepsilon \\ &\le K\int_0^{CA} \sqrt{\log\Big(\frac{r}{\varepsilon}\Big)}\, d\varepsilon \le KA\sqrt{\log\frac{r}{A}}.\end{aligned}$$

It follows from this and Lemma 3.3 that (3.5) holds for all $u \ge KA\log^{1/2}\left(\frac{r}{A}\right)$. This proves Lemma 3.2.

Similarly, Lemma 3.3 implies

Lemma 3.4 [Meerschaert, Wang, X. (2013)] *Assume that (A1) holds. Then there exist positive and finite constants u_0 and C such that for all $t^0 \in T$, and $u \ge u_0$*

$$\mathbb{P}\bigg(\sup_{s:|s_j|\le a_j} |X(t^0+s) - X(t^0)| \ge u\sum_{j=1}^N a_j^{H_j}\bigg) \le e^{-C\,u^2}$$

for all $a = (a_1, \ldots, a_N) \in (0,1]^N$ such that $[t^0 - a, t^0 + a] \subseteq T$.

3.2 Law of the iterated logarithm

Besides (A1), we also need the following condition.

Condition (A2) $\|X(t)\|_{L^2} \ge c > 0$ for all $t \in T$ and

$$\mathbb{E}\big[(X(s) - X(t))^2\big] \ge K\rho(s,\, t)^2 \quad \text{ for all } \; s, t \in T.$$

Theorem 3.5 [Lee and Xiao, 2021] *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies (A1) and (A2). Then for every $t^0 \in \mathbb{R}^N$, there is a constant $\kappa_1 = \kappa_1(t^0) \in (0, \infty)$ such that*

$$\limsup_{|h|\downarrow 0}\, \sup_{s\in[-h,\, h]} \frac{|X(t^0+s) - X(t^0)|}{\varphi_1(s)} = \kappa_1, \qquad a.s., \tag{3.8}$$

where

$$\varphi_1(s) = \rho(0,\, s)\bigg[\log\log\Big(1 + \frac{1}{\prod_{j=1}^N |s_j|^{H_j}}\Big)\bigg]^{\frac{1}{2}}, \qquad \forall\, s \in \mathbb{R}^N.$$

Proof of Theorem 3.1. For any $h \in (0,1)^N$, put

$$M(h) = \sup_{s\in[-h,h]} \frac{|X(t^0+s) - X(t^0)|}{\varphi_1(s)}.$$

We claim that there exist constants $c_{3,1}, c_{3,2} \in (0,\infty)$ such that

$$\limsup_{|h|\to 0} M(h) \le c_{3,1} \qquad \text{a.s.} \tag{3.9}$$

and

$$\limsup_{|h|\to 0} M(h) \ge c_{3,2} \qquad \text{a.s.} \tag{3.10}$$

Before proving (3.9) and (3.10), let us notice that, (3.9), (3.10) and the proof of Lemma 7.1.1 in Marcus and Rosen (2006) imply (3.8) and the constant $\kappa_1 \in [c_{3,2},\, c_{3,1}]$.

Proof of (3.9). Let $\delta > 0$ be a constant whose value will be determined later. For any $\boldsymbol{n} = (n_1, ..., n_N) \in \mathbb{N}^N$, define the event

$$F_{\boldsymbol{n}} = \Big\{ \sup_{s:2^{-n_j}\le|s_j|\le 2^{-n_j+1}} \varphi_1(s)^{-1}|X(t^0+s) - X(t^0)| \ge \delta \Big\}.$$

By Condition (A1), we see that for any $s \in \mathbb{R}^N$ that satisfies $2^{-n_j} \le |s_j| \le 2^{-n_j+1}$ for $j = 1, \ldots, N$, we have

$$\varphi_1(s) \ge \Big(\sum_{j=1}^N 2^{-n_j H_j}\Big)\sqrt{\log\log\Big(1 + \prod_{j=1}^N 2^{(n_j-1)H_j}\Big)}.$$

This and Lemma 3.5 imply

$$\begin{aligned}\mathbb{P}(F_{\boldsymbol{n}}) &\le \exp\bigg(- C\delta^2 \log\log\Big(1 + \prod_{j=1}^N 2^{(n_j-1)H_j}\Big)\bigg)\\ &\le K\bigg(\sum_{j=1}^N n_j\bigg)^{-C\delta^2}.\end{aligned}$$

By taking δ large enough such that $C\,\delta^2 > N$, we see that

$$\sum_{\boldsymbol{n}\in\mathbb{N}^N} \mathbb{P}(F_{\boldsymbol{n}}) \le K \sum_{\boldsymbol{n}\in\mathbb{N}^N} |\boldsymbol{n}|^{-C\delta^2} < \infty.$$

Thus, by the Borel-Cantelli lemma, a.s. only finitely many of the events $F_{\boldsymbol{n}}$ occur. This implies

$$\limsup_{|\boldsymbol{n}|\to\infty} \sup_{\mathbf{s}:2^{-n_j}\le|s_j|\le 2^{-n_j+1}} \frac{|X(t^0+s) - X(t^0)|}{\varphi_1(\mathbf{s})} \le \delta \qquad \text{a.s.}$$

This and a monotonicity argument yield (3.9).

Proof of (3.10). It is sufficient to provide a sequence $\{h_n\} \subset (0,1)^N$ such that $|h_n| \to 0$ and

$$\limsup_{n\to\infty} \frac{\big|X(t^0+h_n) - X(t^0)\big|}{\varphi_1(h_n)} \ge \sqrt{2} \qquad \text{a.s.} \tag{3.11}$$

This will be done by using the Borel-Cantelli lemma.

For $0 < \mu < 1$ and $n \geq 1$, define $h_n = (h_{n,1}, \ldots, h_{n,N})$ by

$$h_{n,j} = \exp\big(-H_j^{-1}\, n^{1+\mu}\big) \quad (j = 1, \ldots, N).$$

Then $\rho(0, h_n) = N\exp(-n^{1+\mu})$.

Let $\beta > 0$ be a constant and let $d_n = \exp(n^{1+\mu})n^{-\beta}$.

For $s \in \mathbb{R}^N$, we write $X(s) = X_n(s) + \widetilde{X}_n(s)$, where

$$X_n(s) = v([d_n, d_{n+1}),\, s) \;\text{ and }\; \widetilde{X}_n(s) = X(s) - v([d_n, d_{n+1}),\, s).$$

Then $\{X_n(s),\, s \in \mathbb{R}^N\}$ and $\{\widetilde{X}_n(s),\, s \in \mathbb{R}^N\}$ are independent. Moreover, the sequence $\{X_n(s),\, s \in \mathbb{R}^N\}$, $n = 1, 2, ...$ are independent.

Notice that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|X(t^0 + h_n) - X(t^0)|}{\varphi_1(h_n)} \\
&\geq \limsup_{n\to\infty} \frac{|X_n(t^0 + h_n) - X_n(t^0)|}{\varphi_1(h_n)} - \limsup_{n\to\infty} \frac{|\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)|}{\varphi_1(h_n)} \\
&:= \limsup_{n\to\infty} I_1(n) - \limsup_{n\to\infty} I_2(n).
\end{aligned}$$

We use (A1) to show that $I_1(n)$ is the main term and $I_2(n)$ is negligible. By (A1) (b), we have

$$\begin{aligned}
&\mathbb{E}\big(\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)\big)^2 \\
&\leq C\bigg(\sum_{j=1}^N d_n^{H_j^{-1}-1} h_{n,j} + d_{n+1}^{-1}\bigg)^2 \\
&= \rho(0, h_n)^2 \cdot CN^{-2}\exp(2n^{1+\mu})\bigg(\sum_{j=1}^N d_n^{H_j^{-1}-1} h_{n,j} + d_{n+1}^{-1}\bigg)^2 \\
&\leq C\rho(0, h_n)^2 \cdot \bigg(\sum_{j=1}^N n^{-\beta(H_j^{-1}-1)} + \exp(-n^{\mu})(n+1)^{\beta}\bigg)^2 \\
&\leq C\rho(0, h_n)^2 \cdot n^{-2\beta(\overline{H}^{-1}-1)},
\end{aligned}$$

for $n \geq n_0$, where $\overline{H} = \max\limits_{1\leq j\leq N}\{H_j\}$.

Hence, for any $\eta \in (0, 1)$,

$$\begin{aligned}
&\mathbb{P}\Big(|\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)| \geq \eta\varphi_1(h_n)\Big\} \\
&\leq \mathbb{P}\Big(|N(0,1)| \geq C\eta\sqrt{\log n}\, n^{\beta(\overline{H}^{-1}-1)}\Big\} \\
&\leq n^{-2}
\end{aligned}$$

for all n large enough. Thus, the Borel-Cantelli lemma and the arbitrariness of η imply $\limsup\limits_{n\to\infty} I_2(n) = 0$. a.s.

On the other hand, by the independence of X_n and $\widetilde{X}_n$, Condition (A2) and (A1)(b), we have

$$\begin{aligned}
&\mathbb{E}\big(X_n(t^0 + h_n) - X_n(t^0)\big)^2 \\
&= \mathbb{E}\big(X(t^0 + h_n) - X(t^0)\big)^2 - \mathbb{E}\big(\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)\big)^2 \\
&\geq C\rho(0, h_n)^2
\end{aligned}$$

for all n large enough. It follows that

$$\begin{aligned}&\mathbb{P}\Big(|X_n(t^0+h_n)-X_n(t^0)|\ge \eta\sqrt{2C}\varphi_1(h_n)\Big\}\\&\ge \mathbb{P}\Big(|N(0,1)|\ge \eta\sqrt{(1+\mu)\log n}\Big\}\\&\ge \frac{1}{\eta\sqrt{2\pi(1+\mu)\log n}}\, n^{-\eta^2(1+\mu)}\end{aligned}$$

for all n large enough.

If the constants η and μ are chosen such that $\eta^2(1+\mu)\le 1$, we have

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(|X_n(t^0+h_n)-X_n(t^0)|\ge \eta\sqrt{2C}\varphi_1(h_n)\Big)=\infty.$$

Since the events in the above are independent, the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty} I_1(n)\ge \eta\sqrt{2C}\quad \text{a.s.}$$

This finishes the proof of Theorem 3.1.

3.3 Chung's law of the iterated logarithm

For studying Chung's LIL at $t^0\in T$, we need the following assumption on the small ball probability of X.

Condition (A3) There is a constant c such that for all $t^0\in T$, $r>0$ and $0<\varepsilon<r$,

$$\mathbb{P}\Big\{\max_{\rho(s,\,t^0)\le r}|X(s)-X(t^0)|\le \varepsilon\Big\}\le \exp\Big(-c\big(\frac{r}{\varepsilon}\big)^Q\Big).$$

A similar lower bound for $\mathbb{P}\big\{\max_{\rho(s,\,t^0)\le r}|X(s)|\le\varepsilon\big\}$ is given in Lemma 3.7 below, which can be proved by applying the following general result due to Talagrand (1993) [cf. p. 257, Ledoux (1996)].

Lemma 3.6 [Talagrand (1993)] *Let $\{Y(t), t\in S\}$ be an $\mathbb{R}$-valued centered Gaussian process indexed by a bounded set S. If there is a decreasing function $\psi:(0,\delta]\to(0,\infty)$ such that $N(S,d_Y,\varepsilon)\le\psi(\varepsilon)$ for all $\varepsilon\in(0,\delta]$ and there are constants $c_{3,4}\ge c_{3,3}>1$ such that*

$$c_{3,3}\psi(\varepsilon)\le\psi(\varepsilon/2)\le c_{3,4}\psi(\varepsilon) \tag{3.12}$$

for all $\varepsilon\in(0,\delta]$, then there is a constant K depending only on $c_{3,3}$ and $c_{3,4}$ such that for all $u\in(0,\delta)$,

$$\mathbb{P}\Big(\sup_{s,\,t\in S}|Y(s)-Y(t)|\le u\Big)\ge\exp\big(-K\psi(u)\big). \tag{3.13}$$

Lemma 3.7 *Under (A1), there is a constant $c'\in(0,\infty)$ such that for every $t^0\in T$, $r>0$ and $0<\varepsilon<r$,*

$$\mathbb{P}\Big\{\max_{\rho(s,\,t^0)\le r}|X(s)-X(t^0)|\le\varepsilon\Big\}\ge\exp\Big(-c'\big(\frac{r}{\varepsilon}\big)^Q\Big). \tag{3.14}$$

Proof . Let $S=\{\mathbf{s}\in T:\rho(s,\,t^0)\le r\}$. It follows from Lemma 2.1 that for all $\varepsilon\in(0,r)$,

$$N(S,d_X,\varepsilon)\le c\prod_{i=1}^{N}\Big(\frac{r}{\varepsilon}\Big)^{\frac{1}{H_i}}=c\Big(\frac{r}{\varepsilon}\Big)^Q:=\psi(\varepsilon).$$

Clearly $\psi(\varepsilon)$ satisfies the condition (3.12) in Lemma 3.6. Hence the lower bound in (3.14) follows from (3.13). □

The following is Chung's law of the iterated logarithm for X. It describes the smallest local oscillation of $X(t)$, which is useful for studying hitting probabilities and fractal properties of X.

Theorem 3.8 [Lee and Xiao, 2021] *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies (A1) and (A3). Then for every $t^0 \in T$, there is a constant $\kappa_2 = \kappa_2(t^0) \in (0,\infty)$ such that*

$$\liminf_{r\to 0} \frac{\max_{s:\rho(s,\,t^0)\le r} |X(t^0+s) - X(t^0)|}{r(\log\log 1/r)^{-1/Q}} = \kappa_2, \qquad a.s., \tag{3.15}$$

where $Q = \sum_{j=1}^N H_j^{-1}$.

Proof . Assumption (A1) implies a 0-1 law for the limit in the left hand-side of (3.15).

We need to prove that $\kappa_2 \in (0,\infty)$. It is sufficient to prove that for some constants $c_{3,5}, c_{3,6} \in (0,\infty)$,

$$\liminf_{r\to 0} \frac{\max_{\mathbf{s}:\rho(s,t^0)\le r} |X(t^0+s) - X(t^0)|}{r(\log\log 1/r)^{-1/Q}} \ge c_{3,5}, \qquad \text{a.s.}, \tag{3.16}$$

and

$$\liminf_{r\to 0} \frac{\max_{\mathbf{s}:\rho(s,t^0)\le r} |X(t^0+s) - X(t^0)|}{r(\log\log 1/r)^{-1/Q}} \le c_{3,6}, \qquad \text{a.s.} \tag{3.17}$$

In fact, (3.16) and (3.17) imply that $\kappa_2 \in [c_{3,5}, c_{3,6}]$.

Proof of (3.16). For any integer $n \ge 1$, let $r_n = e^{-n}$. Let $\eta > 0$ be a constant and consider the event

$$A_n = \left\{ \max_{\rho(s,\,t^0)\le r_n} |X(s) - X(t^0)| \le \eta r_n (\log\log 1/r_n)^{-1/Q} \right\}.$$

By (A3) we have

$$\mathbb{P}(A_n) \le \exp\Big(- \frac{c}{\eta^Q} \log n \Big) = n^{-c/\eta^Q},$$

which is summable if $\eta > 0$ is chosen small enough. Hence, (3.16) follows from the Borel-Cantelli lemma.

Proof of (3.17). For every integer $n \ge 1$, we take $r_n = e^{-(n+n^2)}$ and $d_n = e^{n^2}$. Then it follows that

$$r_n\, d_n = e^{-n} \quad \text{and} \quad r_n\, d_{n+1} > e^n.$$

It's sufficient to prove that there exists a finite constant $c_{3,7}$ such that

$$\liminf_{n\to\infty} \frac{\max_{\rho(s,\,t^0)\le r_n} |X(t^0+s) - X(t^0)|}{r_n(\log\log 1/r_n)^{-1/Q}} \le c_{3,7} \qquad \text{a.s.} \tag{3.18}$$

For proving (3.18) we will use (A1) to decompose X in a way similar to that in the proof of Theorem 3.1.

Define two Gaussian fields X_n and $\widetilde{X}_n$ by

$$X_n(s) = v([d_n, d_{n+1}),\, s) \;\text{ and }\; \widetilde{X}_n(s) = X(s) - X_n(s).$$

Then the Gaussian fields $\{X_n(\mathbf{s}), s \in \mathbb{R}^N\}$ $(n = 1, 2, \cdots)$ are independent and for every $n \ge 1$, X_n and $\widetilde{X}_n$ are independent as well.

Denote $\gamma(r) = r(\log\log 1/r)^{-1/Q}$. We make the following two claims:

(i). There is a constant $\eta > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \max_{\rho(s,t^0)\le r_n} |X_n(t^0+s) - X_n(t^0)| \le \eta\gamma(r_n) \right\} = \infty. \tag{3.19}$$

(ii). For every $\eta_1 > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{\rho(s,\,t^0)\le r_n} |\widetilde{X}_n(t^0+s) - X_n(t^0)| > \eta_1 \gamma(r_n)\right\} < \infty. \tag{3.20}$$

Since the events in (3.19) are independent, we see that (3.18) follows from (3.19), (3.20) and the Borel-Cantelli Lemma.

It remains to verify the claims (i) and (ii) above.

By Lemma 3.7 and Anderson's inequality [see Anderson (1955)], we have

$$\begin{aligned}
&\mathbb{P}\Big\{\max_{\rho(s,\,t^0)\le r_n} |X_n(t^0+s) - X_n(t^0)| \le \eta\gamma(r_n)\Big\}\\
&\ge \mathbb{P}\Big\{\max_{\rho(s,\,t^0)\le r_n} |X(t^0+s) - X(t^0)| \le \eta\gamma(r_n)\Big\}\\
&\ge \exp\Big(-\frac{c'}{\eta^Q}\log(n+n^2)\Big)\\
&= (n+n^2)^{-c'/\eta^Q}.
\end{aligned}$$

Hence (i) holds for $\eta > (2c')^{1/Q}$.

To prove Claim (ii), we let $S = \{s \in T : \rho(s,\, t^0) \le r_n\}$ and consider on S the metric

$$\tilde{d}(s,\,t) = \big\|\widetilde{X}_n(t^0+s) - \widetilde{X}_n(t^0+t)\big\|_{L^2}.$$

By Lemma 3.1 we have $\tilde{d}(s,\,t) \le c\sum_{i=1}^N |s_i - t_i|^{H_i}$ for all $s, t \in T$ and hence

$$N(S, \tilde{d}, \varepsilon) \le c\Big(\frac{r_n}{\varepsilon}\Big)^Q.$$

Now we estimate the $\tilde{d}$-diameter $\widetilde{D}$ of S. By (7.9) in (A1),

$$\tilde{d}(s,\,t) \le C\Big(\sum_{j=1}^N d_n^{H_j^{-1}-1}|s_j - t_j| + d_{n+1}^{-1}\Big) \le C\, e^{-n^2-(\overline{H}^{-1}\wedge 2)n}.$$

Thus $\widetilde{D} \le C\, e^{-n^2-(\overline{H}^{-1}\wedge 2)n}$.

Notice that $\widetilde{D} \le r_n e^{-((\overline{H}^{-1}\wedge 2)-1)n}$. The Dudley's integral is

$$\begin{aligned}
&\int_0^{\widetilde{D}} \sqrt{\log N(S, \tilde{d}, \varepsilon)}\, d\varepsilon \le \int_0^{\widetilde{D}} \sqrt{\log(\frac{r_n}{\varepsilon})^Q}\, d\varepsilon\\
&\le C r_n \sqrt{n}\, e^{-((\overline{H}^{-1}\wedge 2)-1)n}.
\end{aligned}$$

Hence for any $\eta_1 > 0$, it follows from by Lemma 3.3 that for all n large,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{\rho(s,\,t^0)\le r_n} |\widetilde{X}_n(t^0+s) - X_n(t^0)| > \eta_1\gamma(r_n)\right\}\\
&\le \exp\Big(-K\frac{\eta_1^2\gamma(r_n)^2}{\widetilde{D}^2}\Big)\\
&\le \exp\Big(-K\eta_1^2(\log n)^{-2/Q}\, e^{((\overline{H}^{-1}\wedge 2)-1)n}\Big).
\end{aligned}$$

Therefore Claim (ii) holds. The proof of Theorem 3.8 is finished. □

3.4 Exact uniform modulus of continuity

In order to prove an exact uniform modulus of continuity, we will make use of Condition (A1) and the following:

Condition (A4) [sectorial local nondeterminism] There exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}. \tag{3.21}$$

Condition (A4) and the following (A4′) are properties of strong local nondeterminism for Gaussian random fields with certain anisotropy.

Condition (A4′) [strong local nondeterminism] There exists a constant $c > 0$ such that $\forall\ n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X(\mathbf{u}) \,\big|\, X(\mathbf{t}^1), \ldots, X(\mathbf{t}^n)\big) \geq c \min_{1 \leq k \leq n} \rho(\mathbf{u}, \mathbf{t}^k)^2. \tag{3.22}$$

Here are some remarks about (A4) and (A4′).

- The concept of local nondeterminism (LND) of a Gaussian process was first introduced by Berman (1973) for studying local times of Gaussian processes.
- Pitt (1978) extended Berman's definition to the setting of random fields.
- Cuzick and DuPreez (1982) introduced strong local ϕ-nonderterminism for Gaussian processes and showed its usefulness in studying local times.
- The "sectorial local nondeterminism" was first discovered by Khoshnevisan and Xiao (2007) for the Brownian sheet; and extended to fractional Brownian sheets by Wu and Xiao (2007).
- Xiao (2009), Luan and Xiao (2012) proved "strong local nondeterminism" for a large class of Gaussian fields with stationary increments.

Theorem 3.9 [Meerschaert, Wang and Xiao (2013)] *If a centered Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ satisfies*

$$\mathbb{E}\Big[\big(X(s) - X(t)\big)^2\Big] \leq c\rho(s,\, t)^2 \quad \textit{for all} \ \ s, t \in T \tag{3.23}$$

and (A4). Then

$$\lim_{r \to 0} \sup_{t,\, s \in T, \rho(s,\, t) \leq r} \frac{|X(s) - X(t)|}{\rho(s,\, t)\sqrt{\log(1 + \rho(s,\, t)^{-1})}} = \kappa_3, \tag{3.24}$$

where $\kappa_3 > 0$ is a constant.

More precise information on the limit in (3.24) has been obtained by Lee and Xiao (2021) under Condition (A4′).

Due to the monotonicity in r, the limit in the left-hand side of (3.24) exists a.s. We only need to prove that the limit is a positive and finite constant. This is done in three parts:

(a). $\lim_{r \to 0} \sup_{s,\, t \in T, \rho(s,t) \leq r} \frac{|X(s)-X(t)|}{\rho(s,\, t)\sqrt{\log(1+\rho(s,\, t)^{-1})}} \leq c_{3,8} < \infty$, a.s.

(b). $\lim_{r \to 0} \sup_{s,\, t \in T, \rho(s,\, t) \leq r} \frac{|X(s)-X(t)|}{\rho(s,\, t)\sqrt{\log(1+\rho(s,\, t)^{-1})}} \geq c_{3,9} > 0$, a.s.

(c). Eq. (3.24) follows from (a), (b), and a zero-one law.

The proof of (a) relies on the following estimate of the tail probability which follows from Lemma 3.3: For $\varepsilon > 0$ and $x \geq c\varepsilon\sqrt{\log(1+\varepsilon^{-1})}$,

$$\mathbb{P}\Big\{ \sup_{\substack{s,\ t\ \in\ T,\\ \rho(s,\ t)\ \leq\ \varepsilon}} |X(t) - X(s)| \geq x \Big\} \leq \exp\Big(- K\frac{x^2}{\varepsilon^2}\Big).$$

and a standard Borel-Cantelli argument.

Or one can apply Theorem 2.2 (after Dudley's theorem).

Proof of (b). For any $n \geq 2$, we choose a sequence of points $\{t_{n,k}, 1 \leq k \leq L_n\}$ in T such that

$$\rho(t_{n,k}, \mathbf{t}_{n,k-1}) = 2^{-n},$$

and for some direction $i \in \{1, \ldots, N\}$,

$$|t^i_{n,k} - t^i_{n,k-1}| \geq 2^{-n/H_i}, \quad \forall\, 2 \leq k \leq L_n.$$

We take $L_n = \min\{2^{n/H_i}\}$.

We will prove the following stronger statement:

$$\liminf_{n\to\infty} \frac{\max_{2\leq k\leq L_n} |X(t_{n,k}) - X(t_{n,k-1})|}{2^{-n}\sqrt{n}} \geq c_{3,9} > 0, \quad \text{a.s.} \tag{3.25}$$

Let $\eta > 0$ be a constant whose value will be chosen later. Consider the events

$$A_n = \Big\{ \max_{2\leq k\leq L_n} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \leq \eta 2^{-n}\sqrt{n}\Big\}$$

and write

$$\begin{aligned} \mathbb{P}\big(A_n\big) = \mathbb{P}\Big\{ &\max_{2\leq k\leq L_n-1} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \leq \eta 2^{-n}\sqrt{n}\Big\} \\ &\times \mathbb{P}\Big\{ \big|X(t_{n,L_n}) - X(t_{n,L_n-1})\big| \leq \eta 2^{-n}\sqrt{n} \big| \widetilde{A}_{L_n-1}\Big\}, \end{aligned} \tag{3.26}$$

where

$$\widetilde{A}_{L_n-1} = \Big\{ \max_{2\leq k\leq L_n-1} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \leq \eta 2^{-n}\sqrt{n}\Big\}.$$

The conditional distribution of the Gaussian random variable $X(t_{n,L_n}) - X(t_{n,L_n-1})$ under $\widetilde{A}_{L_n-1}$ is still Gaussian and, by (A4), its conditional variance satisfies

$$\mathrm{Var}\big(X(t_{n,L_n}) - X(t_{n,L_n-1})\big|\widetilde{A}_{L_n-1}\big) \geq c\, 2^{-2n}.$$

This and Anderson's inequality (1955) imply

$$\begin{aligned} &\mathbb{P}\Big\{ \big|X(t_{n,L_n}) - X(t_{n,L_n-1})\big| \leq \eta 2^{-n}\sqrt{n}\big|\tilde{A}_{L_n-1}\Big\} \\ &\leq \mathbb{P}\Big\{ \big|N(0,1)\big| \leq c\,\eta\sqrt{n}\Big\} \qquad\qquad \text{(use Mill's ratio)} \\ &\leq 1 - \frac{1}{c\eta\sqrt{n}} \exp\Big(- \frac{c^2\eta^2 n}{2}\Big) \qquad \text{(use } 1 - x \leq e^{-x} \text{ for } x > 0\text{)} \\ &\leq \exp\bigg(- \frac{1}{c\eta\sqrt{n}} \exp\Big(- \frac{c^2\eta^2 n}{2}\Big)\bigg). \end{aligned}$$

Iterating this procedure in (3.26) for L_n times, we obtain

$$\mathbb{P}\big(A_n\big) \le \exp\Big(-\frac{1}{c\eta\sqrt{n}}L_n \exp\Big(-\frac{c^2\eta^2 n}{2}\Big)\Big).$$

By taking $\eta > 0$ small enough, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\big(A_n\big) < \infty.$$

Hence the Borel-Cantelli lemma implies (3.25).

For any $\lambda > 0$, define the set of "fast points"

$$F(\lambda) = \Big\{t \in [0,1]^N : \limsup_{r\to 0} \frac{|X(t+h) - X(t)|}{\rho(0,h)\sqrt{\log \frac{1}{\rho(0,h)}}} \ge \lambda\Big\}.$$

We end this section with the following questions:

- What is the Hausdorff dimension of $F(\lambda)$?
- For a given set $E \subset [0,1]^N$, when is

$$\mathbb{P}\{F(\lambda) \cap E \ne \emptyset\} > 0?$$

3.5 Modulus of non-differentiability

Wang and Xiao (2019) proved the following modulus of non-differentiability for fractional Brownian motion.

For any compact rectangle $T \subseteq \mathbb{R}^N$,

$$\lim_{\varepsilon\to 0+} \inf_{t\in T} \frac{\sup_{s\in B(t,\varepsilon)} |B^H(s) - B^H(t)|}{\varepsilon^H |\log 1/\varepsilon|^{-H/N}} = \kappa_4, \quad \text{a.s.},$$

where $\kappa_4 \in (0,\infty)$ is a constant related to the small ball probability of B^H.

This result was extended to a large class of (approximately isotropic) Gaussian random fields with stationary increments by Wang, Su and Xiao (2020). The following theorem is more general and can be applied to SPDEs.

Theorem 3.10 [Wang and Xiao, 2021] *If a centered Gaussian field $X = \{X(t),\, t \in \mathbb{R}^N\}$ satisfies Condition (A1), (A4′) and a regularity condition on the second order derivative of the covariance function $K(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ on $T \times T\backslash\{(s,s), s \in T\}$. Then*

$$\liminf_{r\to 0} \inf_{t\in T} \frac{\max_{\rho(s,\,t)\le r} |X(s) - X(t)|}{r(\log r^{-1})^{-1/Q}} = \kappa_5, \tag{3.27}$$

where $\kappa_5 \in (0,\infty)$ is a constant. In particular, the sample function of X is almost surely nowhere differentiable in any direction.

The proof of Theorem 3.10 has three parts:

(a). $\liminf_{r\to 0} \inf_{t\in T} \frac{\max\limits_{\rho(s,\,t)\le r} |X(s)-X(t)|}{r(\log r^{-1})^{-1/Q}} \ge c_{3,10} > 0, \quad \text{a.s.}$

(b). $\liminf_{r\to 0} \inf_{t\in T} \frac{\max\limits_{\rho(s,t)\le r} |X(s)-X(t)|}{r(\log r^{-1})^{-1/Q}} \le c_{3,11} < \infty, \quad \text{a.s.}$

(c). A zero-one law for the modulus of non-differentiability [This can be proved under Condition (A1).]

The proof of (a) relies on the following small ball probability estimate and the Borel-Cantelli lemma. Without loss of generality, we assume $T = [0, 1]^N$.

Lemma 3.11 [Xiao, 2009] *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field that satisfies (A1) and (A4′) on $T = [0, 1]^N$. Then there exist constants C and C' such that for every $t \in T$ and $0 < \varepsilon \le r$,*

$$\exp\Big(-C'\big(\frac{r}{\varepsilon}\big)^Q\Big) \le \mathbb{P}\Big\{\max_{s\in T:\rho(s,\,t)\le r} |X(s) - X(t)| \le \varepsilon\Big\} \le \exp\Big(-C\big(\frac{r}{\varepsilon}\big)^Q\Big),$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$.

Proof of (a). Let $\theta > 1$ be a constant. For any $n \ge 2$, let $\varepsilon_n = \theta^{-n}$. Divide T into L_n rectangles of sides ε_n^{1/H_j} $(j = 1, \ldots, N)$. Denote these rectangles by $I_{\boldsymbol{i}}$, where $\boldsymbol{i} = (i_1, \ldots, i_N)$ and $i_j \in \{1, \ldots, \varepsilon_n^{-1/H_j}\}$. Denote the lower-left vertex by $t_{\boldsymbol{i}}$.

Let $\gamma(\varepsilon_n) = \varepsilon_n(\log(1/\varepsilon_n))^{-1/Q}$. By Lemma 3.8, we have

$$\begin{aligned}
&\mathbb{P}\Big(\min_i \max_{s\in I_{\boldsymbol{i}}} |X(s) - X(t_i)| \le \eta\gamma(\varepsilon_n)\Big)\\
&\le \sum_{\boldsymbol{i}} \mathbb{P}\Big(\max_{s\in I_i} |X(s) - X(t_i)| \le \eta\gamma(\varepsilon_n)\Big)\\
&\le L_n \exp\Big(-C\eta^{-Q}\log(1/\varepsilon_n)\Big) = \theta^{n(Q-C\eta^{-Q})},
\end{aligned}$$

which is summable if $\eta > 0$ is chosen small enough. This and the Borel-Cantelli lemma yield (a).

Proof of (b). Using the notation in the last page, it is sufficient to prove that

$$\liminf_{n\to\infty} \min_i \frac{\max\limits_{s,\,t\in I_i} |X(s) - X(t)|}{\varepsilon_n(\log \varepsilon_n^{-1})^{-1/Q}} \le c_{3,11} < \infty, \quad \text{a.s.}$$

This is more difficult to prove. Besides small ball probability estimates, we make use of the following tools:

- due to non-stationarity, a general framework on limsup random fractals that extends that of Khoshnevisan, Peres, and X. (2000) is needed. This was done by Hu, Li, and X. (2021) for studying random covering sets.
- a correlation inequality of Shao (2003).

Since the proof quite lengthy, we omit the details here.

4 Lecture 4. Properties of Strong Local Nondeterminism of Gaussian Random Fields

One of the main difficulties in studying sample path properties of anisotropic Gaussian random fields such as fractional Brownian sheets is the complexity of their dependence structure. For example, unlike fractional Brownian motion which is locally nondeterministic [see Pitt (1978)] or the Brownian sheet which has independent increments, a fractional Brownian sheet has neither of these properties. The same is true for anisotropic Gaussian random fields in general. The main technical tool which we will apply to study anisotropic Gaussian random fields is the properties of strong local nondeterminism [SLND] and sectorial local nondeterminism.

Recall that the concept of local nondeterminism was first introduced by Berman (1973) to unify and extend his methods for studying local times of real-valued Gaussian processes, and then extended by Pitt (1978) to Gaussian random fields.

A Gaussian process $Y = \{Y(t), t \in \mathbb{R}\}$ is called *locally nondeterministic* on $T \subseteq \mathbb{R}$ if for every integer $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{4.1}$$

where V_m is the relative prediction error:

$$V_m = \frac{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1}) | Y(t_1), \ldots, Y(t_{m-1})\big)}{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1})\big)}$$

and the infimum in (4.1) is taken over all ordered points $t_1 < t_2 < \cdots < t_m$ in T with $t_m - t_1 \leq \varepsilon$.

(4.1) is equivalent to the following property: For every integer $m \geq 2$, there exist positive constants $C(m)$ and ε (both may depend on m) such that

$$\begin{aligned} &\mathrm{Var}\Big(\sum_{k=1}^m a_k \left(Y(t_k) - Y(t_{k-1})\right)\Big) \\ &\quad \geq C(m) \sum_{k=1}^m a_k^2 \,\mathrm{Var}\big(Y(t_k) - Y(t_{k-1})\big) \end{aligned} \tag{4.2}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in T with $t_m - t_1 < \varepsilon$ and $a_k \in \mathbb{R}$ $(k = 1, \ldots, m)$.

Pitt (1978) used (4.2) to define local nondeterminism of a Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ by introducing a partial order among $t_1, \ldots, t_m \in \mathbb{R}^N$.

Pitt (1978) proved that fractional Brownian motion $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ has the following property: For any $u \in \mathbb{R}^N \backslash \{0\}$, and and $r \in (0, |u|)$,

$$\mathrm{Var}\big(B^H(u) \,\big|\, B^H(t),\, |t - u| \geq r\big) = c\, r^{2H},$$

where $c > 0$ is a constant. This implies that B^H satisfies the strong local nondeterminism on any compact interval $I \subset \mathbb{R}^N \backslash \{0\}$.

Cuzick and DuPreez (1982) introduced strong local ϕ-nonderterminism and showed its usefulness in studying local times.

The notion of strong local nondeterminism was later developed to investigate the regularity of local times, small ball probabilities and other sample path properties of Gaussian processes and Gaussian random fields. We refer to Xiao (2006, 2007) for more information on the history and applications of the properties of local nondeterminism.

For Gaussian random fields, the aforementioned properties of local nondeterminism can only be satisfied by those with approximate isotropy. It is well-known that the Brownian sheet does not satisfy the properties of local nondeterminism in the senses of Berman or Pitt. Because of this, many problems for fractional Brownian motion and the Brownian sheet have to be investigated using different methods.

Khoshnevisan and Xiao (2007a) have recently proved that the Brownian sheet satisfies the sectorial local nondeterminism [i.e., (C3) with $H = \langle 1/2 \rangle$] and applied this property to study various analytic and geometric properties of the Brownian sheet; see also Khoshnevisan, Wu and Xiao (2006).

Wu and Xiao (2007) extended the result of Khoshnevisan and Xiao (2007a) and proved that fractional Brownian sheet B_0^H satisfies Condition (C3).

Theorem 4.1 *Let $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ be an $(N,1)$-fractional Brownian sheet with index $H = (H_1, \ldots, H_N) \in (0,1)^N$. For any fixed number $\varepsilon \in (0,1)$, there exists a positive constant $c_{3,1}$, depending on ε, H and N only, such that for all positive integers $n \geq 1$, and all $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, we have*

$$\mathrm{Var}\left(B_0^H(u) \,\big|\, B_0^H(t^1), \ldots, B_0^H(t^n)\right) \geq c_{3,1} \sum_{j=1}^N \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j}, \tag{4.3}$$

where $t_j^0 = 0$ for $j = 1, \ldots, N$.

Proof While the argument of Khoshnevisan and Xiao (2007a) relies on the property of independent increments of the Brownian sheet and its connection to Brownian motion, the proof for B_0^H is based on a Fourier analytic argument in Kahane (1985, Chapter 18) and the harmonizable representation of B_0^H. We refer to Wu and Xiao (2007) for details. □

Now we prove a sufficient condition for an anisotropic Gaussian random field with stationary increments to satisfy Condition (C3′).

Theorem 4.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field in $\mathbb{R}$ with stationary increments and spectral density $f(\lambda)$. Assume that there is a vector $H = (H_1, \ldots, H_N) \in (0,1)^N$ such that*

$$f(\lambda) \asymp \frac{1}{\big(\sum_{j=1}^N |\lambda_j|^{H_j}\big)^{2+Q}}, \qquad \forall\, \lambda \in \mathbb{R}^N \backslash \{0\}, \tag{4.4}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$. Then there exists a constant $c_{3,2} > 0$ such that for all $n \geq 1$, and all $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(\, X(u) \mid X(t^1), \ldots, X(t^n)\Big) \geq c_{3,2} \min_{0 \leq k \leq n} \rho(u, t^k)^2, \tag{4.5}$$

where $t^0 = 0$.

Remark 4.3 The following are some comments about Theorem 4.2.

(i) When $H_1 = \cdots = H_N$, (4.5) is of the same form as the SLND of fractional Brownian motion [cf. Pitt (1978)]. As shown by Xiao (1997b, 2007) and Shieh and Xiao (2006), many sample path properties of such Gaussian random fields are similar to those of fractional Brownian motion.

(ii) Condition (4.4) can be significantly weakened. In particular, one can prove that similar results hold for Gaussian random fields with stationary increments and discrete spectrum measures; see Xiao (2008) for details.

Proof of Theorem 4.2 Denote $r \equiv \min\limits_{0 \leq k \leq n} \rho(u, t^k)$. Since the conditional variance in (4.5) is the square of the $L^2(\mathbb{P})$-distance of $X(u)$ from the subspace generated by $\{X(t^1), \ldots, X(t^n)\}$, it is sufficient to prove that for all $a_k \in \mathbb{R}$ $(1 \leq k \leq n)$,

$$\mathbb{E}\bigg(X(u) - \sum_{k=1}^n a_k\, X(t^k)\bigg)^2 \geq c_{3,2}\, r^2 \tag{4.6}$$

and $c_{3,2} > 0$ is a constant which may only depend on H and N.

By the stochastic integral representation (1.4) of X, the left hand side of (4.6) can be written as

$$\mathbb{E}\bigg(X(u) - \sum_{k=1}^n a_k X(t^k)\bigg)^2 = \int_{\mathbb{R}^N} \bigg| e^{i\langle u, \lambda\rangle} - 1 - \sum_{k=1}^n a_k \left(e^{i\langle t^k,\, \lambda\rangle} - 1\right)\bigg|^2 f(\lambda)\, d\lambda. \tag{4.7}$$

Hence, we only need to show

$$\int_{\mathbb{R}^N} \bigg| e^{i\langle u, \lambda\rangle} - \sum_{k=0}^n a_k\, e^{i\langle t^k,\, \lambda\rangle}\bigg|^2 f(\lambda)\, d\lambda \geq c_{3,2}\, r^2, \tag{4.8}$$

where $t^0 = 0$ and $a_0 = -1 + \sum_{k=1}^n a_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0,1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0,1)$ in the metric ρ. Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ as well and $\widehat{\delta}(\lambda)$ decays rapidly as $|\lambda| \to \infty$.

Let E be the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q}\delta(r^{-E}t)$. Then the inverse Fourier transform and a change of variables yield

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r^E\lambda)\, d\lambda. \tag{4.9}$$

Since $\min\{\rho(u, t^k) : 0 \le k \le n\} \ge r$, we have $\delta_r(u - t^k) = 0$ for $k = 0, 1, \ldots, n$. This and (4.9) together imply that

$$\begin{aligned} J &:= \int_{\mathbb{R}^N} \left(e^{i\langle u,\lambda\rangle} - \sum_{k=0}^n a_k\, e^{i\langle t^k,\lambda\rangle} \right) e^{-i\langle u,\lambda\rangle}\, \widehat{\delta}(r^E\lambda)\, d\lambda \\ &= (2\pi)^N \left(\delta_r(0) - \sum_{k=0}^n a_k\, \delta_r(u - t^k) \right) \\ &= (2\pi)^N\, r^{-Q}. \end{aligned} \tag{4.10}$$

On the other hand, by the Cauchy-Schwarz inequality and (4.7), we have

$$\begin{aligned} J^2 &\le \int_{\mathbb{R}^N} \left| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^n a_k\, e^{i\langle t^k,\lambda\rangle} \right|^2 f(\lambda)\, d\lambda \cdot \int_{\mathbb{R}^N} \frac{1}{f(\lambda)} \left| \widehat{\delta}(r^E\lambda) \right|^2 d\lambda \\ &\le \mathbb{E}\left(X(u) - \sum_{k=1}^n a_k X(t^k) \right)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{1}{f(r^{-E}\,\lambda)} \left| \widehat{\delta}(\lambda) \right|^2 d\lambda \\ &\le c\, \mathbb{E}\left(X(u) - \sum_{k=1}^n a_k X(t^k) \right)^2 \cdot r^{-2Q-2}, \end{aligned} \tag{4.11}$$

where $c > 0$ is a constant which may only depend on H and N.

We square both sides of (4.10) and use (4.11) to obtain

$$(2\pi)^{2N}\, r^{-2Q} \le c\, r^{-2Q-2}\, \mathbb{E}\left(X(u) - \sum_{k=1}^n a_k X(t^k) \right)^2.$$

Hence (4.8) holds. This finishes the proof of the theorem. $\square$

Given jointly Gaussian random variables $Z_1, \ldots, Z_n$, we denote by $\mathrm{detCov}(Z_1, \ldots, Z_n)$ the determinant of their covariance matrix. If $\mathrm{detCov}(Z_1, \ldots, Z_n) > 0$, then we have the identity

$$\frac{(2\pi)^{n/2}}{\mathrm{detCov}(Z_1, \ldots, Z_n)} = \int_{\mathbb{R}^n} \mathbb{E} \exp\left(-i \sum_{k=1}^n u_k\, Z_k \right) du_1 \cdots du_n. \tag{4.12}$$

By using the fact that, for every k, the conditional distribution of Z_k given $Z_1, \ldots, Z_{k-1}$ is still Gaussian with mean $\mathbb{E}(Z_k | Z_1, \ldots, Z_{k-1})$ and variance $\mathrm{Var}(Z_k | Z_1, \ldots, Z_{k-1})$, one can evaluate the integral in the right-hand side of (4.12) and thus verify the following formula:

$$\mathrm{detCov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{k=2}^n \mathrm{Var}\big(Z_k \big| Z_1, \ldots, Z_{k-1}\big). \tag{4.13}$$

A little thought reveals that (4.13) still holds when $\mathrm{detCov}(Z_1, \ldots, Z_n) = 0$. Note that the left-hand side of (4.13) is permutation invariant for $Z_1, \ldots, Z_n$, one can represent $\mathrm{detCov}(Z_1, \ldots, Z_n)$ in terms of the conditional variances in $n!$ different ways.

Combined with (4.3) or (4.5), the identity (4.13) can be applied to estimate the joint distribution of the Gaussian random variables $X(t^1), \ldots, X(t^n)$, where $t^1, \ldots, t^n \in \mathbb{R}^N$. This is why the properties of strong local nondeterminism are not only essential in this paper, but will also be useful in studying self-intersection local times [see Meerschaert et al. (2007) for results on fractional Brownian sheets], exact Hausdorff measure of the sample paths and other related problems.

The following simple result will be needed in Section 6.

Lemma 4.4 *Let X be a Gaussian random field satisfying Condition (C3′) [resp., (C3)]. Then for all integers $n \geq 1$ and for all distinct points $t^1, \ldots, t^n \in [\varepsilon, 1]^N$ [resp., all points $t^1, \ldots, t^n \in [\varepsilon, 1]^N$ with distinct coordinates, i.e., $t_i^k \neq t_j^l$ when $(i,k) \neq (j,l)$], the Gaussian random variables $X(t^1), \ldots, X(t^n)$ are linearly independent.*

Proof We assume Condition (C3′) holds and let $t^1, \ldots, t^n \in [\varepsilon, 1]^N$ be n distinct points. Then it follows from (4.13) that $\mathrm{detCov}\big(X(t^1), \ldots, X(t^n)\big) > 0$. This proves the lemma. Similar conclusion holds when Condition (C3) is satisfied. □

4.1 Spectral conditions for strong local nondeterminism

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$.

For any $h \in \mathbb{R}^N$ we have

$$\mathbb{E}\big(X(t+h) - X(t)\big)^2 = 2 \int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\, \Delta(d\lambda),$$

where $\Delta(d\lambda)$ is the spectral measure of X, which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1 + |\lambda|^2}\ \Delta(d\lambda) < \infty.$$

It follows that X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \big(e^{i\langle t, \lambda\rangle} - 1\big)\, \widetilde{W}(d\lambda),$$

where $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Remarks (i). If $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a stationary Gaussian field, let $X(t) = Y(t) - Y(0)$ for all $t \in \mathbb{R}^N$. Then $X = \{X(t), t \in \mathbb{R}^N\}$ has stationary increments and has the same spectral measure as that of Y.

(ii). The spectral measure Δ can be

- absolutely continuous with density $f(\lambda)$, or
- singular with fractal support, or
- singular with a discrete support.

Theorem 4.5 [Xue and Xiao, 2011] *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field with stationary increments and spectral density $f(\lambda)$. If there are constants $H_1, \cdots, H_N \in (0,1]^N$ and $K > 0$ such that*

$$f(\lambda) \asymp \frac{1}{\big(\sum_{j=1}^N |\lambda_j|^{H_j}\big)^{2+Q}}, \qquad \lambda \in \mathbb{R}^N,\ \ |\lambda| \geq K, \tag{4.14}$$

where $Q = \sum_{j=1}^{N} \frac{1}{H_j}$, *then* $\exists$ *a constant* $c > 0$ *such that for all* $n \ge 1$ *and* $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(X(u)\,\big|\,X(t^1), \ldots, X(t^n)\Big) \ge c \min_{0 \le k \le n} \rho(u, t^k)^2, \quad \textit{where } t^0 = 0.$$

Observe from (4.14) that the behavior of $f(\lambda)$ near 0 is not needed.

For proving Theorem 4.5, we need the following lemma.

Lemma 4.6 *Assume (4.14) is satisfied, then for any fixed constant* $L > 0$, *there exists a positive and finite constant* c_1 *such that for all functions* g *of the form*

$$g(\lambda) = \sum_{k=1}^{n} a_k (e^{i\langle t^k, \lambda\rangle} - 1), \tag{4.15}$$

where $a_k \in \mathbb{R}$ *and* $t^k \in [-L, L]^N$, *we have*

$$|g(\lambda)| \le c_1\,|\lambda| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi\Big)^{1/2} \tag{4.16}$$

for all $\lambda \in \mathbb{R}^N$ *that satisfy* $|\lambda| \le K$.

Proof By (4.14), we can find positive constants C and η, such that

$$f(\lambda) \ge \frac{C}{|\lambda|^\eta}, \quad \forall \lambda \in \mathbb{R}^N \text{ with } |\lambda| \text{ large enough.}$$

Let $\mathcal{G}$ be the collection of the functions $g(z)$ defined by (4.15) with $a_k \in \mathbb{R}$, $s^k \in [-L, L]^N$ and $z \in \mathbb{C}^N$. Since each $g \in \mathcal{G}$ is an entire function, it follows from Proposition 1 of Pitt (1975) that for any given constant K,

$$c_1 = \sup_{\substack{g \in \mathcal{G} \\ z \in U(0,K)}} \left\{ |g(z)| : \int_{\mathbb{R}^N} |g(\lambda)|^2 f(\lambda)\, d\lambda \le 1 \right\} < \infty,$$

where $U(0, K) = \{z \in \mathbb{C}^N : |z| < K\}$ is the open ball of radius K in $\mathbb{C}^N$.

Since $g(0) = 0$ and g is analytic in $U(0, K)$, Schwartz's lemma implies

$$\big|g(z)\big| \le c_1 K^{-1}\,|z| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2\, f(\xi) d\xi\Big)^{1/2}$$

for all $z \in U(0, K)$. This finishes the proof of Lemma 4.6. □

Proof of Theorem 4.5. Denote $r \equiv \min_{0 \le k \le n} \rho(u, t^k)$. It is sufficient to prove that for all $a_k \in \mathbb{R}$ $(1 \le k \le n)$,

$$\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k\, X(t^k)\Big)^2 \ge c\, r^2. \tag{4.17}$$

By the stochastic integral representation of X, the left hand side of (4.17), up to a constant, can be written as

$$\begin{aligned}&\mathbb{E}\Big(X(u)-\sum_{k=1}^{n}a_kX(t^k)\Big)^2\\&=\int_{\mathbb{R}^N}\Big|e^{i\langle u,\lambda\rangle}-1-\sum_{k=1}^{n}a_k\,(e^{i\langle t^k,\,\lambda\rangle}-1)\Big|^2 f(\lambda)\,d\lambda.\end{aligned}\tag{4.18}$$

Hence, we only need to show

$$\int_{\mathbb{R}^N}\Big|e^{i\langle u,\lambda\rangle}-\sum_{k=0}^{n}a_k\,e^{i\langle t^k,\,\lambda\rangle}\Big|^2 f(\lambda)\,d\lambda\ge c\,r^2,\tag{4.19}$$

where $t^0=0$ and $a_0=-1+\sum_{k=1}^n a_k$.

Let $\delta(\cdot):\mathbb{R}^N\to[0,1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0)=1$ and it vanishes outside the open ball $B_\rho(0,1)$.

Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot)\in C^\infty(\mathbb{R}^N)$ and decays rapidly as $|\lambda|\to\infty$.

Let A be the diagonal matrix with $H_1^{-1},\dots,H_N^{-1}$ on its diagonal and let $\delta_r(t)=r^{-Q}\delta(r^{-A}t)$. By the inverse Fourier transform,

$$\delta_r(t)=(2\pi)^{-N}\int_{\mathbb{R}^N}e^{-i\langle t,\lambda\rangle}\,\widehat{\delta}(r^A\lambda)\,d\lambda.$$

Since $\min\{\rho(u,t^k):0\le k\le n\}=r$, we have

$$\delta_r(u-t^k)=0\quad\text{for}\quad k=0,1,\dots,n.$$

Hence,

$$\begin{aligned}I&=\int_{\mathbb{R}^N}\Big(e^{i\langle u,\lambda\rangle}-\sum_{k=0}^{n}a_k\,e^{i\langle t^k,\lambda\rangle}\Big)\,e^{-i\langle u,\lambda\rangle}\,\widehat{\delta}(r^A\lambda)\,d\lambda\\&=(2\pi)^N\Big(\delta_r(0)-\sum_{k=0}^{n}a_k\,\delta_r(u-t^k)\Big)\\&=(2\pi)^N\,r^{-Q}.\end{aligned}\tag{4.20}$$

We split the integral in (4.20) over $\{\lambda:|\lambda|<K\}$ and $\{\lambda:|\lambda|\ge K\}$ and denote the two integrals by I_1 and I_2, respectively. It follows from Lemma 4.1 that

$$\begin{aligned}I_1&\le\int_{|\lambda|<K}\Big|e^{i\langle u,\lambda\rangle}-\sum_{k=0}^{n}a_ke^{i\langle t^k,\lambda\rangle}\Big||\hat{\delta}(r^A\lambda)|d\lambda\\&\le c_1\Big[\int_{\mathbb{R}^N}\Big|e^{i\langle u,\lambda\rangle}-\sum_{k=0}^{n}a_ke^{i\langle t^k,\lambda\rangle}\Big|^2 f(\lambda)\,d\lambda\Big]^{1/2}\\&\qquad\times\int_{|\lambda|<K}|\lambda||\hat{\delta}(r^A\lambda)|d\lambda\\&\le c_2\Big[\mathbb{E}\Big(X(u)-\sum_{k=0}^{n}a_kX(t^k)\Big)^2\Big]^{1/2}.\end{aligned}\tag{4.21}$$

On the other hand, the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
I^2 &\le \int_{|\lambda|\ge K} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle}\Big|^2 f(\lambda)\, d\lambda \\
&\qquad \times \int_{|\lambda|\ge K} \frac{\big|\widehat{\delta}(r^A\lambda)\big|^2}{f(\lambda)}\, d\lambda \\
&\le \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{\big|\widehat{\delta}(\lambda)\big|^2}{f(r^{-A}\,\lambda)}\, d\lambda \\
&\le c\, \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-2Q-2}.
\end{aligned}
$$

We square both sides of (4.20) and use the above to obtain

$$
(2\pi)^{2N}\, r^{-2Q} \le c\, r^{-2Q-2}\, \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2.
$$

This proves (4.19) and hence the theorem. □

Remarks

- This method can be modified to prove sectorial local nondeterminism (by choosing appropriate function $\delta(\cdot) : \mathbb{R}^N \to [0,1]$.
- The method is applied in Lan, Marinucci and Xiao (2018) to prove strong local nondeterminism for isotropic Gaussian random fields on the sphere $\mathbb{S}^2$.

4.2 A comparison theorem

Now we consider the case where the spectral measure Δ may be singular.

For any $\lambda \in \mathbb{R}^N$ and $h > 0$, denote by $C(\lambda, h)$ the cube with side-length $2h$ and center λ, i.e.,

$$
C(\lambda, h) = \big\{x \in \mathbb{R}^N : |x_j - \lambda_j| \le h,\ j = 1, \cdots, N\big\}.
$$

Let $L^2(C(0,L))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0,L)$.

Theorem 4.7 [Luan and Xiao, 2012] *Let $\{Y(t), t \in \mathbb{R}^N\}$ be a real, centered Gaussian field with stationary increments and $Y(0) = 0$. If for some $h > 0$ the spectral measure Δ of Y satisfies*

$$
\begin{aligned}
0 &< \liminf_{|\lambda|\to\infty} \rho(0,\lambda)^{Q+2} \Delta(C(\lambda,h)) \\
&\qquad \le \limsup_{|\lambda|\to\infty} \rho(0,\lambda)^{Q+2} \Delta(C(\lambda,h)) < \infty,
\end{aligned}
\tag{4.22}
$$

then for any $L > 0$ such that $LhN < \log 2$, for all $u, t^1, \ldots, t^n \in C(0,L)$,

$$
\mathrm{Var}\Big(Y(u)\,\Big|\, Y(t^1), \ldots, Y(t^n)\Big) \ge c \min_{0\le k\le n} \rho(u,\, t^k)^2.
$$

In order to prove 4.7, we will make use of the following lemma.

Lemma 4.8 Pitt, 1975] *Let $\widetilde{\Delta}(d\lambda)$ be a positive measure on $\mathbb{R}^N$. If, for some constant $h > 0$, $\widetilde{\Delta}(d\lambda)$ satisfies*

$$0 < \liminf_{|\lambda|\to\infty} \widetilde{\Delta}(C(\lambda, h)) \leq \limsup_{|\lambda|\to\infty} \widetilde{\Delta}(C(\lambda, h)) < \infty.$$

Then for every $L > 0$ satisfying $LhN < \log 2$, we have

$$\int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 \widetilde{\Delta}(d\lambda) \asymp \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 d\lambda$$

for all $\psi \in L^2(C(0, L))$.

Lemma 4.9 Luan and Xiao, 2012] *Let $\Delta_1(d\lambda)$ be a measure on $\mathbb{R}^N$ such that for some $h > 0$,*

$$\begin{aligned} 0 < \liminf_{|\lambda|\to\infty} & \rho(0, \lambda)^{Q+2} \Delta_1(C(\lambda, h)) \\ & \leq \limsup_{|\lambda|\to\infty} \rho(0, \lambda)^{Q+2} \Delta_1(C(\lambda, h)) < \infty. \end{aligned}$$

Then for any $L > 0$ with $LhN < \log 2$, $\exists$ constants c_3 and c_4 such that

$$\int_{\mathbb{R}^N} |g(\lambda)|^2 \Delta_1(d\lambda) \asymp \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^N |\lambda_j|^{H_j})^{Q+2}} d\lambda$$

for all $g(\lambda)$ as in Lemma 4.1.

Theorem 4.7 follows from Lemma 4.9 and Theorem 4.5.

Example 4.1. Let $\{\xi_n, n \in \mathbb{Z}^N\}$ and $\{\eta_n, n \in \mathbb{Z}^N\}$ be two independent sequences of i.i.d. $N(0, 1)$ random variables. Let

$$Z(t) = \sum_{n\in\mathbb{Z}^N} a_n \big(\xi_n \cos \langle n, t\rangle + \eta_n \sin \langle n, t\rangle \big), \quad t \in \mathbb{R}^N,$$

where $\{a_n, n \in \mathbb{Z}^N\}$ is a sequence of real numbers such that

$$a_n^2 \asymp \frac{1}{\big(\sum_{j=1}^N |n_j|^{H_j}\big)^{Q+2}}.$$

By Theorem 4.7, the Gaussian field $Y(t) = Z(t) - Z(0)$ has the property of strong local nondeterminism.

Example 4.2. Let μ be the measure on $\mathbb{R}$ obtained by "patching" fractal probability measures on $[n, n + 1]$, and let the spectral measure Δ be given by

$$\frac{d\mu(\lambda)}{|\lambda|^{1+2H}},$$

then Theorem 4.7 implies that any Gaussian process X with spectral measure Δ has the property of SLND which is similar to that of fBm B^H.

4.3 SLND of linear SHE

Consider the linear stochastic heat equation

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma \dot{W}, \quad t \ge 0,\ x \in \mathbb{R}^k,\\ &u(0,x) \equiv 0,\end{aligned} \tag{4.23}$$

where Δ is the Laplacian operator, σ is a constant or a deterministic function, and $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance [Dalang (1999)] given by the Riesz kernel with exponent β if $k \ge 1$ and $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise considered by Walsh (1986).

It follows from Dalang (1999) that the mild solution of (4.23) is the mean zero Gaussian random field $u = \{u(t,x), t \ge 0, x \in \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \widetilde{G}_{t-r}(x-y)\, \sigma W(drdy), \quad t \ge 0, x \in \mathbb{R},$$

where $\widetilde{G}_t(x)$ is the Green kernel given by

$$\widetilde{G}_t(x) = (2\pi t)^{-1/2} \exp\Big(-\frac{|x|^2}{2t}\Big), \qquad \forall\, t > 0, x \in \mathbb{R}^k.$$

Dalang, Khoshnevisan, and Nualart (2007) that for any $0 < a < b < \infty$,

$$\mathbb{E}\Big(|u(t,x) - u(s,y)|^2\Big) \asymp \rho((t,x),(s,y))^2 \tag{4.24}$$

for all $(t,x), (s,y) \in [a,b] \times [-b,b]^k$, where

$$\rho((t,x),(s,y)) = |t-s|^{\frac{2-\beta}{4}} + |x-y|^{\frac{2-\beta}{2}}.$$

Even though the solution $\{u(t,x), t \ge 0, x \in \mathbb{R}\}$ is not stationary nor has stationary increments, by using the following representation in Dalang, Mueller and Xiao (2017):

$$u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{|\xi|^2 - i\tau} |\xi|^{(\beta-k)/2}\, W(d\tau, d\xi),$$

we can prove

Theorem 4.10 [Khoshnevisan, Lee, and Xiao, 2021] *For any $0 < a < b < \infty$, there exists a constant $C > 0$ such that for all integers $n \ge 1$, for all $(t,x), (t^1,x^1), \ldots, (t^n,x^n) \in [a,b] \times [-b,b]^k$,*

$$\mathrm{Var}\big(u_1(t,x)|u_1(t^1,x^1), \ldots, u_1(t^n,x^n)\big) \ge C \min_{1 \le i \le n} \rho((t,x),(t^i,x^i))^2.$$

Consequently, many regularity properties of $\{u(t,x), t \ge 0, x \in \mathbb{R}\}$ can be derived.

4.4 SLND of linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0, \end{cases} \tag{4.25}$$

where $\dot{W}$ is a Gaussian noise as in the previous section with exponent β if $k \geq 1$ and $\beta \in (0, k \wedge 2)$.

The existence of real-valued process solution to (4.25) was studied by Walsh (1986) for the space-time while noise and by Dalang (1999) in the more general setting.

Recall that the fundamental solution of the wave equation G is

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}} \quad \text{if} \ \ k = 1;$$

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2}, \quad \text{if } k \geq 2 \text{ is even};$$

and

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t}, \quad \text{if } \ k \geq 3 \text{ is odd},$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$.

For any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0,\, \xi \in \mathbb{R}^k. \tag{4.26}$$

Dalang (1999) proved that the real-valued process solution of equation (4.25) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{4.27}$$

where W is the martingale measure induced by the noise $\dot{W}$. The range of β has been chosen so that the stochastic integral exists.

Recall from Dalang (1999) that

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^k} H(s,y) W(ds\, dy)\Big)^2\Big] = c \int_0^t ds \int_{\mathbb{R}^k} |\mathscr{F}(H(s,\cdot))(\xi)|^2 \, \frac{d\xi}{|\xi|^{k-\beta}} \tag{4.28}$$

provided that $s \mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} |\mathscr{F}(H(s,\cdot)(\xi)|^2 \, \frac{d\xi}{|\xi|^{k-\beta}} < \infty.$$

In the following, we show that the Gaussian random field $\{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$ satisfies a form of strong local nondeterminism.

Let $0 < a < a' < \infty$ and $0 < b < \infty$ be fixed constants. The following theorem was proved by Lee and Xiao (2019).

Theorem 4.11 *There exists a constant $C > 0$ such that for all integers $n \geq 1$ and $(t, x), (t^1, x^1), \ldots, (t^n, x^n)$ in $[a, a'] \times [-b, b]^k$ with $|t - t^j| + |x - x^j| \leq a/2$, we have*

$$\begin{aligned} &\mathrm{Var}\left(u(t, x) | u(t^1, x^1), \ldots, u(t^n, x^n)\right) \\ &\geq C \int_{\mathbb{S}^{k-1}} \min_{1 \leq j \leq n} |(t - t^j) + (x - x^j) \cdot w|^{2-\beta} \, dw, \end{aligned} \tag{4.29}$$

where dw is the surface measure on the unit sphere $\mathbb{S}^{k-1}$.

When $k = 1$, the surface measure dw in (4.29) is supported on $\{-1, 1\}$. It follows that $u(t, x)$ satisfies sectorial local nondeterminism:

$$\begin{aligned} &\mathrm{Var}(u(t, x) | u(t^1, x^1), \ldots, u(t^n, x^n)) \\ &\geq C\big(\min_{1 \leq j \leq n} |(t - t^j) + (x - x^j)|^{2-\beta} + \min_{1 \leq j \leq n} |(t - t^j) - (x - x^j)|^{2-\beta} \big). \end{aligned}$$

Proof of Theorem 4.11. For each $w \in \mathbb{S}^{k-1}$, let

$$r(w) = \min_{1 \leq j \leq n} |(t^j - t) - (x^j - x) \cdot w|.$$

Since u is a centered Gaussian random field,it suffices to show that there a exist constant $C > 0$ such that for all $(t, x), (t^1, x^1), \ldots, (t^n, x^n)$ in $[a, a'] \times [-b, b]^k$ with $|t - t^j| + |x - x^j| \leq a/2$,

$$\mathbb{E}\Big[\Big(u(t, x) - \sum_{j=1}^n \alpha_j u(t^j, x^j)\Big)^2\Big] \geq C \int_{\mathbb{S}^{k-1}} r(w)^{2-\beta} \, dw \tag{4.30}$$

for all possible choice of real numbers $\alpha_1, \ldots, \alpha_n$.

Using (1.13), (4.28) and spherical coordinate $\xi = \rho\, w$, we have

$$\begin{aligned} \mathbb{E}\Big[\Big(u(t, x) - \sum_{j=1}^n \alpha_j u(t^j, x^j)\Big)^2\Big] &= c \int_0^\infty ds \int_{\mathbb{R}^k} \Big| \sin((t - s)|\xi|) \mathbf{1}_{[0,t]}(s) \\ &\qquad - \sum_{j=1}^n \alpha_j e^{-i(x^j - x) \cdot \xi} \sin((t^j - s)|\xi|) \mathbf{1}_{[0,t^j]}(s) \Big|^2 \frac{d\xi}{|\xi|^{2+k-\beta}} \\ \geq c \int_0^{a/2} ds \int_0^\infty \frac{d\rho}{\rho^{3-\beta}} \int_{\mathbb{S}^{k-1}} &\Big| \sin((t - s)\rho) \\ &\qquad - \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x) \cdot w} \sin((t^j - s)\rho) \Big|^2 dw \\ = c \int_0^{a/2} ds \int_{-\infty}^\infty \frac{d\rho}{|\rho|^{3-\beta}} \int_{\mathbb{S}^{k-1}} &\Big| \big(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\big) \\ &\qquad - \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x) \cdot w} \big(e^{i(t^j - s)\rho} - e^{-i(t^j - s)\rho}\big) \Big|^2 \, dw \\ =: c \int_{\mathbb{S}^{k-1}} A(w) \, dw. \end{aligned}$$

Let $\lambda = \min\{1, a/[2(a' + 2\sqrt{k}b)]\}$ and consider the bump function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(y) = \begin{cases} \exp\left(1 - \frac{1}{1-|\lambda^{-1}y|^2}\right), & |y| < \lambda, \\ 0, & |y| \geq \lambda. \end{cases}$$

Let $\varphi_r(y) = r^{-1}\varphi(y/r)$. For each $w \in \mathbb{S}^{k-1}$ such that $r(w) > 0$, consider the integral

$$\begin{aligned} I(w) = \int_0^{a/2} ds \int_{-\infty}^{\infty} \Big[& \left(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\right) \\ & - \sum_{j=1}^{n} \alpha_j e^{-i\rho(x^j - x)\cdot w}\left(e^{i(t^j-s)\rho} - e^{-i(t^j-s)\rho}\right)\Big] e^{-i(t-s)\rho}\widehat{\varphi}_{r(w)}(\rho) d\rho. \end{aligned}$$

By the inverse Fourier transform, we have

$$\begin{aligned} I(w) = 2\pi \int_0^{a/2} \Big[& \varphi_{r(w)}(0) - \varphi_{r(w)}\big(2(t-s)\big) \\ & - \sum_{j=1}^{n} \alpha_j \big\{\varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) \\ & - \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big)\big\}\Big]\, ds. \end{aligned}$$

Note that $r(w) \leq |t^j - t| + |x^j - x| \leq a' + 2\sqrt{k}b$. For any $s \in [0, a/2]$, we have $2(t-s)/r(w) \geq a/[(a' + 2\sqrt{k}b)]$ and $|(x^j - x)\cdot w - (t^j - t)|/r(w) \geq 1$, thus

$$\varphi_{r(w)}\big(2(t-s)\big) = 0 \quad \text{and} \quad \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) = 0 \quad \text{for } j = 1, \dots, n.$$

Also,

$$[(x^j - x)\cdot w - (t^j - t) + 2(t^j - s)]/r(w) \geq (-\delta + a)/[(a' + 2\sqrt{k}b)] \geq \lambda,$$

we have

$$\varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big) = 0.$$

It follows that

$$I(w) = a\pi\, r(w)^{-1}.$$

On the other hand, by the Cauchy–Schwarz inequality and scaling, we obtain

$$\begin{aligned} (a\pi)^2 r(w)^{-2} = |I(w)|^2 &\leq A(w) \times \int_0^{a/2} ds \int_{-\infty}^{\infty} |\widehat{\varphi}(r(w)\rho)|^2 |\rho|^{3-\beta}\, d\rho \\ &= (a/2)A(w)r(w)^{\beta-4} \int_{-\infty}^{\infty} |\widehat{\varphi}(\rho)|^2|\rho|^{3-\beta}\, d\rho \\ &= CA(w)r(w)^{\beta-4} \end{aligned}$$

for some finite constant C. Hence we have

$$A(w) \geq C' r(w)^{2-\beta} \tag{4.31}$$

and this remains true if $r(w) = 0$. Integrating both sides of (4.31) over $\mathbb{S}^{k-1}$ yields (4.30).

As an application of Theorem 4.4., Lee and Xiao (2019) proved the following uniform modulus of continuity.

Theorem 4.12 [Lee and Xiao, 2019] *There is a positive finite constant K such that*

$$\lim_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ |(t,x)-(t',x')|\le\varepsilon}} \frac{|u(t,x)-u(t',x')|}{\gamma\big[(t,x),(t',x')\big]} = K, \quad \text{a.s.}, \tag{4.32}$$

where

$$\gamma\big[(t,x),(t',x')\big] = \big(|t-t'|+|x-x|\big)^{2-\beta}\sqrt{\log\big[|t-t'|+|x-x|\big]^{-1}}.$$

Remark. Theorems 4.11 and 4.12 have been extended to SWE with fractional-colored noise by Lee (2021).

5 Lecture 5. Fractal Properties of Gaussian Random Fields

Let $X = \{X(t),\, t\in\mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. It generates many random sets, for example,

- Range $\quad X\left([0,1]^N\right) = \left\{X(t):\, t\in[0,1]^N\right\}$
- Graph $\quad \mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)):\, t\in[0,1]^N\right\}$
- Level set $\quad X^{-1}(x) = \left\{t\in\mathbb{R}^N : X(t) = x\right\}$
- Excursion set $\quad X^{-1}(F) = \left\{t\in\mathbb{R}^N : X(t)\in F\right\},\ \forall F\subseteq\mathbb{R}^d$,
- The set of self-intersections, ...

In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.

5.1 Definitions of Hausdorff measure and dimension

Let Φ be the class of functions $\varphi:(0,\delta)\to(0,\infty)$ which are right continuous, monotone increasing with $\varphi(0+)=0$ and such that there exists a finite constant $K>0$ such that

$$\frac{\varphi(2s)}{\varphi(s)}\le K \quad \text{for}\ \ 0<s<\frac{1}{2}\delta.$$

A function φ in Φ is often called a *measure function* or *gauge function*.
For example, $\varphi(s)=s^\alpha$ $(\alpha>0)$ and $\varphi(s)=s^\alpha\log\log(1/s)$ are measure functions.
Given $\varphi\in\Phi$, the *φ-Hausdorff measure* of $E\subseteq\mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon\to 0}\ \inf\left\{\sum_i \varphi(2r_i) : E\subseteq\bigcup_{i=1}^{\infty} B(x_i,r_i),\ r_i<\varepsilon\right\}, \tag{5.1}$$

where $B(x,r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (5.1) is called an ε-covering of E.

It can be shown that φ-m is a metric outer measure and all Borel sets in $\mathbb{R}^d$ is φ-m measurable.

A function $\varphi \in \Phi$ is called *an exact Hausdorff measure function for E* if $0 < \varphi\text{-}m(E) < \infty$. If $\varphi(s) = s^\alpha$, we write $\varphi\text{-}m(E)$ as $\mathcal{H}_\alpha(E)$. The *Hausdorff dimension* of E is defined by

$$\begin{aligned}\dim_{\rm H} E &= \inf\big\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = 0\big\}\\ &= \sup\big\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = \infty\big\},\end{aligned}$$

Convention: $\sup \varnothing := 0$.

Hausdorff dimension has the following properties:

1. $E \subseteq F \subseteq \mathbb{R}^d \ \Rightarrow \dim_{\rm H} E \le \dim_{\rm H} F \le d$.
2. (σ-stability):

$$\dim_{\rm H}\Big(\bigcup_{j=1}^\infty E_j\Big) = \sup_{j\ge 1} \dim_{\rm H} E_j.$$

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\overline{D}{}^{\varphi}_{\mu}(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

The following upper density theorem was proved by Rogers and Taylor (1961), which is useful for proving lower bound for the Hausdorff measure of (random) fractals related to random fields (see Lectures 5 and 6 below).

Lemma 5.1 [Rogers and Taylor, 1961] *Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have*

$$K^{-1}\mu(E) \inf_{x\in E}\big\{\overline{D}{}^{\varphi}_{\mu}(x)\big\}^{-1} \le \varphi\text{-}m(E) \le K\|\mu\| \sup_{x\in E}\big\{\overline{D}{}^{\varphi}_{\mu}(x)\big\}^{-1}.$$

5.2 Packing measure and packing dimension

They were introduced by Tricot (1982), Taylor and Tricot (1985). For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$, define

$$\varphi\text{-}P(E) = \lim_{\varepsilon\to 0} \ \sup\Big\{\sum_i \varphi(2r_i) : \{\overline{B}(x_i, r_i)\} \text{ is an } \ \varepsilon\text{-packing}\Big\}.$$

Here ε-packing means that the balls are disjoint, $x_i \in E$ and $r_i \le \varepsilon$.

The packing measure φ-p of E is defined as:

$$\varphi\text{-}p(E) = \inf\Big\{\sum_n \varphi\text{-}P(E_n) : \quad E \subseteq \bigcup_n E_n\Big\}.$$

A function $\varphi \in \Phi$ is called *an exact packing measure function for E* for E if $0 < \varphi\text{-}p(E) < \infty$. If $\varphi(s) = s^\alpha$, we write $\varphi\text{-}p(E)$ as $\mathcal{P}_\alpha(E)$. The packing dimension of E is defined as:

$$\dim_{\rm P} E = \inf\{\alpha > 0 : \quad \mathcal{P}_\alpha(E) = 0\}.$$

Comparison between $\dim_{\rm H}$ and $\dim_{\rm P}$: For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$,

$$\varphi\text{-}m(E) \le \varphi\text{-}p(E), \qquad \dim_{\rm H} E \le \dim_{\rm P} E.$$

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *lower φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\underline{D}^{\varphi}_{\mu}(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

The following is a dual result to Lemma 5.1 that was proved by Taylor and Tricot (1985).

Lemma 5.2 [Taylor and Tricot, 1985] *Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have*

$$K^{-1}\mu(E) \inf_{x\in E} \big\{\underline{D}^{\varphi}_{\mu}(x)\big\}^{-1} \le \varphi\text{-}p(E) \le K\|\mu\| \sup_{x\in E} \big\{\underline{D}^{\varphi}_{\mu}(x)\big\}^{-1}.$$

Example. Let C denote the standard ternary Cantor set in $[0\,,1]$. At the nth stage of its construction, C is covered by 2^n intervals of length/diameter 3^{-n} each.

It can be proved that

$$\dim_{\mathrm{H}} C = \dim_{\mathrm{P}} C = \log_3 2.$$

By using the upper and lower density theorems, one can prove that

$$0 < \mathcal{H}_{\log_3 2}(C) \le \mathcal{P}_{\log_3 2}(C) < \infty.$$

Example. Let $B([0,1])$ be the image of Brownian motion in $\mathbb{R}^d$. Lévy (1948) and Taylor (1953) proved that

$$\dim_{\mathrm{H}} B([0,1]) = \min\{d,\, 2\} \qquad \text{a.s.}$$

Ciesielski and Taylor (1962), Ray and Taylor (1964) proved that

$$0 < \varphi_d\text{-}m\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where

$$\begin{aligned} &\varphi_1(r) = r,\\ &\varphi_2(r) = r^2 \log(1/r) \log\log\log(1/r),\\ &\varphi_d(r) = r^2 \log\log(1/r), \qquad \text{if } d \ge 3. \end{aligned}$$

Taylor and Tricot (1985) proved that

$$\dim_{\mathrm{P}} B([0,1]) = \min\{d,\, 2\}$$

and, if $d \ge 3$, then

$$0 < \psi\text{-}p\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where $\psi(r) = r^2/\log\log(1/r)$.

LeGall and Taylor (1986) proved that, if $d = 2$, then for any measure function φ, either $\varphi\text{-}p\big(B([0,1])\big) = 0$ or ∞ according to an integral test.

Question: How to extend the above results to Gaussian random fields?

We start with fractional Brownian motion.

For $H \in (0,1)$, the fBm $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ with index H is a centered (N,d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[B^H_i(s)B^H_j(t)\big] = \frac{1}{2}\,\delta_{ij}\,\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right), \tag{5.2}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$ and $H = 1/2$, B^H is Brownian motion.
- B^H is H-self-similar and has stationary increments.

Kahane (1985) proved that

$$\dim_{\rm H} B^H([0,1]^N) = \min\Big\{d, \frac{N}{H}\Big\} \qquad \text{a.s.}$$

5.3 Exact Hausdorff measure functions for $B^H([0,1]^N)$ and Gr$B^H([0,1]^N)$

The following theorem was proved by Talagrand (1995, 1998). The random covering method designed by Talagrand has several important applications.

Theorem 5.3 [Talagrand, 1995, 1998] *Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.*

(i). If $N < Hd$, then

$$K^{-1} \le \varphi_1\text{-}m\big(B^H([0,1]^N)\big) \le K, \qquad a.s.$$

where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N = Hd$, then $\varphi_2\text{-}m\big(B^H([0,1]^N)\big)$ is σ-finite, where $\varphi_2(r) = r^d \log(1/r) \log\log\log(1/r)$.

The following theorem deals with the Hausdorff measure of the graph set of B^H.

Theorem 5.4 [Xiao, 1997] *Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.*

(i). If $N < Hd$, then

$$K^{-1} \le \varphi_1\text{-}m\big(\mathrm{Gr}B^H([0,1]^N)\big) \le K, \qquad a.s.$$

where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N > Hd$, then

$$K^{-1} \le \varphi_3\text{-}m\big(\mathrm{Gr}B^H([0,1]^N)\big) \le K, \qquad a.s.,$$

where $\varphi_2(r) = r^{N+(1-H)d}\big(\log\log(1/r)\big)^{Hd/N}$.

5.4 Exact packing measure functions for fractional Brownian motion

Theorem 5.5 [Xiao, 1996, 2003] *Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$. If $N < Hd$, then there exists a finite constant $K \ge 1$ such that*

$$K^{-1} \le \varphi_4\text{-}p(B^H([0,1]^N)) \le K, \quad \text{a.s.}$$

where $\varphi_4(r) = r^{\frac{N}{H}}\big(\log\log(1/r)\big)^{-N/(2H)}$.

For proving Theorem 5.5, one needs to study the liminf behavior of the sojourn measure

$$T(r) = \int_{\mathbb{R}^N} \mathbf{1}_{\{|B^H(t)| \le r\}} dt.$$

A key ingredient is the following small ball probability estimate for $T(1)$.

Lemma 5.6 [Xiao, 1996, 2003] *Assume that $N < Hd$. Then there exists a positive and finite constant $K \geq 1$, depending only on H, N and d such that for any $0 < \varepsilon < 1$,*

$$\exp\Big(-\frac{K}{\varepsilon^{2H/N}}\Big) \leq \mathbb{P}\{T(1) < \varepsilon\} \leq \exp\Big(-\frac{1}{K\varepsilon^{2H/N}}\Big).$$

This leads to the following Chung's LIL for $T(r)$.

Theorem 5.7 [Xiao, 1996, 2003] *If $N < Hd$, then with probability one,*

$$\liminf_{r\to 0} \frac{T(r)}{\varphi_4(r)} = K, \tag{5.3}$$

where $0 < K < \infty$ is a constant depending on H, N and d only.

By the stationarity of increments of B^H and the lower density theorem, we derive the lower bound in Theorem 5.5.

The proof of upper bound in Theorem 5.5 requires a different argument and is omitted here.

5.5 Exact Hausdorff measure function for the ranges of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by (6.2) whose components $X_1, \ldots, X_d$ are independent copies of a centered Gaussian field X_0. We assume that X_0 satisfies the following conditions from Lecture 3.

Assumption (A1) Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\, t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X_0(t)$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s = (s_1, \ldots, s_N), t = (t_1, \ldots, t_N) \in T$,

$$\begin{aligned} &\big\|v([a,b), s) - X_0(s) - v([a,b), t) + X_0(t)\big\|_{L^2} \\ &\leq C\Big(\sum_{j=1}^N a^{\gamma_j}|s_j - t_j| + b^{-1}\Big), \end{aligned} \tag{5.4}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\big\|v([0,a_0), s) - v([0,a_0), t)\big\|_{L^2} \leq C\sum_{j=1}^N |s_j - t_j|. \tag{5.5}$$

Condition (A4$'$) [strong local nondeterminism]. There exists a constant $c > 0$ such that $\forall\, n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X_0(u) \,\big|\, X_0(t^1), \ldots, X_0(t^n)\big) \geq c \min_{1\leq k\leq n} \rho(u, t^k)^2, \tag{5.6}$$

where $\rho(s,t)$ is the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j},$$

and where $H_j = (\gamma_j + 1)^{-1}$ $(j = 1, \ldots, N)$.

These conditions are weaker than those in Luan and Xiao (2012).

Theorem 5.8 *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian field with values in $\mathbb{R}^d$ such that X_0 satisfies (A1) and (A4′).*

(i). If $Q = \sum_{j=1}^{N} H_j^{-1} < d$, then

$$K^{-1} \le \varphi_5\text{-}m\big(X([0,1]^N)\big) \le K, \qquad a.s., \tag{5.7}$$

where $\varphi_5(r) = r^Q \log\log(1/r)$.

(ii). If $Q > d$, then $X([0,1]^N)$ has positive d-dimensional Lebesgue measure a.s.

The problem to determine the exact Hausdorff measure function for $X([0,1]^N)$ in the "critical case" $Q = d$ is open.

Proof of Theorem 5.8. The lower bound in (5.7) is proved by using the upper density theorem in Lemma 5.1. A natural measure on $X([0,1]^N)$ is the sojourn measure

$$\mu(B) = \lambda_N\big\{t \in [0,1]^N : X(t) \in B\big\}, \qquad \forall\, B \in \mathcal{B}(R^d),$$

where λ_N denotes the Lebesgue measure on $\mathbb{R}^N$.

For any $0 < r < 1$ and $t^0 \in [0,1]^N := I$, we consider

$$\mu\big(B(X(t^0), r)\big) = \int_I \mathbf{1}_{\{|X(t) - X(t^0)| \le r\}}\, dt,$$

which is the sojourn time of X in the ball $B(X(t^0), r)$.

The following moment estimate is essential for determining the asymptotic behavior of $\mu\big(B(X(t^0), r)\big)$ as $r \to 0$.

Lemma 5.9 *If $d > Q$, then there is a finite constant C such that for every $t^0 \in I$ and all integers $n \ge 1$,*

$$\mathbb{E}\left[\mu\big(B(X(t^0), r)\big)^n\right] \le C^n n!\, r^{Qn}.$$

Proof F or $n = 1$, by Fubini's theorem we have

$$\begin{aligned}
\mathbb{E}\big[\mu\big(B(X(t^0), r)\big)\big] &= \int_I \mathbb{P}\left\{|X(t) - X(t^0)| < r\right\} dt \\
&\le \int_I \min\left\{1, c\Big(\frac{r}{\rho(t, t^0)}\Big)^d\right\} dt \\
&= \int_{\{t: \rho(t,t^0) \le cr\} \cap I} dt + c \int_{\{t: \rho(t,t^0) > cr\} \cap I} \Big(\frac{r}{\rho(t, t^0)}\Big)^d dt.
\end{aligned}$$

It is elementary to verify that

$$\mathbb{E}\big[\mu\big(B(X(t^0), r)\big)\big] \le cr^Q.$$

For $n \ge 2$,

$$\mathbb{E}\big[\mu\big(B(X(t^0), r)\big)^n\big] = \int_{I^n} \mathbb{P}\big\{\big|X(t^j) - X(t^0)\big| < r, 1 \le j \le n\big\} dt^1 \cdots dt^n.$$

It is sufficient to consider $t^1, \cdots, t^n \in I$ that satisfy

$$t^j \neq t^0, \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad t^j \neq t^k \quad \text{for} \quad j \neq k.$$

By Condition (A4′), we have

$$\begin{aligned}
&\operatorname{Var}\big(X_0(t^n) - X_0(t^0)\big|X_0(t^1) - X_0(t^0), \cdots, X_0(t^{n-1}) - X_0(t^0)\big) \\
&\ge \operatorname{Var}\big(X_0(t^n)\big|X_0(t^0), X_0(t^1), \cdots, X_0(t^{n-1})\big) \\
&\ge c \min_{0 \le k \le n-1} \rho(t^n, t^k)^2.
\end{aligned} \tag{5.8}$$

Since conditional distributions in Gaussian processes are still Gaussian, it follows from Anderson's inequality and (5.8) that

$$\begin{aligned}
&\int_I \mathbb{P}\Big\{\big|X(t^n) - X(t^0)\big| < r\big|X(t^1) - X(t^0), \cdots, X(t^{n-1}) - X(t^0)\Big\} dt^n \\
&\quad \le c \int_I \sum_{k=0}^{n-1} \min\Big\{1, c\Big(\frac{r}{\rho(t^n, t^k)}\Big)^d\Big\}\, dt^n \\
&\quad \le c\, n \int_I \min\Big\{1, c\Big(\frac{r}{\rho(t^n, 0)}\Big)^d\Big\}\, dt^n \\
&\quad \le c\, n r^Q.
\end{aligned}$$

Iterating the procedure proves Lemma 5.5. □

From Lemma 5.5 and the Borel-Cantelli lemma, we can prove the following law of the iterated logarithm for the sojourn measure of X.

Proposition 5.10 *For every $t^0 \in I$, we have*

$$\limsup_{r \to 0} \frac{\mu\big(B(X(t^0), r)\big)}{\varphi_5(r)} \le C < \infty, \quad a.s.$$

This and Fubini's theorem yield: a.s.

$$\limsup_{r \to 0} \frac{\mu\big(B(X(t^0), r)\big)}{\varphi_5(r)} \le C \quad a.e.\ t^0 \in I.$$

Hence, the lower bound in (5.7) follows from Lemma 5.1.

For proving the upper bound in (5.7), we need the following small ball probability estimates.

Lemma 5.11 [Xiao, 2009] *Under the conditions of Theorem 5.6, There exist constants c and c' such that for all $t^0 \in I = [0,1]^N$ and $0 < \varepsilon < r$,*

$$\exp\Big(-c'\Big(\frac{r}{\varepsilon}\Big)^Q\Big) \le \mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r} |X(t) - X(t_0)| \le \varepsilon\Big\} \le \exp\Big(-c\Big(\frac{r}{\varepsilon}\Big)^Q\Big).$$

The main estimate is given in the following lemma.

Proposition 5.12 *Assume that the conditions of Theorem 5.6 hold. There exist positive constants δ_0 and C such that for any $t^0 \in I$ and $0 < r_0 \le \delta_0$, we have*

$$\begin{aligned}&\mathbb{P}\Big\{\exists\, r \in [r_0^2, r_0],\ \sup_{t\in I:\rho(t,t^0)\le r} |X(t) - X(t^0)| \le Cr\big(\log\log(1/r)\big)^{-1/Q}\Big\}\\ &\qquad \ge 1 - \exp\Big(-\big(\log(1/r_0)\big)^{1/2}\Big).\end{aligned}$$

Proof The method of proof comes form Talagrand (1995). We provide the main steps. Let $U > 1$ be a number whose value will be determined later. For $k \ge 0$, let $r_k = r_0 U^{-2k}$. Consider the largest integer k_0 such that

$$k_0 \le \frac{\log(1/r_0)}{2\log U}.$$

$$\begin{aligned}&\mathbb{P}\Big\{\exists k \le k_0,\ \sup_{t\in I:\rho(t,t^0)\le r_k} |X(t) - X(t^0)| \le c\, r_k\Big(\log\log\frac{1}{r_k}\Big)^{-1/Q}\Big\}\\ &\ge 1 - \exp\Bigg(-\Big(\log\frac{1}{r_0}\Big)^{1/2}\Bigg).\end{aligned} \tag{5.9}$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \cdots$

$$X_{0,k}(t) = v([a_k, a_{k+1}),\, t)$$

and

$$\widehat{X}_k(t) = \big(X_{1,k}(t), \cdots, X_{d,k}(t)\big),$$

where $X_{1,k}(t), \cdots, X_{d,k}(t)$ are independent copies of $X_{0,k}(t)$. It follows that $X_1 - X_{1,k}, \cdots, X_d - X_{d,k}$ are independent copies of $X_0 - X_{0,k}$.

The Gaussian random fields $\widehat{X}_0, \widehat{X}_1, \cdots$ are independent. By Lemma 5.6 we can find a constant $c > 0$ such that, if r_0 is small enough, then for each $k \ge 0$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r_k} \big|\widehat{X}_k(t) - \widehat{X}_k(t^0)\big| \le c\, r_k\big(\log\log(1/r_k)\big)^{-1/Q}\Big\}\\ &\ge \exp\Big(-\frac{1}{4}\log\log(1/r_k)\Big) = \frac{1}{(\log 1/r_k)^{1/4}}\\ &\ge \big(2\log 1/r_0\big)^{-1/4}.\end{aligned}$$

$$\begin{aligned}&\mathbb{P}\Big\{\exists k\le k_0,\ \sup_{t\in I:\rho(t,t^0)\le r_k}\big|\widehat{X}_k(t)-\widehat{X}_k(t^0)\big|\le c\,r_k\Big(\log\log(1/r_k)\Big)^{-1/Q}\Big\}\\&\ge 1-\Big(1-\frac{1}{(2\log 1/r_0)^{1/4}}\Big)^{k_0}\ge 1-\exp\Big(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\Big),\end{aligned}\tag{5.10}$$

where the last inequality follows from $1-x\le e^{-x}$ for all $x\ge 0$.

To deal with $\{X(t)-\widehat{X}_k(t)\}$, we claim that for any $u\ge cr_kU^{-\beta}\sqrt{\log U}$, where $\beta=\min\{{H_N}^{-1}-1,\,1\}$,

$$\mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r_k}\big|X(t)-\widehat{X}_k(t)-(X(t^0)-\widehat{X}_k(t^0))\big|\ge u\Big\}\le\exp\Big(-\frac{u^2}{cr_k^2U^{-2\beta}}\Big).\tag{5.11}$$

To see this, it's enough to prove that (5.11) holds for X_0, by applying Lemma 3.3.

Consider $S=\{t\in I:\rho(t,t^0)\le r_k\}$ and on S the distance

$$d(s,t)=\big\|X_0(s)-X_{0,k}(s)-(X_0(t)-X_{0,k}(t))\big\|_{L^2}.$$

Then $d(s,t)\le c\sum_{i=1}^N|s_i-t_i|^{H_i}$ and $N(S,d,\varepsilon)\le c\big(r_k/\varepsilon\big)^Q$.

Now we estimate the d-diameter D of S. By Condition (A1), we have for any $s,t\in S$,

$$\begin{aligned}&\|X_0(s)-X_{0,k}(s)-(X_0(t)-X_{0,k}(t))\|_{L^2}\\&\le C\bigg(\sum_{j=1}^N a_k^{H_j^{-1}-1}|s_j-t_j|+a_{k+1}^{-1}\bigg)\le Cr_kU^{-\beta},\end{aligned}$$

where $\beta=\min\{{H_N}^{-1}-1,\,1\}$. Therefore, $D\le Cr_kU^{-\beta}$. Notice that

$$\begin{aligned}\int_0^D\sqrt{\log N(S,d,\varepsilon)}d\varepsilon&\le c\int_0^{Cr_kU^{-\beta}}\sqrt{\log r_k/\varepsilon}\,d\varepsilon\\&\le cr_k\int_0^{CU^{-\beta}}\sqrt{\log 1/u}\,du\le cr_kU^{-\beta}\sqrt{\log U}.\end{aligned}$$

Hence (5.11) follows from Lemma 3.3.

Let $U=(\log 1/r_0)^{1/\beta}$. Then for $r_0>0$ small

$$U^\beta\,(\log U)^{-1/2}\ge\bigg(\log\log\frac{1}{r_0}\bigg)^{1/Q}.$$

Take $u=cr_k(\log\log 1/r_0)^{-1/Q}$. It follows from (5.11) that

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r_k}\big|X(t)-\widehat{X}_k(t)-\big(X(t^0)-\widehat{X}_k(t^0)\big)\big|\ge c\,r_k\big(\log\log\frac{1}{r_0}\big)^{-1/Q}\Big\}\\&\le\exp\Big(-\frac{cU^\beta}{(\log\log 1/r_0)^{2/Q}}\Big).\end{aligned}$$

Combining this with (5.10), we get

$$
\begin{aligned}
&\mathbb{P}\Big\{\exists k \le k_0, \sup_{\rho(t,\, t^0)\le r_k} \big|X(t) - X(t^0)\big| \le c\, r_k\big(\log\log(1/r_k)\big)^{-1/Q}\Big\}\\
&\quad \ge 1 - \exp\Big(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\Big) - k_0 \exp\Big(-\frac{cU^\beta}{(\log\log 1/r_0)^{2/Q}}\Big).
\end{aligned}
$$

This proves (5.9) and Proposition 5.12. □

With Proposition 5.12 in hand, we proceed to construction of an economic covering for $X([0,1]^N)$.

Proof of Theorem 5.8. (*continued*) For $k \ge 1$, consider the set

$$
\begin{aligned}
R_k = \Big\{ t \in [0,1]^N : \exists\ r \in [2^{-2k}, 2^{-k}] \text{ such that}\\
\sup_{s\in I:\rho(s,t)\le r} \big|X(s) - X(t)\big| \le c\, r(\log\log \frac{1}{r})^{-1/Q}\Big\}.
\end{aligned}
$$

By Lemma 5.7 we have that for every $t \in [0,1]^N$,

$$
\mathbb{P}\{t \in R_k\} \ge 1 - \exp(-\sqrt{k/2}).
$$

This and Fubini's theorem imply that

$$
\mathbb{E}[\lambda_N(R_k)] \ge 1 - \exp(-\sqrt{k/2}).
$$

Or

$$
\mathbb{E}[\lambda_N(I\backslash R_k)] \le \exp(-\sqrt{k/2}).
$$

By Markov's inequality, we have

$$
\begin{aligned}
\mathbb{P}\Big\{\lambda_N(R_k) < 1 - \exp(-\sqrt{k}/2)\Big\} &= \mathbb{P}\Big\{\lambda_N(I\backslash R_k) > \exp(-\sqrt{k}/2)\Big\}\\
&\le \frac{\mathbb{E}[\lambda_N(I\backslash R_k)]}{\exp(-\sqrt{k}/2)}\\
&\le \exp\Big(-\Big(\frac{1}{\sqrt{2}} - \frac{1}{2}\Big)\sqrt{k}\Big).
\end{aligned}
$$

Hence, by the Borel-Cantelli lemma, we have $\mathbb{P}(\Omega_1) = 1$, where

$$
\Omega_1 = \Big\{\omega :\ \lambda_N(R_k) \ge 1 - \exp(-\sqrt{k}/2) \text{ for all } k \text{ large enough}\Big\}.
$$

On the other hand, by Lemma 3.3, we have $\mathbb{P}(\Omega_2) = 1$, where Ω_2 it the event that for every rectangle I_n of side-lengths $2^{-n/H_i} (i = 1, \cdots, N)$ that meets $[0,1]^N$, we have

$$
\sup_{s,t\in I_n} \big|X(t) - X(s)\big| \le C 2^{-n}\sqrt{n},
$$

where $C > 0$ is a constant.

Now we show that for every $\omega \in \Omega_1 \cap \Omega_2$, we have

$$\varphi_5\text{-}m(X([0,1]^N)) \le K < \infty, \qquad \text{a.s.}$$

For any $n \ge 1$, we divide $[0,1]^N$ into 2^{nQ} disjoint (half open and half closed) rectangles of side-lengths $2^{-n/H_i}$ $(i = 1, \cdots, N)$. Denote by $I_n(x)$ the unique rectangle of side-lengths $2^{-n/H_i}(i = 1, \cdots, N)$ containing x.

Consider $k \ge 1$ such that

$$\lambda_N(R_k) \ge 1 - \exp(-\sqrt{k}/2).$$

For any $x \in R_k$ we can find the smallest integer n with $k \le n \le 2k$ such that

$$\sup_{s,t \in I_n(x)} \big|X(t) - X(s)\big| \le c\, 2^{-n} (\log\log 2^n)^{-1/Q}. \tag{5.12}$$

Thus we have

$$R_k \subseteq V = \bigcup_{n=k}^{2k} V_n$$

and each V_n is a union of rectangles $I_n(x)$ satisfying (5.12).

Notice that $X(I_n(x))$ can be covered by a ball of radius $r_n = c2^{-n}(\log\log 2^n)^{-1/Q}$.

Since $\varphi_5(2r_n) \le c2^{-nQ} = c\lambda_N(I_n)$, we obtain

$$\sum_{n=k}^{2k} \sum_{I_n \in V_n} \varphi_5(2r_n) \le \sum_n \sum_{I_n \in V_n} c\lambda_N(I_n) = C\lambda_N(V) \le C. \tag{5.13}$$

Thus $X(V)$ is contained in the union of a family of balls B_n of radius r_n with $\sum_n \varphi_5(2r_n) \le C$.

On the other hand, $[0,1]^N \backslash V$ is contained in a union of rectangles of side-lengths $2^{-q/H_i}(i = 1, \cdots, N)$ where $q = 2k+1$, none of which meets R_k. There can be at most

$$2^{Qq} \lambda_N([0,1]^N \backslash V) \le c2^{Qq} \exp(-\sqrt{k}/2)$$

such rectangles.

Since $\omega \in \Omega_2$, for each of these rectangles I_q, $X(I_q)$ is contained in a ball of radius $c2^{-q}\sqrt{q}$.

Thus $X([0,1]^N \backslash V)$ can be covered by a sequence $\{B_n\}$ of balls of radius $r_n = c2^{-q}\sqrt{q}$ such that

$$\begin{aligned}\sum_n \varphi_5(2r_n) &\le \Big(c2^{Qq} \exp(-\sqrt{k}/2)\Big)\Big(c2^{-qQ} q^{Q/2} \log\log(c2^q/\sqrt{q})\Big) \\ &\le 1\end{aligned} \tag{5.14}$$

for all k large enough. Since k can be arbitrarily large, it follows from (5.13) and (5.14) that

$$\varphi_5\text{-}m\big(X([0,1]^N)\big) \le K, \quad \text{a.s.}$$

This finishes the proof of Part (i) of Theorem 5.6.

Part (ii) is related to the existence of local times. A proof based on Fourier analysis will be given in Lecture 6. □

If Condition (A4′) in Theorem 5.6 is replaced by (A4), then the exact Hausdorff measure function for $X([0,1]^N)$ is different. See the recent paper of Lee (2021).

6 Lecture 6. Local Times of Gaussian Random Fields

Local times of Brownian motion was first studied by P. Lévy (1948), under a different name. The term of "local times" for general Markov processes was introduced by Blumenthal and Getoor (1964).

In late 1960's, Berman started studying local times of Gaussian processes. His work was extended by Pitt (1978) to random fields, and stimulated a lot of works on local times of random fields. More information can be found in Geman and Horowitz (1980), Dozzi (2002), Xiao (2009), etc.

6.1 Local times: existence and joint continuity

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-random field. For any Borel set $T \subseteq \mathbb{R}^N$, the *occupation measure* of X on T defined by

$$\mu_T(\bullet) = \lambda_N\{t \in T : \, X(t) \in \bullet\}.$$

If $\mu_T \ll \lambda_d$, then X is said to have a *local time* on T, which is defined by

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x),$$

where x is the so-called *space variable,* and T is the *time variable.* We write $L(x, t)$ instead of $L(x, [0, t])$.

$L(x, T)$ satisfies the following *occupation density formula*: For every Borel set $T \subseteq \mathbb{R}^N$ and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_T f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, T)\, dx. \tag{6.1}$$

Suppose we fix an interval $I = \prod_{\ell=1}^N [a_\ell, \, b_\ell]$ in $\mathbb{R}^N$. Let $T = \prod_{\ell=1}^N [a_\ell, \, t_\ell] \subset I$. If we can choose a version of the local time, still denoted by $L(x, T)$, such that it is continuous in $(x, t_1, \ldots, t_N) \in \mathbb{R}^d \times I$, then X is said to have a *jointly continuous local time* on I.

- The smoother the local time, the rougher the sample path (Berman, 1972).
- When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set
$$X^{-1}(x) = \{t \in I : \, X(t) = x\}$$
[cf. Adler, 1981] and is a useful tool for studying fractal properties of $X^{-1}(x)$.

Let $X = \{X(t), \, t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \tag{6.2}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued Gaussian random field X_0.

We study the following questions:

- The existence and joint continuity of local times of X.

- Hölder conditions for the local times of X and apply these results to study its sample path properties of X.

The following result [cf. Geman and Horowitz (1980, Theorem 21.9)] is convenient: X has an $L^2(\mathbb{P} \times \lambda_d)$ local time $L(x, T)$ if and only if

$$\int_{\mathbb{R}^d} \int_T \int_T \mathbb{E}\Big(e^{i\langle \theta, X(s)-X(t)\rangle}\Big)\, dsdtdx < \infty.$$

In particular, we have

Theorem 6.1 *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (6.2) such that*

$$\mathbb{E}\Big[\big(X_0(s) - X_0(t)\big)^2\Big] \asymp \rho(s,t)^2, \quad \textit{for} \ \ s, t \in T. \tag{6.3}$$

Then X has an $L^2(\mathbb{P} \times \lambda_d)$ local time if and only if $Q > d$.

For studying joint continuity of local times, we make use of sectorial local nondeterminism.

Condition (A4) [sectorial local nondeterminism] For a constant vector $H = (H_1, \ldots, H_N) \in (0,\, 1)^N$, there exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X_0(u)\,\big|\, X_0(t^1), \ldots, X(t^n)\big) \geq c \sum_{j=1}^{N} \min_{1\leq k\leq n} \Big|u_j - t_j^k\Big|^{2H_j}. \tag{6.4}$$

Theorem 6.2 [Ayache, Wu, and Xiao (2008), Wu and Xiao, 2011] *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (6.2) such that (6.3) and (A4) hold. If $Q > d$ then the local time of X is jointly continuous on $T \times \mathbb{R}^d$.*

The proof of this theorem relies on Kolmogorov's continuity theorem and the moment estimates for $L(t, D)$ and $L(x, D) - L(y, D)$ in Lemmas 6.1 and 6.2 below. These estimates are more precise than what are needed for proving joint continuity.

We assume $0 < H_1 \leq \ldots \leq H_N < 1$. Under the condition that

$$Q = \sum_{\ell=1}^{N} \frac{1}{H_\ell} > d,$$

there exists a unique $\tau \in \{1, \ldots, N\}$ such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}.$$

We will distinguish three cases:

$$\text{Case 1:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} < d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell},$$

$$\text{Case 2:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} = H_\tau,$$

$$\text{Case 3:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} < H_\tau.$$

Lemma 6.3 *For any $D := \overline{B}_\rho(a,r) \subseteq T$, we have*

$$\mathbb{E}\left[L(x,D)^n\right] \le \begin{cases} c_{6,1}^n\,(n!)^{\eta_\tau}\,r^{n\alpha} & \textit{Cases 1 \& 2,} \\ c_{6,1}^n\,(n!)^{\eta_\tau}\,r^{n\alpha}\,\log^n\big(1+n\big) & \textit{Case 3,} \end{cases} \tag{6.5}$$

where

$$\begin{aligned} \alpha &= Q - d, \\ \eta_\tau &= \tau + H_\tau d - \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell}. \end{aligned} \tag{6.6}$$

For proving Lemma 6.3, we will need the following elementary lemma whose proof is omitted.

Lemma 6.4 *Let β, γ and p be positive constants such that $\gamma\beta > p$. There exists a constant $C > 0$ such that for all $A \in (0,1)$, $r > 0$, $u^* \in \mathbb{R}^p$, all integers $n \ge 1$ and distinct $u_1, \ldots, u_n \in O_p(u^*, r)$ we have*

$$\int_{O_p(u^*,r)} \frac{du}{\Big[A + \min\limits_{1\le j\le n} |u-u_j|^\gamma\Big]^\beta} \le C\, n\, A^{\frac{p}{\gamma}-\beta}, \tag{6.7}$$

where $O_p(u^*,r) \subset \mathbb{R}^p$ denotes the Euclidean ball.

Proof [of Lemma 6.3.] Recall that [e.g., Geman and Horowitz (1980)] for all $x \in \mathbb{R}^d$, any Borel set $D \subseteq \mathbb{R}^N$ and integer $n \ge 1$, l

$$\begin{aligned} \mathbb{E}\Big[L(x,D)^n\Big] = (2\pi)^{-nd} \int_{D^n}\int_{\mathbb{R}^{nd}} &\exp\Big(-i\sum_{j=1}^n \langle u^j, x\rangle\Big) \\ &\times \mathbb{E}\exp\Big(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big)\, d\overline{u}\; d\overline{t}, \end{aligned}$$

where

$$\overline{u} = (u^1,\ldots,u^n) \in \mathbb{R}^{nd}, \quad \overline{t} = (t^1,\ldots,t^n) \in D^n.$$

We see that $\mathbb{E}\big[L(x,D)^n\big]$ is at most

$$\begin{aligned} &\int_{D^n} \prod_{k=1}^d \Big\{ \int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u_k^j\, X_0(t^j)\Big)\Big]\, d\overline{u}_k \Big\}\, d\overline{t} \\ &= \int_{D^n} \Big[\mathrm{detCov}\,\big(X_0(t^1),\ldots,X_0(t^n)\big)\Big]^{-\frac{d}{2}}\, d\overline{t}, \end{aligned}$$

where $\overline{u}_k = (u_k^1,\ldots,u_k^n) \in \mathbb{R}^n$, $\overline{t} = (t^1,\ldots,t^n)$ and the equality follows from the fact that for any positive definite $n\times n$ matrix Γ,

$$\int_{\mathbb{R}^n} \frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}}\, \exp\Big(-\frac{1}{2}x'\Gamma x\Big)\, dx = 1.$$

By using the fact that for any Gaussian vector $(Z_1, \ldots, Z_n)$

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{j=1}^{n} \mathrm{Var}(Z_j | Z_1, \ldots, Z_{j-1}).$$

and Condition (A4) we derive

$$\begin{aligned}
\mathbb{E}\big[L(x, D)^n\big] &\le c^n \int_{D^n} \prod_{j=1}^{n} \left[\sum_{\ell=1}^{N} \min_{0 \le s \le j-1} |t_\ell^j - t_\ell^s|^{2H_\ell} \right]^{-\frac{d}{2}} d\bar{t} \\
&\le c^n \int_{D^n} \prod_{j=1}^{n} \left[\sum_{\ell=1}^{\tau} \min_{0 \le s \le j-1} |t_\ell^j - t_\ell^s|^{2H_\ell} \right]^{-\frac{d}{2}} d\bar{t}.
\end{aligned}$$

To estimate the last integral, we will integrate in the order of $dt_1^n, \ldots, dt_N^n, \ldots, dt_1^1, \ldots, dt_N^1$. In Case 1, if $\tau = 1$, which implies that $H_1 d < 1$, we apply Lemma 6.3 and Lemma 2.3 in Xiao (1997) to derive

$$\begin{aligned}
&\int_D \frac{dt_1^n \cdots dt_N^n}{\Big(\min\limits_{0 \le s \le n-1} |t_1^n - t_1^s|^{2H_1} \Big)^{d/2}} \\
&= (2r)^{\sum_{\ell=2}^N \frac{1}{H_\ell}} \int_{a_1 - r^{\frac{1}{H_1}}}^{a_1 + r^{\frac{1}{H_1}}} \frac{dt_1^n}{\min\limits_{0 \le s \le n-1} |t_1^n - t_1^s|^{H_1 d}} \\
&\le c\, n^{H_1 d}\, r^{\sum_{\ell=1}^N \frac{1}{H_\ell} - d} \\
&= c\, n^{\eta_1}\, r^{\alpha}.
\end{aligned}$$

If $\tau > 1$, since $H_1 d > 1$, we apply Lemma 6.4 with $A = \sum\limits_{\ell=2}^{\tau} \min\limits_{0 \le s \le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}$ and $p = 1$ at first to derive

$$\begin{aligned}
&\int_{a_1 - r^{\frac{1}{H_1}}}^{a_1 + r^{\frac{1}{H_1}}} \frac{dt_1^n}{\Big(\min\limits_{0 \le s \le n-1} |t_1^n - t_1^s|^{2H_1} + \sum\limits_{\ell=2}^{\tau} \min\limits_{0 \le s \le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell} \Big)^{d/2}} \\
&\le \frac{c\, n}{\Big(\sum\limits_{\ell=2}^{\tau} \min\limits_{0 \le s \le n-1} |t_\ell^n - t_\ell^s|^{H_\ell} \Big)^{d - \frac{1}{H_1}}}.
\end{aligned}$$

Since $H_{\tau-1}\big(d - \sum_{\ell=1}^{\tau-2} \frac{1}{H_\ell}\big) > 1$, we can apply Lemma 6.4 repeatedly for $\tau - 1$ many times to get

$$\begin{aligned}
\int_D \frac{dt_1^n \cdots dt_N^n}{\Big(\sum\limits_{\ell=1}^{\tau} \min\limits_{0 \le s \le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell} \Big)^{d/2}} &\le c\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^N \frac{1}{H_\ell}} \\
&\quad \times \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \frac{dt_\tau^n}{\Big(\min\limits_{0 \le s \le n-1} |t_\tau^n - t_\tau^s|^{H_\tau} \Big)^{d - \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell}}}.
\end{aligned}$$

Notice that $H_\tau\big(d - \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell}\big) < 1$, by applying Lemma 2.3 in Xiao (1997), we derive

$$\begin{aligned}&\int_D \frac{dt_1^n \cdots dt_N^n}{\Big(\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}\Big)^{d/2}}\\ &\le c\, n^{\tau-1+H_\tau\left(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\right)}\, r^{\sum_{\ell=1}^{N}\frac{1}{H_\ell}-d} = c\, n^{\eta_\tau} r^{\alpha}.\end{aligned}$$

By iterating the procedure for integrating $dt_1^{n-1}, \ldots, dt_N^{n-1}$ and so on, we obtain (6.5) for Case 1.

The rest of the proof (Cases 2 & 3) of Lemma 6.1 is similar and thus omitted. □

Lemma 6.5 *For $\gamma \in (0,1)$ small and all even number $n \ge 2$, we have*

$$\begin{aligned}&\mathbb{E}\left[\big(L(x,D) - L(y,D)\big)^n\right]\\ &\le \begin{cases} c_{6,2}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)} & \textit{Cases 1 \& 2,}\\ c_{6,2}^n\,(n!)^{\eta_\tau+(2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)}\,\log^n(e+n) & \textit{Case 3.}\end{cases}\end{aligned} \tag{6.8}$$

The proof of Lemma 6.5 is more complicated, and is omitted here (please see Wu and X. 2011)

Finally, Theorem 6.2 follows from Lemmas 6.1, 6.2, and a multiparameter version of Kolmogorov's continuity theorem.

6.2 Hölder conditions for local times

Lemmas 6.3 and 6.5 can be applied to derive local and uniform Hölder conditions for the maximum local time $L^*(D) = \sup_{x\in\mathbb{R}^d} L(x, D)$.

Theorem 6.6 [Wu and Xiao, 2011] *There exists a constant $c_{6,3} > 0$ such that for every $a \in T$,*

$$\begin{aligned}&\limsup_{r\to 0} \frac{L^*(\overline{B}_\rho(a,r))}{\varphi_1^\rho(r)} \le c_{6,3}, \qquad \text{a.s.} \quad \textit{Cases 1 \& 2,}\\ &\limsup_{r\to 0} \frac{L^*(\overline{B}_\rho(a,r))}{\varphi_2^\rho(r)} \le c_{6,3}, \qquad \text{a.s.} \quad \textit{Case 3,}\end{aligned}$$

where $\overline{B}_\rho(a,r) \subset I$ is the ρ-ball and where

$$\begin{aligned}\varphi_1^\rho(r) &= r^\alpha\,\big(\log\log(1/r)\big)^{\eta_\tau},\\ \varphi_2^\rho(r) &= r^\alpha\,\big(\log\log(1/r)\big)^{\tau-1}\log\log\log\big(1/r\big).\end{aligned}$$

To state the uniform Hölder condition, let

$$\begin{aligned}\Phi_1^\rho(r) &= r^\alpha\,\big(\log(1/r)\big)^{\eta_\tau},\\ \Phi_2^\rho(r) &= r^\alpha\,\big(\log(1/r)\big)^{\tau-1}\log\log\big(1/r\big).\end{aligned}$$

The following theorem was proved by Wu and Xiao (2011) and was improved by Lee (2021).

Theorem 6.7 *Under the conditions of Theorem 6.6,*

$$\limsup_{r\to 0}\sup_{a\in T}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_1^\rho(r)} \le c_{6,4}, \qquad \text{a.s.} \quad \textit{Cases 1 \& 2,}$$

$$\limsup_{r\to 0}\sup_{a\in T}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_2^\rho(r)} \le c_{6,4}, \qquad \text{a.s.} \quad \textit{Case 3}$$

Theorems 6.6 and **??** can be applied to derive lower bounds for Chung-type LIL and modulus of non-differentiability for X.

Unless $H_1 = \cdots = H_N$, it is not known whether the Hölder conditions for the local times are optimal (even though we believe they are, at lease in Cases 1 & 2).

6.3 Optimal Hölder conditions

We can establish optimal Hölder conditions for the local times under strong local nondeterminism. This is done in Khoshnevisan, Lee, and Xiao (2021).

Condition (A4′) [strong local nondeterminism] There exists a constant $c > 0$ such that $\forall\, n \ge 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X_0(u)\,\big|\,X_0(t^1), \ldots, X_0(t^n)\big) \ge c \min_{1\le k\le n} \rho(u,\, t^k)^2, \tag{6.9}$$

where $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$.

We will use the following two lemmas.

Lemma 6.8 *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (6.2) such that (6.3) and (A4′) hold. If $Q > d$, then there exists a finite constant C such that for all Borel subsets S of T, for all $x \in \mathbb{R}^d$ and all integers $n \ge 1$, we have*

$$\mathbb{E}\big[L(x,D)^n\big] \le C^n (n!)^{d/Q} \lambda_N(D)^{n(1-d/Q)}.$$

In particular, for all $a \in T$ and $r \in (0,1)$ with $B_\rho(a,r) \subset T$, we have

$$\mathbb{E}\big[L(x, B_\rho(a,r))^n\big] \le C^n (n!)^{d/Q} r^{n(Q-d)}.$$

Lemma 6.9 *Under the conditions of Lemma 6.4, there exist constants C and K such that for all $\gamma \in (0,1)$ small enough, for all Borel sets $D \subseteq T$, for all $x, y \in \mathbb{R}^d$, for all even integers $n \ge 2$, we have*

$$\mathbb{E}\big[(L(x,D) - L(y,D))^n\big] \le C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} \lambda_N(D)^{n(1-(d+\gamma)/Q)}.$$

In particular, for all $a \in T$, $0 < r < 1$ with $B_\rho(a,r) \subset T$, we have

$$\mathbb{E}\big[(L(x, B_\rho(a,r)) - L(y, B_\rho(a,r)))^n\big] \le C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} r^{n(Q-d-\gamma)}.$$

Conditions (A4) and (A4′) have different effects on the Hölder conditions for the local times of X. We can compare the following Hölder conditions with Theorem 6.7.

Theorem 6.10 [Khoshnevisan, Lee, and Xiao, 2021] *Under the conditions of Lemma 6.8, there exist finite constants C and C' such that for any $t \in T$,*

$$\limsup_{r\to 0} \frac{L^*(B_\rho(t,r))}{\varphi_3^\rho(r)} \le C \quad a.s. \tag{6.10}$$

and

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B_\rho(t,r))}{\Phi_3^\rho(r)} \le C' \quad a.s. \tag{6.11}$$

where $\varphi_3^\rho(r) = r^\alpha (\log\log(1/r))^{d/Q}$ and

$$\Phi_3^\rho(r) = r^\alpha (\log(1/r))^{d/Q}.$$

Recall that, under Conditions (A1) and (A4′), Chung's LIL for X holds at $t \in T$ and X has an exact modulus of non-differentiability.

By using these results and the following inequality: For any $t \in T$,

$$\begin{aligned}\lambda_N\big(B_\rho(t,r)\big) &= \int_{\overline{X\left(B_\rho(t,r)\right)}} L(x, B_\rho(t,r))\ dx \\ &\le L^*(B_\rho(t,r)) \cdot \Big(\sup_{s,t\in B_\rho(t,r)} |X(s)-X(t)|\Big)^d,\end{aligned}$$

we derive that the Hölder conditions in Theorem 6.10 are optimal.

Theorem 6.11 *—rm [Khoshnevisan, Lee, and Xiao, 2021] Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (6.2) such that X_0 satisfies Conditions (A1) and (A4′). There exist positive constants K and K' such that for any $t \in T$,*

$$\limsup_{r\to 0} \frac{L^*(B_\rho(t,r))}{\varphi_3^\rho(r)} \ge K \quad a.s. \tag{6.12}$$

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B_\rho(t,r))}{\Phi_3^\rho(r)} \ge K' \quad a.s. \tag{6.13}$$

The results in this lecture can be conveniently applied to the solutions of stochastic heat and wave equations with the Gaussian noise that is white in time and colored in space, as well as some fractional-colored noises.

7 Lecture 7. Hitting Probabilities and Polarity of Points for Gaussian Random Fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. Various intersection problems can be considered:

(1) For Borel sets $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, when is

$$\mathbb{P}\big(X(E) \cap F \ne \emptyset\big) > 0? \tag{7.1}$$

(2) [Multiple intersections] Given disjoint sets $E_1, \ldots, E_k \subseteq \mathbb{R}^N$, when does

$$\mathbb{P}\big(X(E_1) \cap \cdots \cap X(E_k) \cap F \neq \emptyset\big) > 0? \tag{7.2}$$

Question **(1)** is quite general, which includes intersections of the graph set and level sets:

- Let $\mathrm{Gr}X(E) = \{(t, X(t)) : t \in E\}$ be the graph of X on E. Then (1) is equivalent to

$$\mathbb{P}\big(\mathrm{Gr}X(E) \cap (E \times F) \neq \emptyset\big) > 0.$$

- Take $F = \{0\}$, then (1) is equivalent to

$$\mathbb{P}\big(X^{-1}(0) \cap E \neq \emptyset\big) > 0.$$

The following are some known results about Question **(1)**. In the case when $E = [a, b]$, ($a, b \in \mathbb{R}^N$), necessary and sufficient conditions for **(1)** in terms of certain kind of capacity of F have been established for X being

- Brownian motion Lévy processes
- Some multiparameter Markov processes (Fitzsimmons and Salisbury, 1989)
- The Brownian sheet (Khoshnevisan and Shi, 1999)
- Additive Lévy processes (Khoshnevisan and X., 2002, 2003, 2009)
- Hyperbolic SPDEs (Dalang and Nualart, 2004)

In the special case when $F = \{0\}$, Khoshnevisan and Xiao (2002) for a large class of additive Lévy processes.

For general $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, a necessary and sufficient condition in terms of "thermal capacity" of $E \times F$ was established for Brownian motion B by Watson (1978).

The Hausdorff dimension $B(E) \cap F$ was determined by Khoshnevisan and Xiao (2015).

For Gaussian random fields and the solutions of some SPDEs, some necessary conditions and sufficient conditions for the hitting probability in (1) with $E = [a, b]$, ($a, b \in \mathbb{R}^N$) have been obtained by Dalang, Khoshnevisan and Nualart (2007, 2009), Biermé, Lacaux and X. (2009), X. (2009), Dalang and Sanz-Solé (2010), Hinojosa-Calleja and Sanz-Solé (2020, 2021).

In Section 7.1, we will work to extend and strengthen the existing results on the hitting probability in **(1)** for Gaussian random fields.

Question **(2)** is related to existence of self-intersections.

- When $F = \mathbb{R}^d$, then (2) gives existence of k-multiple points.
  - Lévy processes (Khoshnevisan and X., 2005): $F = \mathbb{R}^d$, general $E_1, \ldots, E_k$
  - The Brownian sheet: Dalang et al (2012), Dalang and Mueller (2015), Dalang, Lee, Mueller, and X. (2021): $F = \mathbb{R}^d$, $E_1, \ldots, E_k$ are intervals.
- No results for general F, $E_1, \ldots, E_k$.

In Section 7.1, we will provide some results on the intersection of independent Gaussian random fields, which is technically simpler than Question **(2)**.

7.1 Hitting probabilities of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{7.3}$$

where $X_1, \ldots, X_d$ are independent copies of a centered GF X_0.

Given $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, in order to provide necessary condition and sufficient condition for

$$\mathbb{P}\{X(E) \cap F \neq \emptyset\} > 0,$$

we recall some concepts on fractals.

7.1.1 Hausdorff dimension and Capacity

For any metric $\widetilde{\rho}$ on $\mathbb{R}^p$, any $\beta > 0$ and $E \subseteq \mathbb{R}^p$, the β-dimensional Hausdorff measure in the metric $\widetilde{\rho}$ of E is defined by

$$\mathcal{H}^{\beta}_{\widetilde{\rho}}(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^{\beta} : E \subseteq \bigcup_{n=1}^{\infty} B_{\widetilde{\rho}}(r_n),\, r_n \le \delta \right\},$$

where $B_{\widetilde{\rho}}(r)$ denotes an open ball of radius r in the metric space $(\mathbb{R}^p, \widetilde{\rho})$.

The corresponding Hausdorff dimension of E is defined by

$$\dim^{\widetilde{\rho}}_{\mathrm{H}} E = \inf\big\{\beta > 0 : \mathcal{H}^{\beta}_{\widetilde{\rho}}(E) = 0\big\}.$$

$\widetilde{\rho}$ will be omitted if it is the Euclidean metric.

The Bessel-Riesz type capacity of order α on the metric space $(\mathbb{R}^p, \widetilde{\rho})$ is defined by

$$\mathcal{C}^{\alpha}_{\widetilde{\rho}}(E) = \left[\inf_{\mu \in \mathcal{P}(E)} \int \int f_{\alpha}\big(\widetilde{\rho}(u, v)\big) \mu(du) \mu(dv) \right]^{-1},$$

where $\mathcal{P}(E)$ is the family of probability measures carried by E and the function $f_{\alpha} : (0, \infty) \to (0, \infty)$ is defined by

$$f_{\alpha}(r) = \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log\left(\frac{e}{r \wedge 1}\right) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \tag{7.4}$$

The dimension p and metric $\widetilde{\rho}$ can be chosen appropriately based on the hitting probability problem, as we will show below.

7.1.2 Hitting probabilities

We start by stating the following result which was motivated by Dalang, Khoshnevisan and Nualart (2007).

Theorem 7.1 [Biermé, Lacaux and Xiao, 2009] *If X is defined by (7.3) such that X_0 satisfies:*

$$\mathbb{E}\big[\big(X_0(s) - X_0(t)\big)^2\big] \asymp \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \quad \textit{for all} \;\; s, t \in I (= [\varepsilon, 1]^N), \tag{7.5}$$

where $0 < H_j \le 1$ $(1 \le j \le d)$ *are constants, and* $\exists\, c_{7,1} > 0$ *such that for all* $s, t \in I$,

$$\mathrm{Var}\left(X_0(t)\middle|X_0(s)\right) \ge c_{7,1} \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \tag{7.6}$$

Then $\forall$ *Borel set* $F \subset \mathbb{R}^d$,

$$c_{7,2}\,\mathcal{C}^{d-Q}(F) \le \mathbb{P}\Big\{X(I) \cap F \neq \emptyset\Big\} \le c_{7,3}\,\mathcal{H}^{d-Q}(F),$$

where $Q = \sum_{j=1}^{N} \frac{1}{H_j}$, $\mathcal{C}^{d-Q}$ *is* $(d-Q)$*-dimensional Riesz capacity and* $\mathcal{H}^{d-Q}$ *is* $(d-Q)$*-dimensional Hausdorff measure.*

It is an open problem if $\mathcal{H}^{d-Q}(F)$ in the above can be replaced by $\mathcal{C}^{d-Q}(F)$.

Recently, Dalang, Mueller and X. (2017) proved that, if $d = Q$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\big\{X(I) \cap \{x\} \neq \emptyset\big\} = \mathbb{P}\big\{\exists t \in I : X(t) = x\big\} = 0.$$

We will discuss this result in Section 7.2 below.

For any Borel set $F \subseteq \mathbb{R}^d$, consider the inverse image

$$X^{-1}(F) = \big\{t \in \mathbb{R}^N : X(t) \in F\big\}.$$

Theorem 7.2 [Biermé, Lacaux and Xiao, 2009] *Let* X *be as in Theorem 7.1 and let* $F \subseteq \mathbb{R}^d$ *be a Borel set such that* $\sum_{j=1}^{N} \frac{1}{H_j} > d - \dim_{\mathrm{H}} F$. *Then with positive probability,*

$$\begin{aligned} &\dim_{\mathrm{H}}\big(X^{-1}(F) \cap I\big) \\ &\qquad = \min_{1 \le k \le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\mathrm{H}} F) \right\} \\ &\qquad = \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\mathrm{H}} F), \\ &\qquad\qquad \textit{if} \quad \sum_{j=1}^{k-1} \frac{1}{H_j} \le d - \dim_{\mathrm{H}} F < \sum_{j=1}^{k} \frac{1}{H_j}. \end{aligned}$$

The following extension of Theorem 7.1 from Chen and Xiao (2012) is useful.

Theorem 7.3 *Assume that (7.5) and (7.6) hold. Then for all compact sets* $E \subseteq I$ *and* $F \subseteq \mathbb{R}^d$,

$$c_{7,4}\,\mathcal{C}^d_{\rho_1}(E \times F) \le \mathbb{P}\Big\{X(E) \cap F \neq \emptyset\Big\} \le c_{7,5}\,\mathcal{H}^d_{\rho_1}(E \times F),$$

where $\mathcal{C}^d_{\rho_1}$ *and* $\mathcal{H}^d_{\rho_1}$ *denote respectively the d-dimensional Riesz capacity and d-dimensional Hausdorff measure in the metric space* $(\mathbb{R}^{N+d}, \rho_1)$*, and where*

$$\rho_1\big((s,x),(t,y)\big) = \max\Big\{ \sum_{j=1}^{N} |s_j - t_j|^{H_j},\ \|x - y\| \Big\}.$$

Theorem 7.3 implies the following result on hitting probability of $X^{-1}(\{a\})$:
For every $a \in \mathbb{R}^d$ and Borel set $E \subseteq I$, $\exists\, c_{7,6} \geq 1$, such that

$$c_{7,6}^{-1}\, \mathcal{C}_\rho^d(E) \leq \mathbb{P}\big\{X^{-1}(\{a\}) \cap E \neq \varnothing\big\} \leq c_{7,6}\, \mathcal{H}_\rho^d(E).$$

In the above, $\mathcal{C}_\rho^d$ is the Bessel-Riesz capacity of order d in the metric ρ, and $\mathcal{H}_\rho^d(E)$ is the d-dimensional Hausdorff measure of E in the metric ρ defined as before by

$$\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}.$$

The proof of Theorem 7.3 makes use of the following two lemmas.

Lemma 7.4 [Biermé, Lacaux and Xiao, 2009] *Assume the conditions of Theorem 7.1 hold. For any constant $M > 0$, there exist positive constants c and δ_0 such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$,*

$$\mathbb{P}\bigg\{ \inf_{s \in B_\rho(t,r)\cap I} \big\|X(s) - x\big\| \leq r \bigg\} \leq c\, r^d. \tag{7.7}$$

In the above $B_\rho(t,r) = \{s \in \mathbb{R}^N : \rho(s,t) \leq r\}$ denotes the closed ball of radius r in the metric ρ in $\mathbb{R}^N$.

Lemma 7.5 [Biermé, Lacaux and Xiao, 2009] *There exists a positive and finite constant c such that for all $\varepsilon \in (0,1)$, $s, t \in I$ and $x, y \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^{2d}} \exp\bigg(-\frac{1}{2}(\xi,\, \eta)\, \big(\varepsilon I_{2d} + \mathrm{Cov}\big(X(s), X(t)\big)\big)\, (\xi,\, \eta)^T \bigg) \\ e^{-i(\langle \xi,\, x\rangle + \langle \eta,\, y\rangle)} d\xi d\eta \quad \leq \frac{c}{\rho_1\big((s,x),(t,y)\big)^d}.$$

In the above, I_{2d} denotes the identity matrix of order $2d$, $\mathrm{Cov}\big(X(s), X(t)\big)$ denotes the covariance matrix of the random vector $(X(s), X(t))$, and $(\xi,\eta)^T$ is the transpose of the row vector (ξ,η).

Proof of Theorem 7.3. The upper bound in (7.3) can be proved by a covering argument using Lemma 7.4. The lower bound in (7.3) can be proved by using Lemma 7.5 and a capacity argument. We omit the details. □

7.2 Intersections of independent Gaussian fields

Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian fields with values in $\mathbb{R}^d$ such that the associate random fields X_0^H and X_0^K satisfy (7.5) and (7.6) respectively on $I_1 \subseteq \mathbb{R}^{N_1}$ with $H = (H_1, \ldots, H_{N_1})$ and on $I_2 \subseteq \mathbb{R}^{N_2}$ with $K = (K_1, \ldots, K_{N_2})$.

Theorem 7.6 [Chen and Xiao (2012)] *There exists a constant $C \geq 1$ such that*

$$C^{-1}\, \mathcal{C}_{\rho_2}^d(E_1 \times E_2) \leq \mathbb{P}\big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\big\} \leq C\, \mathcal{H}_{\rho_2}^d(E_1 \times E_2),$$

where

$$\rho_2\big((s,t),(s',t')\big) = \sum_{i=1}^{N_1} |s_i - s_i'|^{H_i} + \sum_{j=1}^{N_2} |t_j - t_j'|^{K_j}.$$

When $E_1 = I_1$ and $E_2 = I_2$ are two intervals, Theorem 7.4 implies that

(i) If $d > \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}$, then

$$\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} = 0.$$

(ii) If $d < \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}$, then

$$\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} > 0.$$

- What happens in the critical case of l

$$d = \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}? \tag{7.8}$$

Theorem 7.7 [Chen and Xiao (2012)] *If X^H (or X^K) satisfies the conditions of Theorem 5.8, then, in the critical case (7.8), $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} = 0$.*

Proof By Theorem 5.8, the exact Hausdorff measure function for $X^H(I_1)$ is

$$\varphi(r) = r^{\sum_{j=1}^{N_1} \frac{1}{H_j}} \log\log \frac{1}{r}.$$

This implies that

$$\mathcal{H}_{d-\sum_{j=1}^{N_2} \frac{1}{K_j}}(X^H(I_1)) = 0 \quad \text{a.s.}$$

Therefore, the conclusion follows from Theorem 7.1. $\square$

7.3 Polarity of points (the critical case)

In Dalang, Mueller and Xiao (2017), the following assumptions are made.

Condition (A1) Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\, t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X_1(t)$, and $v(A,\cdot)$ and $v(B,\cdot)$ are independent whenever A and B are disjoint.

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s,t \in T$,

$$\begin{aligned} &\big\|v([a,b),s) - X_1(s) - v([a,b),t) + X_1(t)\big\|_{L^2} \\ &\leq C\Big(\sum_{j=1}^{N} a^{\gamma_j}|s_j - t_j| + b^{-1}\Big), \end{aligned} \tag{7.9}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\big\|v([0,a_0),s) - v([0,a_0),t)\big\|_{L^2} \leq C\sum_{j=1}^{N} |s_j - t_j|. \tag{7.10}$$

Recall that $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$, where $H_j = (\gamma_j + 1)^{-1}$.

We will need another condition.

Condition (A5) (a). There is a constant $c > 0$ such that $\|X_1(t)\|_{L^2} \geq c$ for all $t \in T$.

(b). For $I \subset T$ and $\varepsilon > 0$ small, let I^{ε} be the ε-neighborhood of I. For every $t \in I$, there is $t' \in \partial I^{(\varepsilon)}$ such that for all $x, \bar{x} \in I$ with $\rho(t,x) \leq 2\varepsilon$ and $\rho(t,\bar{x}) \leq 2\varepsilon$,

$$\left|\mathbb{E}((X_1(x) - X_1(\bar{x}))X_1(t'))\right| \leq C \sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j},$$

where $\delta_j \in (H_j, 1]$, $(j = 1, \ldots, N)$ are constants.

The following is the main result of Dalang, Mueller and Xiao (2017).

Theorem 7.8 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies Conditions (A1) and (A5). Assume that $Q = d$. Then for every $z \in \mathbb{R}^d$, $\mathbb{P}\{\exists t \in T : X(t) = z\} = 0$.*

Theorem 7.8 is proved by constructing an economic covering for the image $X\big(B_\varepsilon(t^0)\big)$ (where $\varepsilon > 0$ and $t^0 \in T$ are fixed) by using the method of Talagrand (1998). See also Xiao (1997), where the exact Hausdorff measure of the level set $L_z = \{t \in T : X(t) = z\}$ was determined.

The main ingredient for proving Theorem 7.8 is the following proposition, which was proved as Proposition 5.2 in Lecture 5.

Proposition 7.9 *Let Assumption (A1) hold. Then there are constants K_1 and δ_0 such that for every $0 < r_0 < \delta_0$ and $t^0 \in T$, we have*

$$\mathbb{P}\left\{\exists r \in [r_0^2, r_0]\colon \sup_{t:\rho(t,t^0)<r} |X(t) - X(t^0)| \leq K_1 \frac{r}{(\log\log \frac{1}{r})^{1/Q}}\right\}$$
$$\geq 1 - \exp\left[-\left(\log \frac{1}{r_0}\right)^{\frac{1}{2}}\right].$$

For $t^0 \in T$ and $\varepsilon > 0$, set

$$B_\varepsilon(t^0) = \{t \in T : \rho(t,t^0) \leq \varepsilon\},$$
$$B'_\varepsilon(t^0) = \{t \in T : \rho(t,t^0) \leq 2\varepsilon\}.$$

For proving Theorem 7.8, it is sufficient to prove the following

Proposition 7.10 *Assume that (A1) holds and $Q = d$. Fix $t^0 \in T$, and consider the following (random) subset of $\mathbb{R}^d$:*

$$M(\varepsilon, t^0) = X\big(B_\varepsilon(t^0)\big),$$

which is the image of $B_\varepsilon(t^0)$ under the mapping $t \mapsto X(t)$. Then for every $z \in \mathbb{R}^d$,

$$\mathbb{P}\{z \in M(\varepsilon, t^0)\} = \mathbb{P}\big\{\exists t \in B_\varepsilon(t^0) : X(t) = z\big\} = 0.$$

Now we work to prove Proposition 7.10.

Let $t' \in B_\varepsilon(t^0)$ be given by (A5) (b). We define two $\mathbb{R}^d$-valued Gaussian random fields

$$X^2(t) = \mathbb{E}(X(t)\big|X(t')), \qquad X^1(t) = X(t) - X^2(t).$$

Notice that the random fields X^1 and X^2 are independent. Further, X^1 is independent of the random vector $X(t')$. The following lemma shows that X^2 can be viewed as a perturbation part.

Lemma 7.11 *There is a finite constant C such that for $t, \bar{t} \in B'_\varepsilon(t^0)$,*

$$|X^2(t) - X^2(\bar{t})| \le C|X(t')| \sum_{j=1}^{N} |t_j - \bar{t}_j|^{\delta_j}.$$

For $p \ge 1$, consider the random set

$$R_p = \Big\{ x \in B'_\varepsilon(t^0) : \exists r \in [2^{-2p}, 2^{-p}) \text{ with}$$
$$\sup_{\bar{x}:\, \rho(\bar{x},x) \le r} |X(\bar{x}) - X(x)| \le \frac{K_1\, r}{(\log\log \frac{1}{r})^{\frac{1}{Q}}} \Big\},$$

and the event

$$\Omega_{p,1} = \Big\{ \lambda_N(R_p) \ge \lambda(B'_\varepsilon(t^0)) \Big(1 - \exp\big(-\sqrt{p}/4\big)\Big) \Big\}.$$

which can be described as the event "a large portion of $B'_\varepsilon(t^0)$ consists of points at which X has minimal oscillation." As in the proof of the upper bound in Theorem 5.6, we have

$$\mathbb{P}\big(\Omega^c_{p,1}\big) \le \frac{\mathbb{E}(\lambda_N(B'_\varepsilon(t^0) \setminus R_p))}{\lambda_N(B'_\varepsilon(t^0)) \exp\big(-\sqrt{p}/4\big)} \le \exp\left(-\frac{3}{4}\sqrt{p}\right). \tag{7.11}$$

This gives

$$\sum_{p=1}^{\infty} \mathbb{P}(\Omega^c_{p,1}) < +\infty. \tag{7.12}$$

Fix $\beta \in]0, \min(\min_{j=1,\ldots,N}(\delta_j H_j^{-1} - 1), 1)[$ (which is possible since $\delta_j > H_j$, $j = 1, \ldots, N$) and set

$$\Omega_{p,2} = \{|X(t')| \le 2^{\beta p}\}.$$

Then $\sum_{p \ge 1} \mathbb{P}(\Omega^c_{p,2}) < +\infty$.

By Lemma 7.11, we have that, on the event $\Omega_{p,2}$,

$$|X^2(x) - X^2(\bar{x})| \le C 2^{\beta p} \sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j} \le \tilde{C}\, 2^{\beta p} \sum_{j=1}^{N} r^{\delta_j H_j^{-1}}$$

for all for $x, \bar{x} \in B'_\varepsilon(t^0)$ that satisfy $\rho(x, \bar{x}) \le cr$.

Therefore, there is a constant $K_2 > K_1$ such that on the event $\Omega_{p,3} \stackrel{\text{def}}{=} \Omega_{p,1} \cap \Omega_{p,2}$, for each $x \in R_p$, there exists $r \in [2^{-2p}, 2^{-p}]$ such that

$$\sup_{\bar{x}:\, \rho(\bar{x},x) \le r} |X^1(\bar{x}) - X^1(x)| \le K_2 \frac{r}{(\log\log \frac{1}{r})^{1/Q}}. \tag{7.13}$$

An "anisotropic dyadic cubes" of order ℓ in $\mathbb{R}^N$ is of the form

$$\prod_{j=1}^{N} \Big[\frac{m_j}{2^{\ell H_j^{-1}}}, \frac{m_j + 1}{2^{\ell H_j^{-1}}}\Big],$$

where $m_j \in \mathbb{N}$. For $x \in \mathbb{R}^N$, let $C_\ell(x)$ denote the anisotropic dyadic cube of order ℓ that contains x.

The cube $C_\ell(x)$ is called "good" if

$$\sup_{\bar{x} \in C_\ell(x) \cap B_\varepsilon(t^0)} |X^1(y) - X^1(\bar{x})| \leq d_\ell, \tag{7.14}$$

where

$$d_\ell = K_2 \, \frac{2^{-\ell}}{(\log\log 2^\ell)^{1/Q}}$$

By (7.13), when $\Omega_{p,3}$ occurs, we can find a family $\mathcal{H}_{1,p}$ of non-overlapping good anisotropic dyadic cubes (they may have intersecting boundaries) of order $\ell \in [p, 2p]$ that covers R_p. This family only depends on the random field X^1.

Let $\mathcal{H}_{2,p}$ be the family of non-overlapping dyadic cubes of order $2p$ that meet $B_\varepsilon(t^0)$ but are not contained in any cube of $\mathcal{H}_{1,p}$. For p large enough, these cubes are contained in $B'_\varepsilon(t^0)$, hence in $B'_\varepsilon(t^0) \setminus R_p$.

Therefore, when $\Omega_{p,3}$ occurs, the number of cubes in $\mathcal{H}_{2,p}$ is at most N_p, where

$$N_p \, 2^{-2pQ} \leq \lambda_N(B'_\varepsilon(t^0)) \exp\big(-\sqrt{p}/4\big),$$

so

$$N_p \leq K \, 2^{2pQ} \exp\left(-\sqrt{p}/4\right), \tag{7.15}$$

where K does not depend on p.

Let $\Omega_{p,4}$ be the event "the inequality

$$\sup_{x, \bar{x} \in C} |X(x) - X(\bar{x})| \leq K_3 \, 2^{-2p} \sqrt{p} \tag{7.16}$$

holds for each dyadic cube C of order $2p$ of $\mathbb{R}^N$ that meets $B_\varepsilon(t^0)$."

We choose K_3 large enough so that $\sum_{p \geq 1} \mathbb{P}\big(\Omega^c_{p,4}\big) < +\infty$. This is possible by Lemma 3.3 in Lecture 3 [it is Lemma 2.1 from Talagrand (1995)].

Set $\mathcal{H}_p = \mathcal{H}_{1,p} \cup \mathcal{H}_{2,p}$. This family is well-defined for all $p \geq 1$, and it is a non-overlapping cover of $B_\varepsilon(t^0)$.

Set

$$\begin{aligned} r_A &= 4d_\ell = 4K_2 2^{-\ell}(\log \ell)^{-1/Q} && \text{if } A \in \mathcal{H}_{1,p} \text{ and } A \text{ is of order } \ell \in [p, 2p], \\ r_A &= K_3 2^{-2p} \sqrt{p} && \text{if } A \in \mathcal{H}_{2,p}. \end{aligned}$$

Let $f(r) = r^d \log\log \frac{1}{r}$. If $\Omega_{p,3} \cap \Omega_{p,4}$ occurs, then we can verify that for p large enough,

$$\sum_{A \in \mathcal{H}_p} f(r_A) \leq K \lambda_N(B_\varepsilon(t^0)). \tag{7.17}$$

For each $A \in \mathcal{H}_p$, we pick a distinguished point p_A in A (say the lower left corner). Let B_A be the Euclidean ball in $\mathbb{R}^d$ centered at $X(p_A)$ with radius r_A.

Let $\mathcal{F}_p$ be the family of balls $\{B_A,\ A \in \mathcal{H}_p\}$. For p large enough, on $\Omega_{p,3} \cap \Omega_{p,4}$, $\mathcal{F}_p$ covers $M(\varepsilon, t^0)$.

Since $f(r)/r^d \to 0$ as $r \to 0+$, it follows from (7.17) that $\lambda_d\big(M(\varepsilon, t^0)\big) = 0$ a.s. This and Fubini's theorem imply that for a.e. $z \in \mathbb{R}^d$, $\mathbb{P}\big(z \in M(\varepsilon, t^0)\big) = 0$.

To prove that for every $z \in \mathbb{R}^d$, $\mathbb{P}\big(z \in M(\varepsilon, t^0)\big) = 0$, we introduce the random field X^3 defined by

$$X^3(t) = \frac{1}{\alpha(t)}(z - X^1(t)), \quad \forall\, t \in \mathbb{R}^N,$$

where

$$\alpha(t) = \frac{\mathbb{E}[X_1(t)X_1(t')]}{\mathbb{E}[X_1(t')^2]}.$$

Notice that $\mathbb{E}[X(t)|X(t')] = \alpha(t)X(t')$.

It can be verified that $1/2 \le \alpha(t) \le 3/2$ for all $t \in B_\varepsilon(t^0)$ when ε is small enough. Moreover, the function $t \mapsto \alpha(t)$ is Hölder continuous by Condition (A5)(b).

For any $z \in \mathbb{R}^d$, by the decomposition

$$X(x) = X^1(x) + \alpha(x)X(t'),$$

we have

$$X(x) = z \qquad \Longleftrightarrow \qquad X^3(x) = X(t'). \tag{7.18}$$

Denote by $g_{X(t')}(w)$ the density function of $X(t')$. By the independence of $X^1(x)$ and $X(t')$, we have l

$$\begin{aligned} \mathbb{P}\big\{z \in M(\varepsilon, t^0)\big\} &= \mathbb{P}\big\{\exists x \in B_\varepsilon(t^0) : X^3(x) = X(t')\big\} \\ &= \int_{\mathbb{R}^d} dw\, g_{X(t')}(w)\mathbb{P}\{\exists x \in B_\varepsilon(t^0) : X^3(x) = w\}. \end{aligned} \tag{7.19}$$

It can be proved as on the previous page that $\lambda_d\big[X^3(B_\varepsilon(t^0))\big] = 0$ a.s. This implies that for a.e. $w \in \mathbb{R}^d$,

$$\mathbb{P}\{\exists x \in B_\varepsilon(t^0) : X^3(x) = w\} = 0.$$

Therefore, (7.19) yields $\mathbb{P}\big\{z \in M(\varepsilon, t^0)\big\} = 0$.

This proves Proposition 7.10 and thus Theorem 7.8.

7.4 Polarity of points for systems of linear stochastic heat and wave equations

Let $\hat{u} = \{\hat{u}(t,x),\, t \in \mathbb{R}_+,\, x \in \mathbb{R}^k\}$ be the mild solution of a linear system of d uncoupled heat equations:

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}_j(t,x) &= \Delta \hat{u}_j(t,x) + \dot{W}_j(t,x), \qquad j = 1, \dots, d, \\ u(0,x) &= 0, \qquad x \in \mathbb{R}^k. \end{cases} \tag{7.20}$$

Here, $\hat{u}(t,x) = (\hat{u}_1(t,x), \dots, \hat{u}_d(t,x))$ and Δ is the Laplacian in the spatial variables. The Gaussian noise $\dot{W}$ is white in time and has a spatially homogeneous covariance given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}_j(t,x)\dot{W}_j(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time white noise.

Theorem 7.12 *Suppose $(4+2k)/(2-\beta) = d$. Then d is the critical dimension for hitting points and points are polar for $\hat{u}$. That is, for all $z \in \mathbb{R}^{(4+2k)/(2-\beta)}$,*

$$\mathbb{P}\big\{\exists (t,x) \in\ (0,+\infty)\ \times \mathbb{R}^k : \hat{u}(t,x) = z\big\} = 0.$$

In particular, in the case when $\hat{W}$ is the space-time white noise and $d = 6$, all points are polar for $\hat{u}$.

Now let $\hat{v}$ be the solution of the stochastic wave equation in spatial dimension k driven by W with $\beta \geq 1$.

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat{v}_j(t,x) & = & \Delta \hat{v}_j(t,x) + \dot{W}_j(t,x), \qquad & j = 1,\ldots,d, \\ \hat{v}(0,x) & = & 0, \qquad \frac{\partial}{\partial t}\hat{v}(0,x) = 0, \qquad & x \in \mathbb{R}^k. \end{cases}$$

Theorem 7.13 *Suppose $k = 1 = \beta$ or $1 < \beta < k \wedge 2$, and $d = \frac{2(k+1)}{2-\beta}$. Then d is the critical dimension for hitting points and points are polar for $\hat{v}$, that is, for all $z \in \mathbb{R}^d$,*

$$\mathbb{P}\big\{\exists (t,x) \in\ (0,+\infty)\ \times \mathbb{R}^k : \hat{v}(t,x) = z\big\} = 0.$$

In particular, in the case when W is the space-time white noise and $d = 4$, all points are polar for $\hat{v}$.

8 Lecture 8. Multiple Points of Gaussian Random Fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \quad \text{ where } X_1, \ldots, X_d \text{ are i.i.d.}$$

Let $m \geq 2$ be fixed. We consider the question: when is

$$\mathbb{P}\Big\{\exists \text{ distinct } t^1, \ldots, t^m \text{ s.t. } X(t^1) = \cdots = X(t^m)\Big\} > 0?$$

Studies on self-intersection have a long history. For Gaussian random fields we mention two approaches:

- Potential-theoretical approach for the Brownian sheet: R. Dalang, D. Khoshnevisan, E. Nualart, D. Wu and X. (2012), Dalang and Mueller (2015).
- Covering argument for fractional Brownian motion: Talagrand (1998).

8.1 Non-existence of multiple points in the critical dimension

We apply an approach which is based on Talagrand (1998). Our setting is the same as in Dalang, Mueller and Xiao (2017).

Lemma 8.1 *Consider $b > a > 1$ and $r > 0$ small. Set*

$$A = \sum_{j=1}^{N} a^{H_j^{-1}-1} r^{H_j^{-1}} + b^{-1}.$$

There are constants A_0, K and c such that if $A \le A_0 r$ and

$$u \ge KA \log^{1/2}\Big(\frac{r}{A}\Big), \tag{8.1}$$

then for $S(t^0, r) = \{t \in T : \rho(t, t^0) \le r\}$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in S(t^0,r)} |X(t) - X(t^0) - (v([a,b],t) - v([a,b],t^0))| \ge u\Big\}\\ &\le \exp\Big(-\frac{u^2}{cA^2}\Big).\end{aligned}$$

In addition, we will make use of the following conditions.

Assumption (A5′) (a) $\|X_1(t)\|_{L^2} \ge c > 0$ for all $t \in T$ and

$$\mathbb{E}\big[(X_1(s) - X_1(t))^2\big] \ge K\rho(s,t)^2 \quad \text{for all} \quad s, t \in T.$$

(b) For $I \subset T$ and $\varepsilon > 0$ small, let I^ε be the ε-neighborhood of I. For every $t \in I$, there is $t' \in \partial I^{(\varepsilon)}$ such that for all $x, \bar{x} \in I$ with $\rho(t,x) \le 2\varepsilon$ and $\rho(t,\bar{x}) \le 2\varepsilon$,

$$\big|\mathbb{E}((X_1(x) - X_1(\bar{x}))X_1(t'))\big| \le C\sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j},$$

where $\delta_j \in (H_j, 1]$, $(j = 1, \dots, N)$ are constants.

Assumption (A6) For any m distinct points $t^1, \dots, t^m \in T$, $X_1(t^1), \dots, X_1(t^m)$ are linearly independent random variables, or equivalently, the Gaussian vector $(X_1(t^1), \dots, X_1(t^m))$ is non-degenerate.

This is equivalent to $\det\mathrm{Cov}(X_1(t^1), \dots, X_1(t^m)) > 0$, which holds if $\{X_1(t), t \in \mathbb{R}^N\}$ has a property of local nondeterminism: for all $k \le m$ and all $t, t^1, \dots, t^k \in T$,

$$\mathrm{Var}\big(X_1(t)\big|X_1(t^1), \dots, X_1(t^k)\big) \ge c\min\{\rho(t,t^j)^2 : 1 \le j \le k\}.$$

Hence (A6) is weaker than the property of strong local nondeterminism.

Here is the main theorem of this section.

Theorem 8.2 [Dalang, Lee, Mueller, and Xiao, 2021] *Let $T \subset \mathbb{R}^N$ (or, say $(0,\infty)^N$) be a compact interval such that (A1), (A5′) and (A6) hold. If $mQ \le (m-1)d$, then $\{X(t), t \in T\}$ has no m-multiple points almost surely.*

Remark 8.3 : (i). The proof for $mQ < (m-1)d$ is easy. The case when $mQ = (m-1)d$, which is called the critical (dimension) case, is more difficult.

(ii). If $mQ > (m-1)d$, then we can show that $\{X(t), t \in T\}$ has m-multiple points with positive probability.

(iii) Theorem 8.2 is applicable to the Brownian sheet, fractional Brownian sheets, solutions of systems of stochastic heat and wave equations.

In the following, we provide a sketch of the proof of Theorem 8.2.

Consider m distinct points $t^1, \ldots, t^m \in T$ such that

$$\rho(t^i, t^j) \geq \eta \quad \text{for} \quad i \neq j,$$

where $\eta > 0$. For $\varepsilon > 0$ small (say $\varepsilon < \eta/4$), let

$$B_\varepsilon^i = \bigg(\prod_{j=1}^N [t_j^i - \varepsilon^{1/H_j}, t_j^i + \varepsilon^{1/H_j}]\bigg) \cap T.$$

Consider the random set $M_\varepsilon := M_{t^1,\ldots,t^m;\varepsilon}$ defined by

$$\begin{aligned} M_\varepsilon = \Big\{ z \in \mathbb{R}^d : \exists\, (s^1, \ldots, s^m) \in B_\varepsilon^1 \times \cdots \times B_\varepsilon^m \\ \text{such that } z = X(s^1) = \cdots = X(s^m) \Big\}. \end{aligned}$$

We will prove that under the conditions of Theorem 8.2, $M_\varepsilon = \emptyset$ a.s. By applying the small ball probability estimate due to Talagrand (1993) [cf. Ledoux, 1994], we have

Lemma 8.4 *There exist constants K and $0 < \delta_0 < 1$ such that for all $(s^1, \ldots, s^m) \in B_{2\varepsilon}^1 \times \cdots \times B_{2\varepsilon}^m$, $0 < a < b$, and $0 < u < r < \delta_0$, we have*

$$\begin{aligned} &\mathbb{P}\Big(\sup_{1 \leq i \leq m} \sup_{x^i \in S(s^i, r)} |v([a,b), x^i) - v([a,b), s^i)| \leq u \Big) \\ &\geq \exp\Big(-K \frac{r^Q}{u^Q} \Big). \end{aligned}$$

Recall that $S(s, r) = \{x \in T : \rho(x, s) \leq r\}$.

The key component for proving Theorem 8.2 is the following:

Proposition 8.5 *Suppose (A1) holds for $B_{2\rho}^1, \ldots, B_{2\rho}^m$. Then there are constants K and $0 < \delta < 1$ such that for all $0 < r_0 < \delta$ and $(s^1, \ldots, s^m) \in B_{2\rho}^1 \times \cdots \times B_{2\rho}^m$, we have*

$$\begin{aligned} &\mathbb{P}\Big(\exists\, r \in [r_0^2, r_0],\ \sup_{1 \leq i \leq m} \sup_{x^i \in S(s^i, r)} |X(x^i) - X(s^i)| \leq K r \big(\log\log 1/r\big)^{-1/Q} \Big) \\ &\geq 1 - \exp\Big(-\big(\log 1/r_0\big)^{1/2} \Big). \end{aligned}$$

Proof Let $1 < a < b$, $r > 0$ small and

$$A = \sum_{j=1}^{N} a^{1/H_j - 1} r^{1/H_j} + b^{-1}.$$

Lemma 8.1 and (A1) imply that there are constants A_0, K and c such that for all $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$ if $A \le A_0 r$ and $u \ge K\, A(\log(r/A))^{1/2}$, then

$$\begin{aligned}&\mathbb{P}\Big(\sup_{1\le i\le m} \sup_{x^i \in S(s^i,r)} |(X(x^i) - X(s^i)) - (v([a,b),x^i) - v([a,b),s^i))| \ge u\Big)\\ &\le m \exp\Big(- \frac{u^2}{cA^2}\Big).\end{aligned}$$

Applying Lemma 8.4 to $\{v([a,b),x)\}$, we can prove the proposition in a similar way to the proof of Proposition 7.9 in Lecture 7. The details are omitted here. □

For any integer $p \ge 1$, let

$$\begin{aligned} R_p = \Big\{ &(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho} : \exists\, r \in [2^{-2p}, 2^{-p}] \text{ s. t.}\\ &\sup_{1\le i\le m} \sup_{x^i \in S(s^i,r)} |X(x^i) - X(s^i)| \le Kr\Big(\log\log\frac{1}{r}\Big)^{-1/Q}\Big\}.\end{aligned}$$

Proposition 8.5 can be re-stated as $\forall\, (s^1, \ldots, s^m) \in B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$,

$$\mathbb{P}\big\{(s^1, \ldots, s^m) \in R_p\big\} \ge 1 - \exp\big(-\sqrt{p}/2\big).$$

This and Fubini's theorem imply that with probability 1,

$$\lambda(R_p) \ge \lambda\big(B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}\big)\Big(1 - \exp\big(-\sqrt{p}/4\big)\Big) \tag{8.2}$$

for all p large enough, where λ denotes Lebesgue measure.

This means that for most of points $(s^1, \ldots, s^m) \in B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$, the oscillations of $X(x^i)$ in $S(s^i, r)$ $(i = 1, \ldots, m)$ are characterized by $r\big(\log\log\frac{1}{r}\big)^{-1/Q}$ along a sequence of $r_p \to 0$.

The points in $B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$ where the oscillation is large can be covered by much fewer balls of ρ-radius r.

The largest such oscillation can be bounded by the uniform modulus of continuity of X on T.

The effect of these points can be shown to be negligible and so we will focus on dealing with the points in R_p that satisfies (8.2).

Suppose that for each small $\varepsilon > 0$, (A5′) holds for the rectangles $B^1_{2\varepsilon}, \ldots, B^m_{2\varepsilon}$ and there are $(\hat{t}^1, \ldots, \hat{t}^m)$ on the boundary of $B^1_{3\varepsilon} \times \cdots \times B^m_{3\varepsilon}$ such that for every $i = 1, \ldots, m$ and all $x, y \in B^i_{2\varepsilon}$,

$$\big|\mathbb{E}\big[(X(x) - X(y))X(\hat{t}^i)\big]\big| \le C \sum_{j=1}^{N} |x_j - y_j|^{\delta_j}.$$

Let Σ_2 denote the σ-field generated by $X(\hat{t}^1), \ldots, X(\hat{t}^m)$. Define

$$X^2(x) = \mathbb{E}(X(x)|\Sigma_2), \quad X^1(x) = X(x) - X^2(x).$$

The processes X^1 and X^2 are independent.

Lemma 8.6 *There is a constant K depending on $\hat{t}^1, \dots, \hat{t}^m$ such that for all $i = 1, \dots, m$, for all $x, y \in B^i_{2\varepsilon}$,*

$$\left|X^2(x) - X^2(y)\right| \le K \sum_{j=1}^{N} |x_j - y_j|^{\delta_j} \max_{1\le i\le m} \left|X(\hat{t}^i)\right|.$$

This shows that the process X^1 is a small perturbation of X, provided $\max_{1\le i\le m} |X(\hat{t}^i)|$ is not too big. (We can control is easily using Condition (A5′) (a).)

Lemma 8.7 *Suppose (A6) is satisfied. There exists a constant K (depending on $t^1, \dots, t^m$) such that for all ε small, $a_2, \dots, a_m \in \mathbb{R}^d$, $r > 0$, and $(x^1, \dots, x^m) \in B^1_\varepsilon \times \cdots \times B^m_\varepsilon$,*

$$\mathbb{P}\Big\{ \sup_{2\le i\le m} |X^2(x^1) - X^2(x^i) - a_i| \le r \Big\} \le K r^{(m-1)d}.$$

This is proved by showing $X^2(x^1), \dots, X^2(x^m)$ are linearly dependent.

Next, we construct a random covering for M_ε.

Denote by $\mathscr{C}$ the family of "generalized dyadic cubes" of the form $C = I_{q,1} \times \cdots \times I_{q,m}$ of order q. We say that such a cube C is *good* if

$$\sup_{1\le i\le m} \sup_{x,y\in I_{q,i}} |X^1(x) - X^1(y)| \le d_q,$$

where $d_q = K 2^{-q} (\log\log 2^q)^{-1/Q}$.

For each $x \in R_p$, we can find a good dyadic cube C containing x of smallest order q, where $p \le q \le 2p$.

We obtain a family $\mathscr{G}^1_p$ of disjoint good dyadic cubes of order between p and $2p$ that meet R_p.

Let $\mathscr{G}^2_p$ be the family of dyadic cubes of order $2p$ that meet $B^1_\varepsilon \times \cdots \times B^m_\varepsilon$ but are not contained in any cube of $\mathscr{G}^1_p$. (These are the *bad* cubes.)

Let $\mathscr{G}_p = \mathscr{G}^1_p \cup \mathscr{G}^2_p$, which covers $B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$.

Note that for each $C \in \mathscr{C}$, the events $\{C \in \mathscr{G}^1_p\}$ and $\{C \in \mathscr{G}^2_p\}$ are in the σ-field $\Sigma_1 := \sigma(X^1(x) : x \in T)$.

For each $p \ge 1$, we construct a family $\mathscr{F}_p$ of balls in $\mathbb{R}^d$ (depending on ω).

For each $C \in \mathscr{C}$, we choose a distinguished point $x_C = (x^1_C, \dots, x^m_C)$ in $C \cap (B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon})$. Let the ball $B_{p,C}$ be defined as follows:

(i) If $C \in \mathscr{G}^1_p$, take $B_{p,C}$ as the ball of center $X^1(x^1_C)$ of radius $r_{p,C} = 4d_q$.

(ii) If $C \in \mathscr{G}^2_p$, take $B_{p,C}$ as the ball of center $X^1(x^1_C)$ of radius $r_{p,C} = K 2^{-2p} p^{1/2}$.

(iii) Otherwise, take $B_{p,C} = \varnothing$ and $r_{p,C} = 0$.

Note that for each $p \ge 1, C \in \mathscr{C}$, the random variable $r_{p,C}$ is Σ_1-measurable. Consider the event

$$\Omega_{p,C} = \Big\{ \omega \in \Omega : \sup_{2\le i\le m} |X(x^1_C, \omega) - X(x^i_C, \omega)| \le r_{p,C}(\omega) \Big\}.$$

Define $\mathscr{F}_p(\omega) = \{B_{p,C} : C \in \mathscr{G}_p(\omega), \omega \in \Omega_{p,C}\}$.

Claim 1: There is an event Ω^* of probability one such that for all p large enough and $\omega \in \Omega^* \cap \Omega_{p,C}$, the family $\mathscr{F}_p(\omega)$ covers M_ε.

This is proved by making use of Lemma 8.6.

Claim 2: $\mathbb{P}\big\{\Omega_{p,C}\,|\Sigma_1\big\} \le K\, r_{p,C}^{(m-1)d}$.

This is proved by making use of Lemma 8.7.

To finish the proof, we make use of an argument of geometric flavor. Let

$$\phi(r) = r^{mQ-(m-1)d}(\log\log(1/r))^m.$$

We consider the following quantity related to M_ε:

$$\phi\text{-}m\big(M_\varepsilon\big) = \liminf_{p\to\infty} \sum_{B_{p,C}\in\mathscr{F}_p} \phi(r_{p,C}).$$

Notice that, if $mQ-(m-1)d > 0$, then $\phi\text{-}m(M_\varepsilon)$ gives an upper bound for the ϕ-Hausdorff measure of M_ε. However, $\phi\text{-}m(M_\varepsilon)$ is well-defined even if $mQ-(m-1)d \le 0$.

By applying Fatou's lemma, Claims 1 and 2, we derive

$$\begin{aligned}
\mathbb{E}\big[\phi\text{-}m(M_\varepsilon)\big] &\le \liminf_{p\to\infty} \mathbb{E}\Big\{\sum_{B_{p,C}\in\mathscr{F}_p} \phi(r_{p,C})\Big\}\\
&= \liminf_{p\to\infty} \mathbb{E}\Big\{\sum_{C\in\mathscr{G}_p} \phi(r_{p,C})\mathbf{1}_{\Omega_{p,C}}\Big\}\\
&= \liminf_{p\to\infty} \mathbb{E}\Big\{\mathbb{E}\Big[\sum_{B_{p,C}\in\mathscr{G}_p} \phi(r_{p,C})\mathbf{1}_{\Omega_{p,C}}\big|\Sigma_1\Big]\Big\}\\
&\le K\varepsilon^{mQ}.
\end{aligned}$$

This implies that $\phi\text{-}m(M_\varepsilon) < \infty$ a.s.

However, if $mQ \le (m-1)d$, then $\phi(r) \to \infty$ as $r \to 0$. This implies that M_ε is empty and finishes the proof of Theorem 8.2.

8.2 Self-intersection local times and Hausdorff dimension of the set of multiple times

In this section, we consider the case when $mQ > (m-1)d$. We can show that $\{X(t), t \in T\}$ has m-multiple points with positive probability and determine the Hausdorff dimension of the set $\mathcal{L}_m$ of m-multiple times

$$\begin{aligned}
\mathcal{L}_m = \big\{(t^1,\ldots,t^m) \in T^m :\, & t^1,\ldots,t^m \text{ are distinct such that}\\
& X(t^1) = \ldots = X(t^m)\big\}.
\end{aligned} \tag{8.3}$$

The tool for studying this question is the self-intersection local time.

9 Lecture 9. Propagation of Singularities of the Linear Stochastic Wave Equation

We consider the linear stochastic wave equation

$$
\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0, \ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0. \end{cases} \tag{9.1}
$$

Here, $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$
\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.
$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise.

Dalang (1999) proved that the real-valued process solution of equation (9.1) is given by

$$
u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{9.2}
$$

where G is the fundamental solution of the wave equation and W is the martingale measure induced by the noise $\dot{W}$.

We only consider the case of $k = 1$. Hence $0 < \beta \leq 1$ and

$$
G(t,x) = \frac{1}{2} \mathbf{1}_{\{|x|<t\}}.
$$

The mild solution of (9.1) is

$$
u(t,x) = \frac{1}{2} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-y| \leq t-s\}}(s,y)\, W(ds\, dy) = \frac{1}{2} W(\Delta(t,x)), \tag{9.3}
$$

where $\Delta(t,x) = \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : 0 \leq s \leq t, \ |x-y| \leq t-s\}$, see Figure 1.

Consider a new coordinate system (τ, λ) obtained by rotating the (t,x)-coordinates by $-45°$. In other words,

$$
(\tau, \lambda) = \Big(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}} \Big) \quad \text{and} \quad (t,x) = \Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}} \Big).
$$

For $\tau \geq 0$, $\lambda \geq 0$, denote

$$
\tilde{u}(\tau, \lambda) = u\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}} \Big).
$$

We will study the simultaneous LIL and propagation of singularities for $\{\tilde{u}(\tau,\lambda), \tau \geq 0, \lambda \geq 0\}$.

9.1 The simultaneous law of the iterated logarithm and singularities

Recall that, if $B = \{B(t), t \geq 0\}$ is standard Brownian motion, then for every $t \geq 0$, the law of the iterated logarithm states:

$$
\limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log\log 1/h}} = 1, \qquad \text{a.s.}
$$

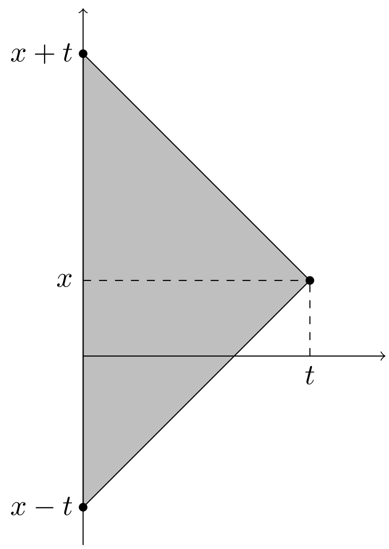

Figure 1: **1**

By Fubini's theorem, we have

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log\log 1/h}} = 1 \ \text{ for almost all } t \ge 0 \right) = 1. \tag{9.4}$$

In the above, "for almost all $t \ge 0$" can not be strengthened to "for all $t \ge 0$".

In fact, Orey and Taylor (1974) proved that the set

$$\mathscr{S} = \left\{ t \ge 0 : \limsup_{h\to 0+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log\log 1/h}} = \infty \right\}$$

is dense in $[0,\infty)$, even though it follows from (9.4) that the Lebesgue measure of $\mathscr{S}$ equals 0.

The points in $\mathscr{S}$ are called singularities of Brownian motion.

Some geometric properties of $\mathscr{S}$ and the λ-fast sets

$$\mathscr{F}(\lambda) = \left\{ t \ge 0 : \limsup_{h\to 0+} \frac{|B(t+h)-B(t)|}{\sqrt{2h\log 1/h}} \ge \lambda \right\}$$

were studied by Orey and Taylor (1974), Khoshnevisan, Peres and X. (2000), among others.

For a random field $X = \{X(t), t \in \mathbb{R}^N\}$, the set of its singularities may have interesting topological and geometric properties.

This was first studied by Walsh (1982) for the Brownian sheet $W = \{W(s,t), s \ge 0, t \ge 0\}$, which is a centered Gaussian random field with covariance function

$$\mathbb{E}\big(W(s_1,t_1)W(s_2,t_2)\big) = (s_1 \wedge s_2)(t_1 \wedge t_2).$$

For each fixed $s > 0$, $\{\frac{1}{\sqrt{s}}W(s,t), t \ge 0\}$ is standard Brownian motion. Hence the LIL states that for every $t \ge 0$,

$$\limsup_{h\to 0+} \frac{|W(s,t+h)-W(s,t)|}{\sqrt{2h\log\log 1/h}} = \sqrt{s}, \quad \text{ a.s.}$$

Zimmerman (1972) proved the following simultaneous LIL: For any $t \geq 0$ fixed,

$$\mathbb{P}\bigg(\limsup_{h \to 0+} \frac{|W(s,t+h) - W(s,t)|}{\sqrt{2h \log\log 1/h}} = \sqrt{s} \text{ for all } s \geq 0 \bigg) = 1.$$

However, by the result of Orey and Taylor (1974), for any $s > 0$ fixed, there is a random time τ such that

$$\limsup_{h \to 0+} \frac{|W(s,\tau+h) - W(s,\tau)|}{\sqrt{2h \log\log 1/h}} = \infty, \qquad \text{a.s.}$$

In this case, we say that (s, τ) is a singularity in the t-direction.

Similarly, we say that (s,t) is a *singular point* of W in the s-direction if

$$\limsup_{h \to 0+} \frac{|W(s+h,t) - W(s,t)|}{\sqrt{h \log\log(1/h)}} = \infty.$$

Based on the simultaneous LIL of Zimmerman (1972), Walsh (1982) proved the following surprising result.

Let $s_0 > 0$ be fixed. If τ is any positive and finite $\sigma(W(s_0,t) : t \geq 0)$-measurable random variable, then on an event of probability 1, we have

$$\begin{aligned} &\limsup_{h \to 0+} \frac{|W(s_0,\tau+h) - W(s_0,\tau)|}{\sqrt{h \log\log(1/h)}} = \infty \\ &\qquad \Longleftrightarrow \limsup_{h \to 0+} \frac{|W(s,\tau+h) - W(s,\tau)|}{\sqrt{h \log\log(1/h)}} = \infty \end{aligned}$$

for all $s > s_0$ simultaneously.

The existence of a positive and finite $\sigma(W(s_0,t) : t \geq 0)$-measurable random variable τ is guaranteed by Meyer's section theorem. Walsh's theorem shows that the singularities of the Brownian sheet W propagate in directions parallel to the coordinate axis.

Walsh (1986) studied the singularities of the linear SWE with $k = \beta = 1$ and their propagation. Carmona and Nualart (1988) extended the results of Walsh (1982, 1986) to one-dimensional nonlinear stochastic wave equations driven by the space-time white noise.

The method of Carmona and Nualart (1988) is based on the general theory of semimartingales and two-parameter strong martingales. They showed that, in the white noise case, their solution $X(t,x)$ has the following important properties:

(i). For any $x \in \mathbb{R}$, $\big\{X(\frac{h}{\sqrt{2}}, x + \frac{h}{\sqrt{2}}), h \geq 0\big\}$ is a continuous semimartingale.

(ii). The increments of $X(t,x)$ over a certain class of rectangles form a two-parameter strong martingale.

Carmona and Nualart (1988) proved the law of the iterated logarithm for a semimartingale by the LIL of Brownian motion and a time change.

They also proved that, for a class of two-parameter strong martingales, the law of the iterated logarithm in one variable holds simultaneously for all values of the other variable.

By applying these results and properties (i) and (ii), Carmona and Nualart proved the existence and propagation of singularities of the solution.

In the context of Gaussian random fields, Blath and Martin (2008) extended the result of Walsh (1982) to the semi-fractional Brownian sheets.

Due to the scaling property of the semi-fractional Brownian sheets, Blath and Martin (2008) was able to use the following large deviation result to prove their simultaneous LIL: If $\{Z(t), t \in T\}$ is a continuous centered Gaussian random field which is a.s. bounded, then

$$\lim_{\gamma\to\infty} \frac{1}{\gamma^2} \log \mathbb{P}\left(\sup_{t\in T} Z(t) > \gamma\right) = -\frac{1}{2\sup_{t\in T}\mathbb{E}(Z(t)^2)}. \tag{9.5}$$

However, this large deviation result is not enough for proving the following analogous LIL for $\{\tilde{u}(s,y), s \geq 0, y \geq 0\}$, where

$$\tilde{u}(s,y) = u\Big(\frac{s+y}{\sqrt{2}}, \frac{-s+y}{\sqrt{2}}\Big).$$

Theorem 9.1 [Lee and Xiao, (2021] *For any $y > 0$ fixed, we have*

$$\mathbb{P}\bigg(\limsup_{h\to 0+} \frac{|\tilde{u}(s,y+h) - \tilde{u}(s,y)|}{\sqrt{(s+y)h^{2-\beta}\log\log(1/h)}} = K_\beta \text{ for all } s \in [0,\infty)\bigg) = 1, \tag{9.6}$$

where K_β is

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)}\right)^{1/2}.$$

To prove the simultaneous LIL, we make use of more precise results on the tail probability for the supremum of Gaussian random fields based on the metric entropy obtained by Talagrand (1994).

Lemma 9.2 [Talagrand, 1994] *Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process and $\sigma_T^2 = \sup_{t\in T}\mathbb{E}[Z(t)^2]$. Let d_Z be the canonical metric defined by $d_Z(s,t) = \mathbb{E}[(Z(s)-Z(t))^2]^{1/2}$. Assume that for some constant $M > \sigma_T$, $\alpha > 0$ and $0 < \varepsilon_0 \leq \sigma_T$,*

$$N(T, d_Z, \varepsilon) \leq \left(\frac{M}{\varepsilon}\right)^{\alpha} \quad \text{for all } \varepsilon < \varepsilon_0,$$

Then for any $\gamma > \sigma_T^2[(1+\sqrt{\alpha})/\varepsilon_0]$, we have

$$\mathbb{P}\Big\{\sup_{t\in T} Z(t) \geq \gamma\Big\} \leq \Big(\frac{KM\gamma}{\sqrt{\alpha}\,\sigma_T^2}\Big)^{\alpha} \Phi\Big(\frac{\gamma}{\sigma_T}\Big), \tag{9.7}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_x^\infty \exp(-z^2/2)\,dz$ and K is a universal constant.

The upper bound in (9.7) is more precise than (9.5) if M/σ_T is not too large. However, the upper bound in (9.7) may not be useful when M/σ_T becomes very large (which will be the case in one part of the proof of Theorem 9.1).

To deal with the latter case, we use the following lemma, which is more efficient if the variance of $Z(t)$ attains its maximum at a unique point because the size of the set T_ρ can be very small.

Lemma 9.3 *Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process. For $\rho > 0$, set*

$$T_\rho = \{t \in T : \mathbb{E}[Z(t)^2] \geq \sigma_T^2 - \rho^2\}.$$

Assume that there exist constants $v \geq w \geq 1$ such that for all $\rho > 0$, and $0 < \varepsilon \leq \rho(1+\sqrt{v})/\sqrt{w}$, we have

$$N(T_\rho, d_Z, \varepsilon) \leq A\rho^w \varepsilon^{-v}.$$

Then for any $\gamma > 2\sigma_T\sqrt{w}$, we have

$$\mathbb{P}\Big\{\sup_{t\in T} Z(t) \geq \gamma\Big\} \leq \frac{Aw^{w/2}}{v^{v/2}} K^{v+w} \Big(\frac{\gamma}{\sigma_T^2}\Big)^{v-w} \Phi\Big(\frac{\gamma}{\sigma_T}\Big).$$

We will also need the following estimates on the variance of two types of increments.

Lemma 9.4 *For any $\tau, \lambda, h > 0$,*

$$\begin{aligned}&\mathbb{E}[(\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda))^2] \\ &= \frac{1}{2} K_\beta^2 \left[(\tau+\lambda)h^{2-\beta} + (3-\beta)^{-1}h^{3-\beta}\right],\end{aligned}$$

where K_β is the constant defined by

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)}\right)^{1/2}. \tag{9.8}$$

Lemma 9.5 *Fix $\lambda \geq 0$. Then, for any $0 \leq \tau \leq \tau'$ and $0 \leq h \leq h'$,*

$$\begin{aligned}&\mathbb{E}\big[(\tilde{u}(\tau', \lambda+h') - \tilde{u}(\tau', \lambda+h) - \tilde{u}(\tau, \lambda+h') + \tilde{u}(\tau, \lambda+h))^2\big] \\ &= \begin{cases} \frac{1}{2}K_\beta^2 (h'-h)^{2-\beta}\big[(\tau'-\tau) - \frac{1-\beta}{3-\beta}(h'-h)\big] & \text{if } h'-h \leq \tau'-\tau, \\ \frac{1}{2}K_\beta^2 (\tau'-\tau)^{2-\beta}\big[(h'-h) - \frac{1-\beta}{3-\beta}(\tau'-\tau)\big] & \text{if } h'-h > \tau'-\tau. \end{cases}\end{aligned}$$

9.2 Proof of the upper bound in Theorem 9.1

First we prove that for any fixed $\lambda > 0$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \leq K_\beta \text{ for all } \tau \in [0,\infty)\Big) = 1,$$

where K_β is the constant in Lemma 9.3.

It suffices to show that for any $0 \leq a < b < \infty$ and any $0 < \varepsilon < 1$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \leq (1+\varepsilon)K_\beta,\ \forall\, \tau \in [a,b]\Big) = 1. \tag{9.9}$$

Let $\delta = (c+\lambda)\varepsilon/2$. Since we can cover $[a,b]$ by finitely many intervals $[c,d]$ of length δ, we only need to show that (9.9) holds for all $\tau \in [c,d]$, where $[c,d] \subset [a,b]$ and $d = c+\delta$.

Choose a real number q such that $1 < q < (1+\varepsilon)^{1/(2-\beta)}$. For every integer $n \geq 1$, consider the event

$$A_n = \Big\{ \sup_{\tau \in [c,d]} \sup_{h \in [0, q^{-n}]} \big| \tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda) \big| > \gamma_n \Big\}, \tag{9.10}$$

where

$$\gamma_n = (1+\varepsilon) K_\beta \sqrt{(c+\lambda)(q^{-n-1})^{2-\beta} \log\log q^n}.$$

To estimate $\mathbb{P}(A_n)$, we will apply Lemma 9.2.

Define $T = [c,d] \times [0, q^{-n}]$ and $Z(\tau, h) = \tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)$ for $(\tau, h) \in T$. It follows from Lemma 9.3 that $\mathbb{E}\big[Z(\tau,h)^2\big]$ attains its unique maximum σ_T^2 at (d, q^{-n}), where

$$\sigma_T^2 = \frac{1}{2} K_\beta^2 \big[(d+\lambda) q^{-n(2-\beta)} + (3-\beta)^{-1} q^{-n(3-\beta)} \big].$$

For any $(\tau, h), (\tau', h') \in T$, without loss of generality, we may assume that $\tau \leq \tau'$. Then by Lemma 9.3 and 9.4, we have

$$\begin{aligned}
&d_Z((\tau,h),(\tau',h')) \\
&\leq \mathbb{E}\big[(Z(\tau,h) - Z(\tau,h'))^2\big]^{1/2} + \mathbb{E}\big[(Z(\tau',h') - Z(\tau,h'))^2\big]^{1/2} \\
&= \mathbb{E}\big[(\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda+h'))^2\big]^{1/2} \\
&\quad + \mathbb{E}\big[(\tilde{u}(\tau',\lambda+h') - \tilde{u}(\tau',\lambda) - \tilde{u}(\tau,\lambda+h') + \tilde{u}(\tau,\lambda))^2\big]^{1/2} \\
&\leq C(q^{-n(2-\beta)/2} |\tau - \tau'|^{1/2} + |h-h'|^{(2-\beta)/2}).
\end{aligned} \tag{9.11}$$

Next, in order to apply Lemma 9.2, we estimate $N(T_\rho, d_Z, \varepsilon)$, where

$$T_\rho = \{(\tau,h) \in T : \sigma_T^2 - \mathbb{E}[Z(\tau,h)^2] \leq \rho^2\}.$$

It can be shown that

$$T_\rho \subset [d - C_1 q^{n(2-\beta)} \rho^2, d] \times [q^{-n} - C_2 \rho^{2/(2-\beta)}, q^{-n}]$$

for some constants C_1 and C_2. This and (9.11) imply that

$$N(T_\rho, d_Z, \varepsilon) \leq C_0 (\rho/\varepsilon)^{2 + \frac{2}{2-\beta}}.$$

By Lemma 9.2 with $v = w = 2 + \frac{2}{2-\beta}$, we have

$$\mathbb{P}(A_n) \leq C \exp\Big(- \frac{\gamma_n^2}{2\sigma_T^2} \Big) = (n \log q)^{-p_n},$$

where

$$p_n = \frac{(1+\varepsilon)^2}{q^{2-\beta} \big[\frac{d+\lambda}{c+\lambda} + (3-\beta)^{-1} (c+\lambda)^{-1} q^{-n} \big]}$$

which is eventually bigger than 1. Hence $\sum_{n=1}^\infty \mathbb{P}(A_n) < \infty$. This is enough for proving (9.9).

9.3 Proof of the lower bound in Theorem 9.1

Next, we prove the corresponding lower bound: For any $\lambda > 0$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h)-\tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \geq K_\beta,\ \forall \tau \in [a,b]\Big) = 1, \tag{9.12}$$

where K_β is the constant in (9.8).

Similarly to the previous section, we only need to show that (9.12) holds for all $\tau \in [c,d]$, where $[c,d] \subset [a,b]$ and $d = c + \delta$.

The following are the main ingredients.

Lemma 9.6 *Let $\tau > 0$, $\lambda > 0$ and $q > 1$. Then for all $0 < \varepsilon < 1$,*

$$\mathbb{P}\Big(\frac{\tilde{u}(\tau,\lambda+q^{-n})-\tilde{u}(\tau,\lambda+q^{-n-1})}{\tilde{\sigma}[(\tau,\lambda+q^{-n}),(\tau,\lambda+q^{-n-1})]} \geq (1-\varepsilon)\sqrt{2\log\log q^n}\ \text{ i.o.}\Big) = 1,$$

where

$$\tilde{\sigma}[(\tau,\lambda),(\tau',\lambda')] = \mathbb{E}[(\tilde{u}(\tau,\lambda)-\tilde{u}(\tau',\lambda'))^2]^{1/2}.$$

This is proved by an extended Borel-Cantelli lemma.

For all $\tau \in [c,d]$ we write

$$\begin{aligned}\tilde{u}(\tau,\lambda+q^{-n}) - \tilde{u}(\tau,\lambda) &= \tilde{u}(d,\lambda+q^{-n}) - \tilde{u}(d,\lambda+q^{-n-1}) \\ &\quad + \tilde{u}(\tau,\lambda+q^{-n-1}) - \tilde{u}(\tau,\lambda) \\ &\quad - \Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]),\end{aligned}$$

where the last term is the the increment of $\tilde{u}$ over the rectangle $(\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]$.

The first difference in the right hand side of (9.3) is dealt by Lemma 9.5.

For the second difference, (9.9) says that for all $\tau \in [c,d]$ simultaneously,

$$\begin{aligned}&|\tilde{u}(\tau,\lambda+q^{-n-1})-\tilde{u}(\tau,\lambda)| \\ &\leq K_\beta\sqrt{(\tau+\lambda+q^{-n-1})(q^{-n-1})^{2-\beta}\log\log q^n}.\end{aligned}$$

eventually for all large n.

To derive a bound for the term $\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])$, we consider the event

$$A_n = \Big\{\sup_{\tau\in[c,d]} |\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])| > \gamma_n\Big\},$$

where

$$\gamma_n = K_\beta\,\phi_n(d)\sqrt{(q^{-n})^{2-\beta}\log\log q^n}$$

and

$$\begin{aligned}\phi_n(\tau) = (1-\varepsilon/4)\Big(\frac{q-1}{q}\Big)^{\frac{2-\beta}{2}}(d+\lambda)^{1/2} \\ - q^{-\frac{2-\beta}{2}}(\tau+\lambda+q^{-n-1})^{1/2} - (1-\varepsilon)(\tau+\lambda)^{1/2}.\end{aligned}$$

Consider n large enough such that $q^{-n} - q^{-n-1} \le d - c$. Then

$$\mathbb{P}(A_n) \le \mathbb{P}(A_n^1) + \mathbb{P}(A_n^2),$$

where

$$A_n^1 = \Big\{ \sup_{\tau \in [c, d-(q^{-n}-q^{-n-1})]} |\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \Big\},$$
$$A_n^2 = \Big\{ \sup_{\tau \in [d-(q^{-n}-q^{-n-1}), d]} |\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \Big\}.$$

By Lemma 9.3,

$$\mathbb{P}(A_n^1) \le C \exp\left(-\frac{\gamma_n^2}{2\sigma_T^2}\right) \le (n \log q)^{-p_n},$$

where

$$p_n = \frac{1}{d-c}\left(\frac{q}{q-1}\right)^{2-\beta} \phi_n(d)^2.$$

We can check that $\sum_{n=1}^\infty \mathbb{P}(A_n^1) < \infty$.

Since the size of $[d - (q^{-n} - q^{-n-1}), d]$ is small, we can apply Lemma 9.1 to see that for n large,

$$\mathbb{P}(A_n^2) \le C\phi_n(d)^2 (q^n \log n) \exp(-C'\phi_n(d)^2 q^n \log n)$$

which also yields $\sum_{n=1}^\infty \mathbb{P}(A_n^2) < \infty$.

Combing the three parts, we derive that for all $\tau \in [c, d]$ simultaneously,

$$\begin{aligned}
&\big|\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda)\big| \\
&\ge \big|\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})\big| \\
&\quad - \big|\tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda)\big| \\
&\quad - \big|\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])\big| \\
&\ge (1-\varepsilon) K_\beta \sqrt{(\tau + \lambda)(q^{-n})^{2-\beta} \log\log q^n},
\end{aligned}$$

where the last inequality holds infinitely often in n. This concludes the proof.

9.4 Propagation of singularities

For $s_0 > 0$, denote by $\mathscr{F}_{s_0}$ be the σ-field generated by $\{W\big(B \cap \Pi(s_0)\big) : B \in \mathscr{B}(\mathbb{R}^2)\}$ and the $\mathbb{P}$-null sets, where $\Pi(s_0) = \big\{(s, y) : 0 \le s < s_0/\sqrt{2}, y \in \mathbb{R}\big\}$.

The following theorem shows that the singularities of $u(t, x)$ propagate along the straight lines curves $s + y = c$ and $s - y = -c$.

Theorem 9.7 [Lee and Xiao, 2021] *Let $s_0 > 0$. The following statements hold.*

(i) There exists a positive and finite $\mathscr{F}_{s_0}$-measurable r.v. Λ such that

$$\limsup_{h \to 0+} \frac{|\tilde{u}(s_0, \Lambda + h) - \tilde{u}(s_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \quad a.s.$$

(ii) *For any positive and finite $\mathscr{F}_{s_0}$-measurable r.v. Λ, with probability 1,*

$$\limsup_{h\to 0+} \frac{|\tilde{u}(s_0, \Lambda+h) - \tilde{u}(s_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty$$
$$\iff \limsup_{h\to 0+} \frac{|\tilde{u}(s, \Lambda+h) - \tilde{u}(s, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty$$

for all $s > s_0$ simultaneously.

Part (i) of Theorem 9.7 is proved by using Meyer's section theorem [Dellacherie (1972, p.18)]: Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a complete probability space and S be a $\mathscr{B}(\mathbb{R}_+) \times \mathscr{G}$-measurable subset of $\mathbb{R}_+ \times \Omega$. Then there exists a $\mathscr{G}$-measurable random variable Λ with values in $(0, \infty]$ such that

(a) the graph of Λ, denoted by $[\Lambda] := \{(t,\omega) \in \mathbb{R}_+ \times \Omega : \Lambda(\omega) = t\}$, is contained in S;

(b) $\{\Lambda < \infty\}$ is equal to the projection $\pi(S)$ of S onto Ω.

For fixed $s_0 > 0$, we decompose $\tilde{u}$ into $\tilde{u}_1 + \tilde{u}_2$, where

$$\tilde{u}_i(\tau, \lambda) = u_i\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big), \quad i = 1, 2,$$

and

$$u_1(t,x) = \frac{1}{2} W\big(\Delta(t,x) \cap \Pi(s_0)\big),$$
$$u_2(t,x) = \frac{1}{2} W\big(\Delta(t,x) \cap \Pi(s_0)^c\big).$$

It can be proven that there exists a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that

$$\limsup_{h\to 0+} \frac{|\tilde{u}_1(s_0, \Lambda+h) - \tilde{u}_1(s_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty \quad \text{a.s.}$$

This is proved by taking

$$S = \Big\{(\lambda, \omega) : \limsup_{h\to 0+} \frac{|\tilde{u}_1(s_0, \lambda+h)(\omega) - \tilde{u}_1(s_0, \lambda)(\omega)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty\Big\}$$

and applying Meyer's section theorem.

Moreover, for $\lambda > 0$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}_2(\tau, \lambda+h) - \tilde{u}_2(\tau, \lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\tau - s_0 + \lambda)^{1/2} \text{ for all } \tau \geq s_0\Big)$$
$$= 1.$$

Combining the above ingredients yields Theorem 9.7.

10 Lecture 10. Local Properties of A Nonlinear Stochastic Heat Equation

Consider a parabolic SPDE of the following form:

$$\frac{\partial}{\partial t} u_t(x) = \tfrac{1}{2}\Delta_{\alpha/2} u_t(x) + \sigma(u_t(x))\, \dot{W}_t(x), \tag{10.1}$$

subject to $u_0(x) := U_0$ for all $x \in \mathbb{R}$, for some non-negative constant U_0.

In the above $\Delta_{\alpha/2} = -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian of index $\alpha/2$, $\sigma : \mathbb{R} \to \mathbb{R}$ is non random and Lipschitz continuous, and $\dot{W}$ denotes space-time white noise.

We assume $1 < \alpha \leq 2$. According to Dalang (1999), this is a sufficient and necessary condition for (10.1) to have a mild solution that is a random field.

Let $p_t(x)$ denote the fundamental solution to the fractional heat operator $(\partial/\partial t) - \frac{1}{2}\Delta_{\alpha/2}$. Then

$$\widehat{p}_t(\xi) = \exp\left(-t|\xi|^{\alpha}/2\right) \qquad (t \geq 0,\, \xi \in \mathbb{R}). \tag{10.2}$$

The Plancherel theorem implies that: For all $t > 0$,

$$\|p_t\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\widehat{p}_t\|^2_{L^2(\mathbb{R})} = \frac{\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}}. \tag{10.3}$$

Let us mention also the following variation: By the symmetry of the heat kernel, $\|p_t\|^2_{L^2(\mathbb{R})} = (p_t * p_t)(0) = p_{2t}(0)$. The Fourier inversion theorem shows that

$$p_t(0) = \sup_{x \in \mathbb{R}} p_t(x) = \frac{2^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}} \qquad (t > 0).$$

When $\sigma \equiv 1$ and $U_0 \equiv 0$, the mild solution to (10.1) is given by

$$v_t(x) = \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\, W(ds\, dy). \tag{10.4}$$

Then $\{v_t(x)\}$ is a continuous, centered Gaussian random field. Many of its properties (such as LIL, Chungs's LIL, variation, etc) can be established (e.g., when $\alpha = 2$, Swanson (2007), Tudor and X. (2007), Lai and Nualart (2009), etc).

We will relate some of the local properties of $\{v_t(x)\}$ to those of the solution to (10.1).

10.1 General case: moment estimates

In general (10.1) is interpreted as

$$u_t(x) = U_0 + \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))\, W(ds\, dy). \tag{10.5}$$

When $t > 0$ is fixed, Foondun, Khoshnevisan and Mahboubi (2015) have studied some properties of the function $x \mapsto u_t(x)$ by relating it to a fractional Brownian motion through the Gaussian process $\{v_t(x), x \in \mathbb{R}\}$.

Khoshnevisan, Swanson, Xiao, and Zhang (2014) have considered some properties of the function $t \mapsto u_t(x)$, when $x \in \mathbb{R}$ is fixed.

By combining the results of the two papers, one can derive some local properties of the sample function $(t, x) \mapsto u_t(x)$.

In the following, we will focus on the behavior of $t \mapsto u_t(x)$, when $x \in \mathbb{R}$ is fixed, and present some results in Khoshnevisan, Swanson, Xiao and Zhang (2014).

We will make use of the following Lemmas 10.1 and 10.2

Lemma 10.1 [Dalang, 1999; Foondun and Khoshnevisan, 2009] *For all $k \in [2\,, \infty)$ there exists a constant $A_{k,T}$ such that:*

$$\begin{aligned} &\mathbb{E}\left(|u_t(x)|^k\right) \le A_{k,T}; \qquad \text{and} \\ &\mathbb{E}\left(\left|u_t(x) - u_{t'}(x')\right|^k\right) \le A_{k,T}\left(|x - x'|^{(\alpha-1)k/2} + |t - t'|^{(\alpha-1)k/(2\alpha)}\right) \end{aligned} \tag{10.6}$$

uniformly for all $t, t' \in [0\,, T]$ and $x, x' \in \mathbb{R}$.

For every $t \ge 0$, let $\mathscr{F}_t^0$ denote the σ-algebra generated by $\int_{(0,t)\times\mathbb{R}} \varphi_s(y)\, W(ds\, dy)$ as φ ranges over all elements of $L^2(\mathbb{R}_+ \times \mathbb{R})$. We complete every such σ-algebra, and make the filtration $\{\mathscr{F}_t\}_{t\ge 0}$ right continuous.

The following BDG Inequality is Proposition 4.4 in Khoshnevisan (2014).

Lemma 10.2 *If $h \in L^2([0, t] \times \mathbb{R})$ for all $t > 0$ and $\Phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$. Then, for every real number $k \in [2\,, \infty)$, we have*

$$\left\| \int_{(0,t)\times\mathbb{R}} h_s(y)\Phi_s(y)\, W(ds\, dy) \right\|_k^2 \le 4k \int_0^t ds \int_{-\infty}^{\infty} dy\ [h_s(y)]^2 \|\Phi_s(y)\|_k^2. \tag{10.7}$$

10.2 Approximation theorems

Notation $\mathcal{D}_\varepsilon$: For any $\varepsilon > 0$ and random field $\{X_t(x)\}_{t\ge 0, x\in\mathbb{R}}$, denote

$$(\mathcal{D}_\varepsilon X)_t(x) := X_{t+\varepsilon}(x) - X_t(x) \qquad (t \ge 0,\, x \in \mathbb{R}).$$

The following theorem was proved in Khoshnevisan, Swanson, Xiao, and Zhang (2014).

Theorem 10.3 *For every $k \in [2\,, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,, 1)$, $x \in \mathbb{R}$, and $t \in [0\,, T]$,*

$$\mathbb{E}\Big(\left|(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))\,(\mathcal{D}_\varepsilon v)_t(x)\right|^k\Big) \le A_{k,T}\, \varepsilon^{\mathcal{G}_\alpha k}, \tag{10.8}$$

where

$$\mathcal{G}_\alpha := \frac{2(\alpha - 1)}{3\alpha - 1}. \tag{10.9}$$

Proof For $x \in \mathbb{R}$ fixed, we write the increment of $t \mapsto u_t(x)$ as

$$u_{t+\varepsilon}(x) - u_t(x) := \mathscr{J}_1 + \mathscr{J}_2, \tag{10.10}$$

where

$$\begin{aligned}
\mathscr{J}_1 &:= \int_{(0,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, \sigma(u_s(y))\, W(ds\, dy);\\
\mathscr{J}_2 &:= \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)\sigma(u_s(y))\, W(ds\, dy).
\end{aligned} \tag{10.11}$$

Define

$$\widetilde{\mathscr{J}_2} := \sigma(u_t(x)) \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)\, W(ds\, dy). \tag{10.12}$$

Lemma 10.4 *For every $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that for all $\varepsilon \in (0\,,1)$,*

$$\sup_{x\in\mathbb{R}} \sup_{t\in[0,T]} \mathbb{E}\left(\left| \mathscr{J}_2 - \widetilde{\mathscr{J}_2}\right|^k\right) \le A_{k,T}\, \varepsilon^{(\alpha-1)k/\alpha}. \tag{10.13}$$

To prove (10.13), we first consider

$$\mathscr{J}_2 - \mathscr{J}_2' = \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)[\sigma(u_s(y)) - \sigma(u_s(x))]\, W(ds\, dy).$$

By the BDG inequality and Lemma 10.1, we have

$$\begin{aligned}
&\left\| \mathscr{J}_2 - \mathscr{J}_2'\right\|_{L^k(\Omega)}^2\\
&\le 4k \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy\ \ [p_{t+\varepsilon-s}(y-x)]^2 \left\|\sigma(u_s(y)) - \sigma(u_s(x))\right\|_k^2\\
&\le A \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy\ \ [p_{t+\varepsilon-s}(y-x)]^2 \left\|u_s(y) - u_s(x)\right\|_k^2\\
&\le A \int_0^{\varepsilon} ds \int_{-\infty}^{\infty} dy\ \ [p_s(y)]^2 \left(|y|^{\alpha-1} \wedge 1\right)\\
&\le A\varepsilon^{2(\alpha-1)/\alpha}.
\end{aligned}$$

Next we consider $\mathscr{J}_2' - \widetilde{\mathscr{J}_2}$. The same argument gives:

$$\begin{aligned}
&\left\| \mathscr{J}_2' - \widetilde{\mathscr{J}_2}\right\|_{L^k(\Omega)}^2\\
&\le A \int_t^{t+\varepsilon} ds \int_{-\infty}^{\infty} dy\ \ [p_{t+\varepsilon-s}(y)]^2 \left\|u_s(x) - u_t(x)\right\|_{L^k(\Omega)}^2\\
&\le A \int_t^{t+\varepsilon} \left\|p_{t+\varepsilon-s}\right\|_{L^2(\mathbb{R})}^2 |s-t|^{(\alpha-1)/\alpha}\, ds\\
&\le A\varepsilon^{2(\alpha-1)/\alpha}.
\end{aligned}$$

Let $a = 2\alpha/(3\alpha - 1) \in (0\,,1)$, and write

$$\mathscr{I}_1 = \mathscr{I}_{1,a} + \mathscr{I}'_{1,a},$$

where

$$\mathscr{I}_{1,a} := \int_{(0,t-\varepsilon^a)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_s(y))\,W(ds\,dy),$$

$$\mathscr{I}'_{1,a} := \int_{(t-\varepsilon^a,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_s(y))\,W(ds\,dy).$$

By applying the BDG inequality and Lemma 10.1, one can verify that

$$\sup_{x\in\mathbb{R}}\sup_{t\in[0,T]} \mathrm{E}\Big(\,|\mathscr{I}_{1,a}|^k\,\Big) \le A\,\varepsilon^{\left(\mathcal{G}_\alpha + \frac{1}{3\alpha-1}\right)k},$$

which is a lot smaller than $A\,\varepsilon^{\mathcal{G}_\alpha k}$.

To estimate $\mathscr{I}'_{1,a}$, we use the same strategy and introduce

$$\mathscr{I}''_{1,a} := \int_{(t-\varepsilon^a,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_{t-\varepsilon^a}(y))\,W(ds\,dy),$$

$$\widetilde{\mathscr{I}}_{1,a} := \sigma(u_{t-\varepsilon^a}(x)) \int_{(t-\varepsilon^a,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,W(ds\,dy).$$

Note that the moments of $u_{t-\varepsilon^a}(x) - u_t(x)$ is negligible compared with the main term.

By the BDG inequality and Lemma 10.1, we can prove

Lemma 10.5 *For every $T > 0$ and $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,,1)$, $x \in \mathbb{R}$, and $t \in [0\,,T]$,*

$$\mathrm{E}\left(\left|\mathscr{I}_1 - \sigma(u_t(x)) \int_{(0,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,W(ds\,dy)\right|^k\right) \le A_{k,T}\,\varepsilon^{\mathcal{G}_\alpha k}.$$

Theorem 10.3 follows from Lemmas 10.4 and 10.5. □

From Theorem 10.3 and an interpolation argument, we can derive the following result, which is useful for deriving local properties of $t \mapsto u_t(x)$ from those of the Gaussian process $t \mapsto v_t(x)$.

Theorem 10.6 *For all $T > 0$, $M > 0$, and $q \in (0\,,\mathcal{G}_\alpha)$,*

$$\mathrm{E}\bigg(\sup_{t\in[0,T]}\sup_{\varepsilon\in[0,\eta]}\sup_{x\in[-M,M]} \left|(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))(\mathcal{D}_\varepsilon v)_t(x)\right|^k\bigg) = o\left(\eta^{kq}\right),$$

as $\eta \to 0^+$.

By a Borel-Cantelli argument, we obtain the following a.s. uniform approximation bound:

Corollary 10.7 *For all $T > 0$, $M > 0$, and $q \in (0\,, \mathcal{G}_\alpha)$,*

$$\sup_{t\in[0,T]} \sup_{\varepsilon\in[0,\eta]} \sup_{x\in[-M,M]} |(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))(\mathcal{D}_\varepsilon v)_t(x)| = o\left(\eta^q\right),$$

as $\eta \to 0^+$, almost surely.

Notice that $\mathcal{G}_\alpha > (\alpha-1)(2\alpha)$, we derive from Corollary 10.7 and local properties of the Gaussian process $t \mapsto v_t(x)$ the following results.

Theorem 10.8 [Law of the iterated logarithm] *Let $x \in \mathbb{R}$ be fixed. The following hold almost surely:*

1. *If $t > 0$, then*

$$\limsup_{\varepsilon\to 0^+} \frac{u_{t+\varepsilon}(x) - u_t(x)}{\varepsilon^{(\alpha-1)/(2\alpha)}\sqrt{2\log|\log\varepsilon|}} = \sigma(u_t(x))\sqrt{\frac{2^{1/\alpha}\Gamma(1/\alpha)}{(\alpha-1)\pi}}.$$

2. *If $t = 0$, then*

$$\limsup_{\varepsilon\to 0^+} \frac{u_{\varepsilon}(x) - U_0}{\varepsilon^{(\alpha-1)/(2\alpha)}\sqrt{2\log|\log\varepsilon|}} = \sigma(U_0)\sqrt{\frac{\Gamma(1/\alpha)}{(\alpha-1)\pi}}.$$

Next we consider the weighted variation of the solution.

For any $t > 0$ fixed and some integer $n > 1$, consider a partition $\{t_{j:n}\}_{j=0}^{k_n}$ of $[0\,, t]$ by letting

$$t_{j:n} := jt\varepsilon_n \quad (0 \le j < k_n := \lfloor \varepsilon_n^{-1} \rfloor),\ t_{k_n:n} := t,$$

with "mesh size" ε_n.

For a fixed $x \in \mathbb{R}$, we consider the following function

$$V_t^{(n,\varphi)}(x) := \sum_{j=0}^{k_n-1} \varphi\left(u_{t_{j:n}}(x)\right) \cdot \left|u_{t_{j+1:n}}(x) - u_{t_{j:n}}(x)\right|^{2\alpha/(\alpha-1)}.$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is a non random and Lipschitz continuous function. When $\varphi \equiv 1$, $V_t^{(n,\varphi)}(x)$ is the "β-variation" of the function $s \mapsto u_s(x)$, in $[0\,, t]$, where $\beta := 2\alpha/(\alpha-1)$.

Theorem 10.9 [KSXZ, 2018] *Choose and fix $x \in \mathbb{R}$, $t > 0$, and a non random and Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Then,*

$$\lim_{n\to\infty} V_t^{(n,\varphi)}(x) = \mathfrak{V}(\alpha) \int_0^t \varphi(u_s(x))\, |\sigma(u_s(x))|^{2\alpha/(\alpha-1)}\, ds$$

in $L^2(\Omega)$ as $n \to \infty$, where $\mathfrak{V}(\alpha)$ is an explicit constant depending on α. Moreover, if

$$\sum_{n=1}^{\infty} \varepsilon_n^{(\alpha-1)/(3\alpha-1)} < \infty,$$

then the preceding can be strengthened to almost-sure convergence.

10.3 Polarity of a.e. point

Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be a matrix function. We are dealing with solutions $u(t,x)$ to the system of d nonlinear stochastic heat equations

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x)) \dot{W}(t,x), \qquad t > 0,\ x \in \mathbb{R}, \tag{10.14}$$

where $\dot{W}(t,x) = (\dot{W}_1(t,x), \dots, \dot{W}_d(t,x))$ is a d-dimensional space-time white noise (cf. Khoshnevisan (2009)) defined on a probability space $(\Omega, \mathcal{F}, P)$, with i.i.d. components, subject to the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{10.15}$$

where $u_0 : \mathbb{R} \to \mathbb{R}^d$ is Borel. We associate to the white noise its natural filtration $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$, where $\mathcal{F}_t$ is the σ-field generated by the white noise on $[0,t] \times \mathbb{R}$ (and completed with P-null sets).

Building on the methods of Talagrand (1995, 1998), Dalang, Mueller, and Xiao (2021) proved that for a broad class of Gaussian random fields, points are not hit in the critical dimension.

For the linear heat equation, where $\sigma \equiv 1$ and the solution of (10.14) is a Gaussian random field, the extra step that allows to go from "almost all points are polar" to "all points are polar" involves taking the conditional expectation of the random field given its value at a specific point (see Lecture 7). In the Gaussian case, conditional expectations can be computed explicitly, but in the nonlinear SPDE where $\sigma \not\equiv 1$, this is no longer true and a new argument seems to be needed.

In this section, we extend the main result in Dalang, Mueller, and Xiao (2021) to the solution of the nonlinear heat equation of the form (10.1).

We use the same notation for a matrix $\sigma_0 \in \mathbb{R}^{d \times d} \cong \mathbb{R}^{d^2}$.

Assumption 10.1 (a) The function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is Lipschitz continuous with Lipschitz constant L: for all $v_1, v_2 \in \mathbb{R}^d$,

$$|\sigma(v_1) - \sigma(v_2)| \le L |v_1 - v_2|.$$

(b) There is a finite constant $\sigma_1 \in \mathbb{R}$ such that for all $v \in \mathbb{R}^d$,

$$|\sigma(v)| \le \sigma_1.$$

(c) The initial function u_0 is bounded: there is $K_0 \in \mathbb{R}_+$ such that, for all $x \in \mathbb{R}$,

$$|u_0(x)| \le K_0.$$

We note that (10.14) has a rigorous formulation in terms of the mild form, $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ is a jointly measurable and $(\mathcal{F}_t)$-adapted process such that, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$u(t,x) = \int_{-\infty}^{\infty} G(t, x-y) u_0(y) dy + \int_0^t \int_{-\infty}^{\infty} G(t-s, x-y) \sigma(u(s,y)) W(dy, ds),$$

where

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is the heat kernel on $\mathbb{R}$. Existence and uniqueness is proved in Chapter 3 of Walsh (1986) in the case $d = 1$, and this proof extends directly to $d \geq 1$ (see Section 2 of Dalang, Khoshnevisan, and Nualart (2009)). The random field $(u(t,x))$ has a continuous version on $]0,\infty[\times\mathbb{R}$, and if the initial condition u_0 is continuous (which we do not assume here), then this version of $(u(t,x))$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$, see Theorem 3.1 of Chen and Dalang (2014). We will work only with this continuous version.

The main result of this paper is the following.

Theorem 10.10 *Assume that $d \geq 6$. Almost surely, the range of $u = (u(t,x), (t,x) \in]0,\infty[\times\mathbb{R})$ has 6-dimensional Hausdorff-measure 0. In particular, if $d \geq 6$, then (Lebesgue) almost all points in $\mathbb{R}^d$ are polar for u.*

10.4 Local decomposition

In this section, our goal is to study the range of $(t,x) \mapsto u(t,x)$ when (t,x) belongs to a small rectangle with center $(t_0,x_0) \in R_0 := [1,2] \times [0,1]$, where t_0 and x_0 are fixed. Throughout most of the paper, we will be working on sub-rectangles of R_0.

For $\rho \in (0,\, 1/2]$, define

$$R_\rho = R_\rho(t_0,x_0) := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : |t - t_0| < \rho^4,\ |x - x_0| < \rho^2\}. \tag{10.16}$$

This rectangle has side-lengths that are compatible with the metric

$$d((t,x);(s,y)) = \Delta(t-s, x-y) := \max(|t-s|^{1/4}, |x-y|^{1/2}).$$

We define a first stopping time $\tau_{K,1}$ that will help with Hölder-continuity properties of the solution, then a stopping time $\tau_{K,2}$ that will deal with growth as $x \to \pm\infty$, and a third stopping time $\tau_{K,3}$ that will help with an associated Gaussian process.

First stopping time $\tau_{K,1}$

Fix $T_0 > 3$ and a large constant $K > 0$. From Theorem 3.1 of Chen and Dalang (2014), we know that $u(t,x)$ is locally $(1-\delta)/4$-Hölder continuous in t and $(1-\delta)/2$-Hölder continuous in x on $(0,\infty) \times R$. More precisely, for each $\delta \in (0,1)$, there is an almost surely finite positive random variable Z such that for all $s,t \in [1/2,\, T_0]$ and $x,y \in [-2,\, 2]$,

$$|u(t,x) - u(s,y)| \leq Z\Delta(t-s, x-y)^{1-\delta}. \tag{10.17}$$

Now we define the stopping time $\tau_{K,1}$ to be the first time $t \in [1/2,\, T_0]$ such that there exist $s < t$, $x,y \in [-2,2]$ with

$$|u(t,x) - u(s,y)| \geq K\Delta(t-s, x-y)^{1-\delta};$$

if there is no such time t, let $\tau_{K,1} = T_0$.

Now (10.17) shows that

$$\lim_{K\to\infty} \mathbb{P}\{\tau_{K,1} < T_0\} = 0. \tag{10.18}$$

Also note that $u(t \wedge \tau_{K,1}, x)$ satisfies

$$|u(t_1 \wedge \tau_{K,1}, x_1) - u(t_2 \wedge \tau_{K,1}, x_2)| \leq K\Delta\big(t_1 - t_2, x_1 - x_2\big)^{1-\delta}, \tag{10.19}$$

for $(t_i, x_i) \in [1/2,\, T_0] \times [-2,\, 2]$.

Modified solution $\tilde{u}$

We will modify the random field u using $\tau_{K,1}$. We define $\tilde{u}(t,x) = \tilde{u}_K(t,x)$ as the (continuous version on $(0,\infty[\times\mathbb{R}$ of the) solution of

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{u}(t,x) &= \frac{\partial^2}{\partial x^2}\tilde{u}(t,x) + \sigma(u(t\wedge\tau_{K,1},x))\dot{W}(t,x), \qquad t>0,\ x\in\mathbb{R},\\ \tilde{u}(0,x) &= u_0(x), \qquad x\in\mathbb{R}.\end{aligned}$$

Note that on the right-hand side of the equation for $\tilde{u}$, σ is evaluated at u, not at $\tilde{u}$. In terms of the mild form,

$$\begin{aligned}\tilde{u}(t,x) = &\int_{-\infty}^{\infty} G(t,x-y)u_0(y)dy\\ &+ \int_0^t\int_{-\infty}^{\infty} G(t-s,x-y)\sigma(u(s\wedge\tau_{K,1},y))W(dy,ds).\end{aligned} \tag{10.20}$$

Finally, note that on $\{\tau_{K,1} = T_0\}$, we have that $u(t,x) = \tilde{u}(t,x)$ for all $(t,x)\in[0,T_0]\times\mathbb{R}$. Thus,

$$\lim_{K\to\infty}\mathbb{P}\{u(t,x) = \tilde{u}_K(t,x) \text{ for all } (t,x)\in[0,T_0]\times\mathbb{R}\} = 1. \tag{10.21}$$

For the time being, we will work with $\tilde{u}$.

Second stopping time $\tau_{K,2}$

We also want to control the growth of our solution $\tilde{u}$ as $x\to\pm\infty$. Let $\tau_{K,2}$ be the first time $t\in[0,T_0]$ such that there exists $x\in\mathbb{R}$ with

$$|\tilde{u}(t,x)| \geq K(1+|x|).$$

If there is no such time t, let $\tau_{K,2} = T_0$.

Since we are assuming that σ and our initial function $u_0(x)$ are bounded (Assumption 10.1 (b) and (c)), it is a consequence of Lemma 10.12 below (taking $\phi(r,z) = \sigma(u(r\wedge\tau_{K,1},z))$ in (10.24) and $\phi_1 = \sigma_1$ in (10.25)) that

$$\lim_{K\to\infty}\mathbb{P}\{\tau_{K,2} < T_0\} = 0. \tag{10.22}$$

Third stopping time $\tau_{K,3}$

We also work with the (continuous version on $(0,\infty[\times\mathbb{R}$ of the) following linear system of stochastic heat equations with additive noise:

$$\begin{aligned}\frac{\partial}{\partial t}v(t,x) &= \frac{\partial^2}{\partial x^2}v(t,x) + \dot{W}(t,x), \qquad t>0,\ x\in\mathbb{R},\\ v(0,x) &= u_0(x), \qquad x\in\mathbb{R}.\end{aligned} \tag{10.23}$$

Now we define $\tau_{K,3}$ in the same way as $\tau_{K,2}$, but with respect to v rather than $\tilde{u}$:

$$\tau_{K,3} = \inf\{t\in[0,T_0] : \exists x\in\mathbb{R} \text{ with } |v(t,x)| \geq K(1+|x|)\}.$$

As with the stopping time $\tau_{K,2}$, since we are assuming that our initial function $u_0(x)$ is bounded, it is a consequence of Lemma 10.12 (taking $\phi(r,z)\equiv 1$ in (10.24) and $\phi_1 = 1$ in (10.25)) below that

$$\lim_{K\to\infty}\mathbb{P}\{\tau_{K,3} < T_0\} = 0.$$

We will need some lemmas.

Probability bounds for the modulus of continuity

In this section, we get the probability bound in Lemma 10.11 below. For this section, let

$$N^{(3)}(t,x) = N^{(3)}(t,x,\phi) = \int_0^t \int_{-\infty}^{\infty} G(t-r, x-z)\phi(r,z) W(dz,dr), \tag{10.24}$$

where $\phi(r,z)$ is a jointly measurable and $(\mathcal{F}_t)$-adapted $\mathbb{R}^{d\times d}$-valued process, and for some $\phi_1 \in \mathbb{R}_+$,

$$\sup_{r,z} |\phi(r,z)| \le \phi_1, \qquad \text{a.s.} \tag{10.25}$$

We will be using the jointly continuous version of $N^{(3)}$ (which exists by Propositions 4.3 & 4.4 of Chen and Dalang (2014)).

Lemma 10.11 *Fix $\lambda_0 > 0$. There exist constants C_0 and C_1 such that the following holds. For $\rho \in (0, 1]$ and $\lambda \ge \lambda_0$, for each rectangle $R \subset R_0 = [1,2] \times [0,1]$ of dimensions $\rho^4 \times \rho^2$, let $A_\lambda(R)$ be the event that for all $p^{(1)}$, $p^{(2)} \in R$,*

$$\left|N^{(3)}(p^{(1)}) - N^{(3)}(p^{(2)})\right| \le \lambda \Delta(p^{(1)} - p^{(2)}) \log_+ \left(1/\Delta(p^{(1)} - p^{(2)})\right),$$

where for $\gamma > 0$, $\log_+(\gamma) := \max(1, \log_2(\gamma))$. Then

$$\mathbb{P}(A_\lambda(R)^c) \le C_0 \exp\left(-C_1 \lambda^2 \phi_1^{-2} \log_+^2(1/\rho)\right).$$

Let $N^{(3)}(t,x)$ be the jointly continuous version of the process defined in (10.24).

Lemma 10.12 *Fix $T > 0$. There exists an almost surely finite random variable Z such that with probability one, for all $x \in \mathbb{R}$,*

$$\sup_{t\le T} \left|N^{(3)}(t,x)\right| \le Z(|x|+1).$$

For each $A \in \mathcal{H}_q$, we pick a distinguished point $(s_A, y_A) \in A$ (say, the lower left corner). Let B_A be the Euclidean ball in $\mathbb{R}^d$ centered at $\tilde{u}(s_A, y_A)$ with radius r_A.

Lemma 10.13 *Let $\mathcal{F}_q$ be the family of balls (B_A, $A \in \mathcal{H}_q$). For q large enough, on $\Omega_q \cap \{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$, $\mathcal{F}_q$ covers the random set*

$$\tilde{M} = \{\tilde{u}(s,y) : (s,y) \in R_0\}.$$

Notice that $\tilde{M} \subset \mathbb{R}^d$ is the range of $\tilde{u}$ as (s,y) varies in $R_0 = [1,2] \times [0,1]$.

Proposition 10.14 *Let λ_6 denote 6-dimensional Hausdorff measure. Then $\lambda_6(\tilde{M}) = 0$ a.s.*

Proof of Theorem 10.10. We first prove that $\lambda_6(M) = 0$, where

$$M = \{u(s,y) : (s,y) \in R_0\}.$$

On the event $\{\tau_{K,1} = T_0\}$, u and $\tilde{u}$ coincide on $[0,T_0] \times \mathbb{R}$, so $M = \tilde{M}$, where $\tilde{M}$ is defined in Lemma 10.13, and therefore, by Proposition 10.14,

$$\lambda_6(M) = 0 \qquad \text{a.s. on } \{\tau_{K,1} = T_0\}.$$

Since $\lim_{K\uparrow\infty} \mathbb{P}\{\tau_{K,1} = T_0\} = 1$, we conclude that $\lambda_6(M) = 0$ a.s.

Let $u(]0,\infty[\times\mathbb{R})$ denote the random set $\{u(s,y) : (s,y) \in (0,\infty) \times \mathbb{R}\}$. Since in the entire paper, the rectangle R_0 could have been replaced by any other compact rectangle in $(0,\infty) \times \mathbb{R}$, we deduce that $\lambda_6(u((0,\infty) \times \mathbb{R})) = 0$. Therefore, for $d \geq 6$, $\lambda_d(u((0,\infty) \times \mathbb{R})) = 0$, where λ_d denotes Lebesgue-measure on $\mathbb{R}^d$. By Fubini's theorem,

$$0 = \mathbb{E}\left[\int_{\mathbb{R}^d} 1_{u((0,\infty)\times\mathbb{R})}(z)\,\lambda_d(dz)\right] = \int_{\mathbb{R}^d} \mathbb{P}\{z \in u((0,\infty) \times \mathbb{R})\}\,\lambda_d(dz),$$

that is, for Lebesgue-almost all $z \in \mathbb{R}^d$, $\mathbb{P}\{z \in u((0,\infty) \times \mathbb{R})\} = 0$. This proves Theorem 10.10. $\square$

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Yimin Xiao. Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.

E-mail: `xiao@stt.msu.edu`

URL: `http://www.stt.msu.edu/~xiaoyimi`