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Ramanujan's Ranks,
Mock Theta Functions, and Beyond
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LECTURE 2
THE COMBINATORICS OF PARTITION CONGRUENCES

RAMANUJAN'S IDENTITY IMPLIES ATKIN AND SWINNERTON-DYER'S RESULT

RECALL

$$\begin{aligned} R(\zeta_5, q) &= A(q^5) + (3 - \zeta_5^2 - \zeta_5^3)\phi(q^5) \\ &+ qB(q^5) \\ &+ q^2(\zeta_5 + \zeta_5^4)C(q^5) \\ &+ q^3\left((1 + \zeta_5^2 + \zeta_5^3)D(q^5) + (1 + 2\zeta_5^2 + 2\zeta_5^3)\frac{\psi(q^5)}{q^5}\right) \end{aligned}$$

where

$$A(q) = \frac{1 - q - q^3 + q^9 + \cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$\phi(q) = -1 + \frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \cdots$$

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Next we define

$$r_a(d) = r_a(d,t) = \sum_{n=0}^{\infty} N(a,t,tn+d)q^n$$

and

$$\begin{aligned} r_{a,b}(d) = r_{a,b}(d,t) &= r_a(d) - r_b(d) \\ &= \sum_{n=0}^{\infty} \left(N(a,t,tn+d) - N(b,t,tn+d)\right) q^n \end{aligned}$$

Theorem (ATKIN AND SWINNERTON-DYER)

Let $t = 5$. Then

$$\begin{aligned} r_{1,2}(0) &= \frac{q}{(q^5;q^5)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+1}}, \\ r_{0,2}(0) + 2r_{1,2}(0) &= A(q), \\ r_{0,2}(1) &= B(q), \\ r_{1,2}(1) &= r_{0,2}(2) = 0, \\ r_{1,2}(2) &= C(q), \\ r_{0,2}(3) &= \frac{-q}{(q^5;q^5)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+2}}, \\ r_{0,1}(3) + r_{0,2}(3) &= D(q), \\ r_{0,2}(4) &= r_{1,2}(4) = 0 \end{aligned}$$

WATSON-WHIPPLE

$$-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(z;q)_{n+1}(z^{-1}q;q)_n}=\frac{z}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n}$$

$$\phi(q)=-1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}=\frac{q}{(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+1}}$$

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$$\frac{\psi(q)}{q}=\frac{1}{q}\left(-1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}\right)$$
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$$R(\zeta_5, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{4} N(r,5,n)\zeta_5^r q^n$$

$$= \sum_{n=0}^{\infty} \big((N(0,5,n) - N(1,5,n) + (\zeta_5^2 + \zeta_5^3)(N(2,5,n) - N(1,5,n)\big) \, q^n$$

Considering coefficients of q^{5n} in Ramanujan's identity gives

$$A(q) - (\zeta_5^2 + \zeta_5^3 + 3)\phi(q) = r_{0,1}(0) - (\zeta_5^2 + \zeta_5^3) r_{1,2}(0)$$

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$$r_{0,1}(0) = A(q) - 3\phi(q), \quad r_{1,2}(0) = \phi(q)$$
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RAMANUJAN's function

$$F(q) = \frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}$$

$$C(z,q) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

$\mathcal{P}$ denote the set of partitions

$\mathcal{D}$ the set of partitions into distinct parts

$|\pi|$ denote the sum of parts of partition π

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$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$$

$$|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|,$$
$$\omega(\vec{\pi}) = (-1)^{\#(\pi_1)},$$
$$\text{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).$$

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$$\begin{aligned} &\sum_{\vec{\pi} \in V} \omega(\vec{\pi}) z^{\text{crank}(\vec{\pi})} q^{|\vec{\pi}|} \\ &= \sum_{\pi_1 \in \mathcal{D}} (-1)^{\#(\pi_1)} q^{|\pi_1|} \sum_{\pi_2 \in \mathcal{P}} z^{\#(\pi_2)} q^{|\pi_2|} \sum_{\pi_3 \in \mathcal{P}} z^{-\#(\pi_2)} q^{|\pi_3|} \\ &= (q;q)_\infty \frac{1}{(zq;q)_\infty} \frac{1}{(z^{-1}q;q)_\infty} \\ &= C(z,q) \end{aligned}$$

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$$N_V(m,n) = \sum_{\substack{\vec{\pi}\in V \\ |\vec{\pi}|=n \\ \text{crank}(\vec{\pi})=m}} \omega(\vec{\pi})$$

so that

$$C(z,q) = \sum_{n=0}^{\infty} \left(\sum_{m} N_V(m,n) z^m \right) q^n$$

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$$N_V(k,t,n) = \sum_{m\equiv k \pmod{t}} N_V(m,n)$$

$$C(1,q)=\sum_{n=0}^{\infty}\left(\sum_{m}N_V(m,n)\right)q^n=\frac{1}{(q)_\infty}=\sum_{n=0}^{\infty}p(n)q^n$$

$$\sum_{m}N_V(m,n)=p(n)$$

Theorem (G.)

$$N_V(k,5,5n+4)=\frac{1}{5}p(5n+4),\quad 0\le k\le 4$$
$$N_V(k,7,7n+5)=\frac{1}{7}p(7n+5),\quad 0\le k\le 6$$
$$N_V(k,11,11n+6)=\frac{1}{11}p(11n+6),\quad 0\le k\le 10$$

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$$F(q) - \text{Identity in RLN}$$

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$$\text{Coeff of } q^{5n+4} \text{ in } C(\zeta_5, q) \text{ equals } 0$$

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EXAMPLE The 41 vector partitions of 4 are given in the table below. From the this table we have

$$\begin{aligned} N_V(0,5,4) &= \omega(\vec{\pi}_6) + \omega(\vec{\pi}_9) + \omega(\vec{\pi}_{12}) + \omega(\vec{\pi}_{13}) + \omega(\vec{\pi}_{24}) \\ &\qquad + \omega(\vec{\pi}_{26}) + \omega(\vec{\pi}_{33}) + \omega(\vec{\pi}_{40}) + \omega(\vec{\pi}_{41}) \\ &= 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 + 1 \\ &= 1. \end{aligned}$$

Similarly

$$N_V(0,5,4) = N_V(1,5,4) = \cdots = N_V(4,5,4) = 1 = \frac{p(4)}{5},$$

	Weight	Crank		Weight	Crank
$\vec{\pi}_1 = (\phi, \phi, 4)$	$+1$	-1	$\vec{\pi}_{22} = (1, \phi, 2+1)$	-1	-2
$\vec{\pi}_2 = (\phi, \phi, 3+1)$	$+1$	-2	$\vec{\pi}_{23} = (1, \phi, 1+1+1)$	-1	-3
$\vec{\pi}_3 = (\phi, \phi, 2+2)$	$+1$	-2	$\vec{\pi}_{24} = (1, 1, 2)$	-1	0
$\vec{\pi}_4 = (\phi, \phi, 2+1+1)$	$+1$	-3	$\vec{\pi}_{25} = (1, 1, 1+1)$	-1	-1
$\vec{\pi}_5 = (\phi, \phi, 1+1+1+1)$	$+1$	-4	$\vec{\pi}_{26} = (1, 2, 1)$	-1	0
$\vec{\pi}_6 = (\phi, 1, 3)$	$+1$	0	$\vec{\pi}_{27} = (1, 1+1, 1)$	-1	1
$\vec{\pi}_7 = (\phi, 1, 2+1)$	$+1$	-1	$\vec{\pi}_{28} = (1, 3, \phi)$	-1	1
$\vec{\pi}_8 = (\phi, 1, 1+1+1)$	$+1$	-2	$\vec{\pi}_{29} = (1, 2+1, \phi)$	-1	2
$\vec{\pi}_9 = (\phi, 2, 2)$	$+1$	0	$\vec{\pi}_{30} = (1, 1+1+1, \phi)$	-1	3
$\vec{\pi}_{10} = (\phi, 2, 1+1)$	$+1$	-1	$\vec{\pi}_{31} = (2, \phi, 2)$	-1	-1
$\vec{\pi}_{11} = (\phi, 1+1, 2)$	$+1$	1	$\vec{\pi}_{32} = (2, \phi, 1+1)$	-1	-2
$\vec{\pi}_{12} = (\phi, 1+1, 1+1)$	$+1$	0	$\vec{\pi}_{33} = (2, 1, 1)$	-1	0
$\vec{\pi}_{13} = (\phi, 3, 1)$	$+1$	0	$\vec{\pi}_{34} = (2, 2, \phi)$	-1	1
$\vec{\pi}_{14} = (\phi, 2+1, 1)$	$+1$	1	$\vec{\pi}_{35} = (2, 1+1, \phi)$	-1	2
$\vec{\pi}_{15} = (\phi, 1+1+1, 1)$	$+1$	2	$\vec{\pi}_{36} = (3, \phi, 1)$	-1	-1
$\vec{\pi}_{16} = (\phi, 4, \phi)$	$+1$	1	$\vec{\pi}_{37} = (2+1, \phi, 1)$	$+1$	-1
$\vec{\pi}_{17} = (\phi, 3+1, \phi)$	$+1$	2	$\vec{\pi}_{38} = (3, 1, \phi)$	-1	1
$\vec{\pi}_{18} = (\phi, 2+2, \phi)$	$+1$	2	$\vec{\pi}_{39} = (2+1, 1, \phi)$	$+1$	1
$\vec{\pi}_{19} = (\phi, 2+1+1, \phi)$	$+1$	3	$\vec{\pi}_{40} = (4, \phi, \phi)$	-1	0
$\vec{\pi}_{20} = (\phi, 1+1+1+1, \phi)$	$+1$	4	$\vec{\pi}_{41} = (3+1, \phi, \phi)$	$+1$	0
$\vec{\pi}_{21} = (1, \phi, 3)$	-1	-1			

SOLUTION OF DYSON'S CRANK CONJECTURE

$$
\begin{aligned}
C(z,q) &= \sum_{n=0}^{\infty} \left(\sum_{m} N_V(m,n) z^m \right) q^n \\
&= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \\
&= 1 + (z^{-1} - 1 + z)q + (z^{-2} + z^2)q^2 \\
&\quad + (z^{-3} + 1 + z^3)q^3 + +(z^{-4} + z^{-2} + 1 + z^2 + z^4)q^4 + \cdots
\end{aligned}
$$

Theorem (CAUCHY'S q-BINOMIAL THEOREM)

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \zeta^n = \frac{(a\zeta;q)_\infty}{(\zeta;q)_\infty}$$

for $|q| < 1$, $|\zeta| < 1$.

Theorem (ANDREWS AND G.)

$$N_V(m,n) \geq 0, \qquad \textit{for all } n > 1$$

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for $|q| < 1$, $|\zeta| < 1$.

Theorem (ANDREWS AND G.)

$$N_V(m,n) \geq 0, \qquad \textit{for all } n > 1$$

Proof

$$C(z,q) = \frac{(q)_\infty}{(zq)_\infty(z^{-1}q)_\infty} = \frac{(1-q)(q^2;q)_\infty}{(zq)_\infty(z^{-1}q)_\infty}$$

$$= \frac{(1-q)}{(zq)_\infty}\sum_{n=0}^{\infty}\frac{(zq)_n}{(q)_n}(z^{-1}q)^n$$

(by Cauchy with $\zeta = z^{-1}q$, $a = zq$)

$$= \frac{(1-q)}{(zq)_\infty} + \sum_{n=1}^{\infty}\frac{z^{-n}q^n}{(q^2;q)_{n-1}(zq^{n+1};q)_\infty}$$

$$= (1-q)\sum_{n=0}^{\infty}\frac{z^nq^n}{(q)_n} + \sum_{n=1}^{\infty}\frac{z^{-n}q^n}{(q^2;q)_{n-1}(zq^{n+1};q)_\infty}$$

(again by Cauchy with $\zeta = zq$, $a = 0$)

$$= (1-q) + \sum_{n=1}^{\infty} \frac{z^n q^n}{(q^2;q)_{n-1}} + \sum_{n=1}^{\infty} \frac{z^{-n} q^n}{(q^2;q)_{n-1}(zq^{n+1};q)_\infty}$$

$$= 1 + (-1+z)q + \sum_{n=2}^{\infty} \frac{z^n q^n}{(q^2;q)_{n-1}} + \sum_{n=1}^{\infty} \frac{z^{-n} q^n}{(q^2;q)_{n-1}(zq^{n+1};q)_\infty}$$

□

THIS GIVES A CLUE TO THE PARTITION CRANK

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$$\frac{z^n q^n}{(q^2;q)_{n-1}} = \frac{z^n q^n}{(1-q^2)\cdots(1-q^n)}$$

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$$\frac{z^n q^n}{(q^2;q)_{n-1}} = \frac{z^n q^n}{(1-q^2)\cdots(1-q^n)}$$

is the generating function for partitions with no ones (assuming $n > 1$)and whose largest part is n.

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is the generating function for partitions with exactly n one and the coefficient of z^k is the number of partitions in which

$$k = (\#\text{ of parts} > n) - n$$

$$\frac{z^{-n}q^n}{(q^2;q)_{n-1}(zq^{n+1};q)_\infty}$$
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The CRANK of a partition as

$$\begin{cases}\text{largest part if partition has no ones} \\ (\#\text{ of parts larger number of ones}) - (\#\text{ of ones}), & \text{otherwise}\end{cases}$$

Let $M(m, n)$ denote the number of partitions of n with crank m.

Theorem (ANDREWS AND G.)

$$N_V(m, n) = M(m, n),$$

for $n > 1$.

$$\Downarrow$$

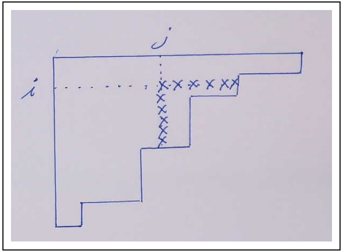
SOLUTION OF CRANK CONJECTURE

Theorem (ANDREWS and G.)

$$M(k,5,5n+4)=\frac{1}{5}p(5n+4),\quad 0\le k\le 4$$
$$M(k,7,7n+5)=\frac{1}{7}p(7n+5),\quad 0\le k\le 6$$
$$M(k,11,11n+6)=\frac{1}{11}p(11n+6),\quad 0\le k\le 10$$

CRANKS AND t-CORES [G., KIM and STANTON]

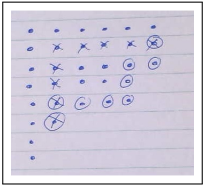
Let (i, j) be a node in the i-th row and j-th column in the diagram of a partition α.



The *hook* H_{ij}^{α} is the set of nodes to the right of (i, j) and below (i, j) in the diagram including (i, j) (marked "x"). A hook of length t is called a *t-hook*. The length of a hook in a partition is called a *hook number*.

<u>EXAMPLE</u> The partition

$$\alpha = 6+6+6+5+5+2+1+1$$



has a 9-hook $H^{\alpha}_{2,2}$. To each t-hook H^{α}_{ij} in a partition α there is a t-rim hook R^{α}_{ij} where

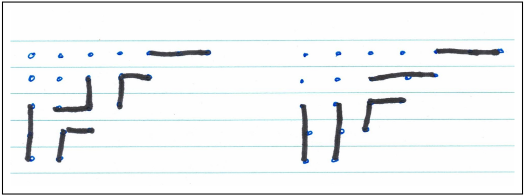
$$|H^{\alpha}_{ij}| = |R^{\alpha}_{ij}| = t$$

$R^{\alpha}_{2,2}$ is marked by $\cdot$. Removal of a rim-hook gives rise to a partition. A partition is a $\boxed{t\text{-core}}$ if it has no hook numbers that are multiples of t.

Theorem

Given any partition α and any integer $t \geq 1$, sucessive removal of t-rim hooks gives rise to a unique t-core partition.

EXAMPLE The 3-core of the partition $\alpha = 7+5+4+3+2$ is $\widetilde{\alpha} = 4+2$.



Let $a_t(n)$ denote the number of partitions of n which are t-cores.

Theorem

Let $t > 1$. Then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n=0}^{\infty} a_t(n)q^n$$

Proof Sketch Let $t > 0$. We construct a bijection

$$\Phi : \mathcal{P} \longrightarrow \mathcal{P} \times \mathcal{P} \times \cdots \mathcal{P} \times \mathcal{P}_{t-\text{core}}$$

where $\mathcal{P}$ is the set of partitions and $\mathcal{P}_{t-\text{core}}$ is the set of t-cores, such that

$$\Phi(\pi) = (\pi_0, \pi_1, \ldots, \pi_{t-1}, \widetilde{\pi})$$

where $\widetilde{\pi}$ is the t-core of π and

$$|\pi| = \sum_{j=0}^{t-1} t|\pi_j| + |\widetilde{\pi}|$$

Given a partition π we label cell in i-row and j-column by $(j-i)$ (mod t) (for each (i,j)). The resulting diagram is called the t-residue diagram. We add an infinite row 0 and column 0 and label in the same way to form the extended t-residue diagram.

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$$t(r-1) \leq j - i < tr$$

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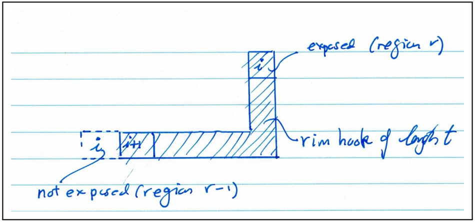
A cell is exposed if it is at the end of a row. A partition is a t-core if and only if any exposed cell labelled i in region r has exposed cells labelled i in each region $< r$.



We construct t bi-infinite words

$$W_0, W_1, \ldots, W_{t-1}$$

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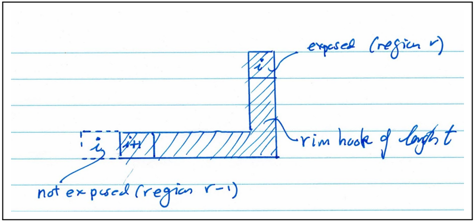

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$$j\text{-th element of } W_i = \begin{cases} \text{N} & i \text{ is not exposed in region } j \\ \text{E} & i \text{ is exposed in region } j \end{cases}$$

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We now describe the bijection. We initialize

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{t-1}, \widetilde{\pi})(-, -, \ldots, -, -)$$

For each i we do the following steps:

STEP 1. Find the right most E.

STEP 2. Find the right most N the left of E.

STEP 3. Remove the rim hook whose head is at E and whose tail is one cell to the right of N. Place a part of size $\frac{1}{t}$(length of rim hook removed) into π_i.

STEP 4. Go to STEP 1 and repeat untill t-core is obtained.

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NOTE: The operation in STEP 3 changes W_i from a string of the form

$$\cdots \mathrm{N\,E\,E} \cdots \mathrm{E\,E\,N} \cdots$$

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i.e. N is pushed to the right and the other W_j are unchanged by the removal of this rim hook.

Corollary

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q)_\infty}$$

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$$\sum_{n=0}^{\infty} a_t(n) q^n = \sum_{\substack{\vec{n}\in\mathbb{Z}^t \\ \vec{n}\cdot\vec{1}=0}} q^{\frac{t}{2}\|\vec{n}\|^2+\vec{b}\cdot\vec{n}}$$

where

$$\vec{n} = (n_0, n_1, \ldots, n_{t-1}), \quad \vec{1} = (1, 1, \ldots, 1), \quad \vec{b} = (0, 1, \ldots, t-1)$$

Proof Sketch Let $t > 1$. There is a bijection

$$\Psi : \mathcal{P}_{t-\text{core}} \longrightarrow \left\{ \vec{n} \in \mathbb{Z}^t : \vec{n}\cdot\vec{1} = 0 \right\}$$

such that

$$\Psi(\widetilde{\pi}) = (n_0, n_1, \ldots, n_{t-1})$$

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Theorem

Let $t = 5, 7, 11$, and $\delta_t = 4, 5, 6$ (resp.). Then

$$\sum_{n=0}^{\infty} p(tn+\delta_t)q^n = \frac{1}{(q)_\infty^t} \sum_{\substack{\vec{a}\in\mathbb{Z}^t \\ \vec{n}\cdot\vec{1}=1}} q^{Q(\vec{a})}$$

where

$$Q(\vec{a}) = Q(a_1, a_2, \ldots, a_t)$$
$$= a_1^2 + a_2^2 + \cdots a_t^2 - (a_1a_2 + a_2a_3 + \cdots + a_ta_1) - 1$$

Corollary

For $t = 5, 7, 11$

$$p(tn + \delta_t) \equiv 0 \pmod{t}$$

Proof Sketch for $t = 5$ For a t-core $\widetilde{\pi}$ we call

$$\vec{n} = (n_0, n_1, \ldots, n_{t-1}) = \Psi(\widetilde{\pi})$$

the n-vector of $\widetilde{\pi}$. The 5 partitions of 4 are 5-cores:

$\widetilde{\pi}$	n-vector
$1+1+1+1$	$(1,-1,0,0,0) = \vec{v}_1$
$2+1+1$	$(0,1,-1,0,0) = \vec{v}_2$
$3+1$	$(0,0,1,-1,0) = \vec{v}_3$
4	$(0,0,0,1,-1) = \vec{v}_4$
$2+2$	$(1,1,0,-1,-1) = \vec{v}_5$

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Let

$$\vec{b} = (0, 1, 2, 3, 4)$$
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The five vectors $\vec{v}_j$ for a non-planar pentagon with center $\vec{c}$

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Each $\vec{n} \in \mathbb{Z}^5$ satisifies $\vec{n}\cdot\vec{1} = 0$ and $\vec{n}\cdot\vec{b} \equiv 4 \pmod{5}$
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Under this correspondence

$$\frac{5}{2}\vec{n}\cdot\vec{n} + \vec{b}\cdot\vec{n} = 5Q(\vec{a}) + 4$$
$$= 5\left(a_1^2 + a_2^2 + \cdots a_5^2 - (a_1a_2 + a_2a_3 + \cdots + a_5a_1) - 1\right) + 4$$

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Theorem (G., STANTON and KIM)

(i) *There is a five-cycle with no fixed points that acts on 5-cores of* $5n+4$

(ii) *There is a five-cycle with no fixed points that acts on partitions of* $5n+4$

DEFINITION: Let $t > 1$ be given. For a partition λ we define the r-vector of λ by

$$\vec{r} = (r_0, r_1, \ldots, r_{t-1})$$

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For a statement A define $\chi(A) = 1/2$ is A is true, and $\chi(A) = -1/2$ if A is false.

Proposition

Let $t > 1$ be given and suppose λ is a t-core with r-vector $\vec{r}$ and n-vector $\vec{n}$. Then

(i) *For* $0 < k \leq t - 1$

$$r_k = \sum_{n_i > 0} \left(1/2n_i^2 + \chi(i \geq b)n_i\right) + \sum_{n_j < 0} \left(1/2n_j^2 - \chi(j < b)n_j\right)$$

(ii)

$$r_0 = \sum_i \binom{n_i + 1}{2}$$

(iii) $n_k = r_k - r_{k-1}$ *for* $0 \leq k \leq t - 1$

Corollary

λ is a t-core if and only if

$$r_0 = \sum_i r_i(r_i - r_{i+1})$$

A FIVE-CYCLE AND CRANK FOR 5-CORES OF $5n+4$

Suppose $m = 5n + 4$

$$\mathcal{P}_{5-\text{core}}(m) = \text{set of 5-cores of } m$$

$$N(m) = \left\{ \vec{n} \in \mathbb{Z}^5 \,:\, \vec{n} \cdot \vec{1} = 0,\ \tfrac{5}{2}\|\vec{n}\|^2 + \vec{b} \cdot \vec{n} = m \right\}$$

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$$A(m) = \left\{ \vec{a} \in \mathbb{Z}^5 \; : \; \vec{a} \cdot \vec{1} = 1, \; Q(\vec{a}) = 5(a_1^2 + \cdots a_5^2 - a_1 a_2 - \cdots - a_5 a_1) - 1 = m \right\}$$

A FIVE-CYCLE AND CRANK FOR 5-CORES OF $5n+4$

Suppose $m = 5n+4$

$$\mathcal{P}_{5-\text{core}}(m) = \text{set of 5-cores of } m$$

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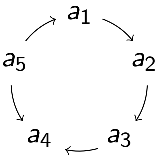
BIJECTIONS:

$$\mathcal{P}_{5-\text{core}}(m) \longrightarrow N(m) \longrightarrow A(m)$$



This five-cycle on $A(m)$ has NO fixed points

This induces a five-cycle on $N(m)$ with NO fixed points

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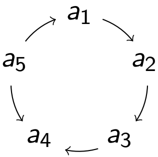

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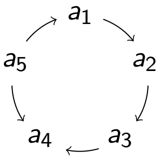

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CONSTRUCTING FIVE-CORE CRANK

For $\vec{a}$ in $A(m)$ define

$$\omega(\vec{a}) = a_1 + 2a_2 + 3a_3 + 4a_4$$

For the five cycle $\sigma = (1\,2\,3\,4\,5)$ we have

$$\begin{aligned}\omega(\sigma(\vec{a})) &= \omega(a_2\, a_3\, a_4\, a_5\, a_1) \\ &= a_2 + 2a_3 + 3a_4 + 4a_5 \\ &= a_1 + 2a_2 + 3a_3 + 4a_4 - (a_1 + a_2 + a_3 + a_4 + a_5) + 5a_5 \\ &\equiv \omega(\vec{a}) - 1 \pmod 5\end{aligned}$$

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$$\begin{aligned}\omega(\vec{a}) &= a_1 + 2a_2 + 3a_3 + 4a_4\\ &= 10n_0 + 9n_1 + 7n_2 + 4n_3 - 15n_4\\ &\equiv 4n_1 + 2n_2 + 4n_3 \pmod 5\\ &\equiv 4(r_1 - r_2) + 2(r_2 - r_3) + 4(r_3 - r_4) \pmod 5\\ &\equiv 4r_1 - 2r_2 + 2r_3 - 4r_4 \pmod 5\\ &\equiv (-1)(r_1 + 2r_2 + 3r_3 + 4r_4) \pmod 5\end{aligned}$$

A crank for 5-cores λ of $5n+4$ is
$\omega(\lambda) \equiv r_1 + 2r_2 + 3r_3 + 4r_4 \pmod 5$
where $\vec{r} = (r_0, r_1, r_2, r_3, r_4)$ is the r-vectorof λ

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CONSTRUCTING FIVE-CORE CRANK FOR PARTITIONS OF $5n+4$

Recall the bijection

$$\Phi : \mathcal{P} \longrightarrow \mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \times \mathcal{P}_{5-\text{core}}$$

where

$$\Phi(\pi) = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \widetilde{\pi})$$

where $\widetilde{\pi}$ is the 5-core of π and

$$|\pi| = 5\sum_{j=0}^{4}|\pi_j| + |\widetilde{\pi}|$$

Let $\mathcal{P}(5n+4)$ denote the set of partitions of $5n+4$. For $\ell \geq 0$ let

$$\mathcal{Q}(\ell) = \left\{ \vec{\pi} \in \mathcal{P}^5 \; : \; \sum_{j=0}^{4} |\pi_j| = \ell \right\}$$

The map Φ gives a bijection

$$\mathcal{P}(5n+4) \longrightarrow \bigcup_{\ell+m=n} \left(\mathcal{Q}(\ell) \times \mathcal{P}_{5-\text{core}}(5m+4) \right)$$

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The map Φ gives a bijection

$$\mathcal{P}(5n+4) \longrightarrow \bigcup_{\ell+m=n} \left(\mathcal{Q}(\ell) \times \mathcal{P}_{5-\text{core}}(5m+4) \right)$$

For a partition π of $5n+4$ we let

$$\vec{r} = (r_0, r_1, r_2, r_3, r_4)$$

be its r-vector, and let

$$\vec{r}' = (r_0', r_1', r_2', r_3', r_4')$$

be the r-vector of its 5-core $\widetilde{\pi}$. Let k be the number of 5-rim hooks removed from π to obtains its 5-core $\widetilde{\pi}$. Then

$$r_j' = r_j - k$$

for $0 \leq j \leq 4$.

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For the partition π we define

$$\omega(\pi) = \sum_{j=1}^{4} j r_j$$

Then

$$\omega(\pi) = \sum_{j=1}^{4} j r_j = \sum_{j=1}^{4} j(r_j' + k) = \sum_{j=1}^{4} j r_j' + 10k \equiv \omega(\widetilde{\pi}) \pmod 5$$

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Since $\omega(\widetilde{\pi}) \pmod 5$ divides the 5-cores of $5m+4$ into 5 equal classes for $m=0,1,\ldots,n$, it is clear that $\omega(\pi) \pmod 5$ divides the partitions of $5n+4$ into 5 equal classes.

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Since $\omega(\widetilde{\pi}) \pmod 5$ divides the 5-cores of $5m+4$ into 5 equal classes for $m = 0, 1, \ldots, n$, it is clear that $\omega(\pi) \pmod 5$ divides the partitions of $5n+4$ into 5 equal classes.

These FIVE CLASSES are

$$\bigcup_{\ell+m=n} \left(\mathcal{Q}(\ell) \times \mathcal{P}_{5-\text{core}}(5m+4, j)\right)$$

where $0 \leq j \leq 4$ and

$$\mathcal{P}_{5-\text{core}}(5m+4, j) = \left\{\widetilde{\pi} \in \mathcal{P}_{5-\text{core}}(5m+4) : \omega(\widetilde{\pi}) \equiv j \pmod{5}\right\}$$

A crank for partitions λ of $5n+4$ is
$\omega(\lambda) \equiv r_1 + 2r_2 + 3r_3 + 4r_4 \pmod{5}$
where $\vec{r} = (r_0, r_1, r_2, r_3, r_4)$ is the r-vector of λ

Theorem

For partitions

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

the function

$$\omega(\lambda) \equiv \sum_{i<j} \lambda_i \lambda_j + \sum_i i\lambda_i \pmod 5$$

divides the partitions of $5n+4$ *into 5 equal classes.*