NSF/CBMS Research Conference
Ramanujan's Ranks,
Mock Theta Functions, and Beyond
May 16-20, 2022
The University of Texas Rio Grande Valley

Frank Garvan
url: qseries.org/fgarvan

University of Florida

May 19, 2022

LECTURE 7
HOLOMORPHIC PROJECTION
(Includes notes of Jonathan Bradley-Thrush)

THE PETERSSON INNER PRODUCT

HOLOMORPHIC PROJECTION

HOLOMORPHIC PROJECTION AND HECKE-ROGERS TYPE SERIES

HOLOMORPHIC PROJECTION WITH RADICALS

HOLOMORPHIC PROJECTION WITHOUT RADICALS

THE PETERSSON INNER PRODUCT

For f, g in $M_k(\Gamma_0(N))$ where at least one of f, g is in $S_k(\Gamma_0(N))$ the $\boxed{\text{Petersson inner product of weight } k}$ is defined by

$$(f,g))k := \iint_{\mathcal{F}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is a fundamental domain of $\Gamma_0(N)$ in $\mathfrak{h}$.

The integral is well-defined due to the fact that the hyperbolic measure

$$\frac{dxdy}{y^2}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

THE PETERSSON INNER PRODUCT

For f, g in $M_k(\Gamma_0(N))$ where at least one of f, g is in $S_k(\Gamma_0(N))$ the Petersson inner product of weight k is defined by

$$(f,g))k := \iint_{\mathcal{F}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is a fundamental domain of $\Gamma_0(N)$ in $\mathfrak{h}$.

The integral is well-defined due to the fact that the hyperbolic measure

$$\frac{dxdy}{y^2}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

The next two theorems illustrate the importance of the Petersson inner product for classical modular forms.

THE PETERSSON INNER PRODUCT

For f, g in $M_k(\Gamma_0(N))$ where at least one of f, g is in $S_k(\Gamma_0(N))$ the Petersson inner product of weight k is defined by

$$(f, g))k := \iint_{\mathcal{F}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is a fundamental domain of $\Gamma_0(N)$ in $\mathfrak{h}$.

The integral is well-defined due to the fact that the hyperbolic measure

$$\frac{dxdy}{y^2}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$.
The next two theorems illustrate the importance of the Petersson inner product for classical modular forms.

Theorem

Let f, g in $M_k(\Gamma_0(N))$ where at least one of f, g is a cusp form. Then

$$(T_n f, g)_k = (f, T_n g)_k$$

for the Hecke operator T_n where $(n, N) = 1$.

Theorem

There exists a basus of the $\mathbb{C}$-vector space $S_k(\Gamma_0(N)$ whose elements are eigenforms of all the Hecke operators T_n with $(n, N) = 1$.

Theorem

Let f, g in $M_k(\Gamma_0(N))$ where at least one of f, g is a cusp form. Then

$$(T_n f, g)_k = (f, T_n g)_k$$

for the Hecke operator T_n where $(n, N) = 1$.

Theorem

There exists a basus of the $\mathbb{C}$-vector space $S_k(\Gamma_0(N)$ whose elements are eigenforms of all the Hecke operators T_n with $(n, N) = 1$.

For integral $k \geq 0$, **the weight k slash operator** is

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = (ad - bc)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$.

Let $\widetilde{M}_k(\Gamma_0(N))$ denote the space of C^∞ functions $f : \mathfrak{h} \longrightarrow \mathbb{C}$ that satisfy

1. $f|_k\gamma = f$ for every $\gamma \in \Gamma_0(N)$,
2. For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a $C > 0$ such that

$$(f|_k\gamma)(z) = O(q^C), \qquad \text{as } y \to \infty$$

For integral $k \geq 0$, the weight k slash operator is

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = (ad - bc)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$.

Let $\widetilde{M}_k(\Gamma_0(N))$ denote the space of C^∞ functions $f : \mathfrak{h} \longrightarrow \mathbb{C}$ that satisfy

1. $f|_k\gamma = f$ for every $\gamma \in \Gamma_0(N)$,
2. For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a $C > 0$ such that

$$(f|_k\gamma)(z) = O(q^C), \qquad \text{as } y \to \infty$$

Let $\boxed{N_k(\Gamma_0(N))}$ be the subspace of $\widetilde{M}_k(\Gamma_0(N))$ consisting of functions f satisfying: there is an $\varepsilon > 0$ such that

$$(f|_k\gamma)(z) = O(y^{-\varepsilon}), \qquad \text{as } y \to \infty$$

HOLOMORPHIC PROJECTION

For $k \geq 2$ we construct a map

$$\pi_{\text{hol}}^{(k)} : N_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(N))$$

such that

$$\pi_{\text{hol}}^{(k)}(f) = f, \qquad \text{for } f \in S_k(\Gamma_0(N))$$

Theorem (THE RIESZ REPRESENTATION THEOREM)

Let H be a Hilbert space with Hermitian inner-product $\langle \cdot, \cdot \rangle$. Then for every continuous linear functional $\phi : H \longrightarrow \mathbb{C}$ there exists a unique $g_\phi \in H$ such that

$$\phi(f) = \langle f, g_\phi \rangle$$

for all $f \in H$.

HOLOMORPHIC PROJECTION

For $k \geq 2$ we construct a map

$$\pi_{\text{hol}}^{(k)} : N_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(N))$$

such that

$$\pi_{\text{hol}}^{(k)}(f) = f, \qquad \text{for } f \in S_k(\Gamma_0(N))$$

Theorem (THE RIESZ REPRESENTATION THEOREM)

Let H be a Hilbert space with Hermitian inner-product $\langle \cdot, \cdot \rangle$. Then for every continuous linear functional $\phi : H \longrightarrow \mathbb{C}$ there exists a unique $g_\phi \in H$ such that

$$\phi(f) = \langle f, g_\phi \rangle$$

for all $f \in H$.

DEFINITION The weight $2k$ Poincaré series $P_{m,s}^{(2k)}(z)$ is defined by

$$P_{m,s}^{(2k)}(z) = y^s \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)/\Gamma_\infty} \frac{\exp\left(2\pi i m \left(\frac{az+b}{cz+d}\right)\right)}{(cz+d)^{2k}|cz+d|^{2s}},$$

for $k, m \geq 1$ and all complex s with $\Re(s) > 1 - k$. Here

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} : \ell \in \mathbb{Z} \right\}$$

$$\left| \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)/\Gamma_\infty} \frac{\exp\left(2\pi i m \left(\frac{az+b}{cz+d}\right)\right)}{(cz+d)^{2k}|cz+d|^{2s}} \right|$$

$$\leq \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)/\Gamma_\infty} \frac{\exp\left(-2\pi i m \Im(Az)\right)}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)/\Gamma_\infty} \frac{\exp\left(-2\pi i m y/|cz+d|^{2}\right)}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)/\Gamma_\infty \\ c \neq 0}} \frac{\exp\left(-2\pi imy/|cz+d|^2\right)}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + \sum_{\substack{c=1 \\ (c,d)=1}}^{\infty} \sum_{d=1}^{\infty} \frac{1}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + y^{-\Re(s)-k} E(z, \Re(s)+k) - 1$$

where $E(z,s)$ is the real analytic Eisenstein series

$$E(z,s) = \frac{1}{2} \sum \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}},$$

which converges for $\Re(s) > 1$.

$$\leq e^{-2\pi my} + \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)/\Gamma_\infty \\ c \neq 0}} \frac{\exp\left(-2\pi imy/|cz+d|^2\right)}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + \sum_{\substack{c=1 \\ (c,d)=1}}^{\infty} \sum_{d=1}^{\infty} \frac{1}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + y^{-\Re(s)-k} E(z, \Re(s)+k) - 1$$

where $E(z,s)$ is the real analytic Eisenstein series

$$E(z,s) = \frac{1}{2} \sum_{(c,d)=1} \sum \frac{y^s}{|cz+d|^{2s}},$$

which converges for $\Re(s) > 1$. Thus the Poincaré series $P_{m,s}^{(2k)}(z)$ converges for $\Re(s) > 1-k$ and

$$\leq e^{-2\pi my} + \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)/\Gamma_\infty \\ c \neq 0}} \frac{\exp\left(-2\pi imy/|cz+d|^2\right)}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + \sum_{\substack{c=1 \\ (c,d)=1}}^{\infty} \sum_{d=1}^{\infty} \frac{1}{|cz+d|^{2k+2\Re(s)}}$$

$$\leq e^{-2\pi my} + y^{-\Re(s)-k} E(z, \Re(s)+k) - 1$$

where $E(z,s)$ is the real analytic Eisenstein series

$$E(z,s) = \frac{1}{2} \sum \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}},$$

which converges for $\Re(s) > 1$. Thus the Poincaré series $P_{m,s}^{(2k)}(z)$ converges for $\Re(s) > 1-k$ and

$$E(z,s) = y^s + O(y^{1-s})$$

which implies

$$P_{m,s}^{(2k)}(z) = O(y^{1-\Re(s)-2k}), \qquad \text{as } y \to \infty.$$

Lemma

Let $k \geq 1$ and $\Re(s) > 1-k$. Then

$$P_{m,s}^{(2k)}(z) \in N_{2k}(\Gamma_0(N)) \quad \text{and} \quad P_{m,0}^{(2k)}(z) \in S_{2k}(\Gamma_0(N)).$$

$$E(z,s) = y^s + O(y^{1-s})$$

which implies

$$P_{m,s}^{(2k)}(z) = O(y^{1-\Re(s)-2k}), \qquad \text{as } y \to \infty.$$

Lemma

Let $k \geq 1$ and $\Re(s) > 1-k$. Then

$$P_{m,s}^{(2k)}(z) \in N_{2k}(\Gamma_0(N)) \quad \text{and} \quad P_{m,0}^{(2k)}(z) \in S_{2k}(\Gamma_0(N)).$$

Lemma (THE UNFOLDING LEMMA)

Let $k \geq 1$ and $f \in N_{2k}(\Gamma_0(N))$. Then for integers $m \geq 1$, and complex s with $\Re(s) > 1 - k$,

$$(f, P_{m,s}^{(2k)})_{2k} = \iint\limits_{\mathfrak{h}/\Gamma_\infty} f(z)\, e^{-2\pi i m\bar{z}}\, y^{2k+\bar{s}-2}\, dxdy.$$

Also if f has a Fourier expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m z}$$

then

$$(f, P_{m,s}^{(2k)})_{2k} = \int_0^\infty a_m(y)\, e^{-4m\pi y}\, y^{2k+\bar{s}-2}\, dxdy$$

for $m \geq 1$.

Lemma (THE UNFOLDING LEMMA)

Let $k \geq 1$ and $f \in N_{2k}(\Gamma_0(N))$. Then for integers $m \geq 1$, and complex s with $\Re(s) > 1 - k$,

$$(f, P_{m,s}^{(2k)})_{2k} = \iint\limits_{\mathfrak{h}/\Gamma_\infty} f(z)\, e^{-2\pi i m\bar{z}}\, y^{2k+\bar{s}-2}\, dxdy.$$

Also if f has a Fourier expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m z}$$

then

$$(f, P_{m,s}^{(2k)})_{2k} = \int_0^\infty a_m(y)\, e^{-4m\pi y}\, y^{2k+\bar{s}-2}\, dxdy$$

for $m \geq 1$.

Proof Sketch

$$(f, P_{m,s}^{(2k)})_{2k} = \sum_{A \in \Gamma_0(N)/\Gamma_\infty} \iint_{\mathfrak{h}/\Gamma_0(N)} \frac{f(z)\, e^{-2\pi i m A\overline{z}}}{(c\overline{z}+d)^{2k} |(c\overline{z}+d|^{2\overline{s}}} \, y^{2k+\overline{s}-2}\, dxdy.$$

$$= \sum_{A \in \Gamma_0(N)/\Gamma_\infty} \iint_{A(D)} f(w)\, e^{-2\pi i m \overline{w}}\, v^{2k+\overline{s}-2}\, dudv$$

(where D is a fundamental region of $\Gamma_0(N)$)

$$= \iint_{\mathfrak{h}/\Gamma_\infty} f(z)\, e^{-2\pi i m \overline{z}}\, y^{2k+\overline{s}-2}\, dxdy,$$

since

$$\mathfrak{h}/\Gamma_\infty = \bigcup_{A \in \Gamma_0(N)/\Gamma_\infty} A(D)$$

If

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi imz} = \sum_{m=-\infty}^{\infty} \left(a_m(y) e^{-2\pi my}\right) e^{2\pi imx}$$

then

$$a_m(y) e^{-2\pi my} = \int_0^1 f(z)\, e^{-2\pi imx}\, dx$$

for $m \geq 1$. The vertical strip $0 < \Re(z) < 1$ is a fundamental region for $\mathfrak{h}/\Gamma_\infty$ so that

If

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m z} = \sum_{m=-\infty}^{\infty} \left(a_m(y) e^{-2\pi m y}\right) e^{2\pi i m x}$$

then

$$a_m(y) e^{-2\pi m y} = \int_0^1 f(z)\, e^{-2\pi i m x}\, dx$$

for $m \geq 1$. The vertical strip $0 < \Re(z) < 1$ is a fundamental region for $\mathfrak{h}/\Gamma_\infty$ so that

$$(f, P_{m,s}^{(2k)})_{2k} = \iint\limits_{\mathfrak{h}/\Gamma_\infty} f(z)\, e^{-2\pi i m(x-iy)}\, y^{2k+\bar{s}-2}\, dxdy,$$

$$= \int_0^\infty \int_0^1 f(z) e^{-2\pi m y}\, e^{-2\pi i m x}\, y^{2k+\bar{s}-2}\, dxdy$$

$$= \int_0^\infty a_m(y)\, e^{-2\pi m y} y^{2k+\bar{s}-2}\, dxdy$$

If

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m z} = \sum_{m=-\infty}^{\infty} \left(a_m(y) e^{-2\pi m y}\right) e^{2\pi i m x}$$

then

$$a_m(y) e^{-2\pi m y} = \int_0^1 f(z)\, e^{-2\pi i m x}\, dx$$

for $m \geq 1$. The vertical strip $0 < \Re(z) < 1$ is a fundamental region for $\mathfrak{h}/\Gamma_\infty$ so that

$$(f, P_{m,s}^{(2k)})_{2k} = \iint_{\mathfrak{h}/\Gamma_\infty} f(z)\, e^{-2\pi i m(x-iy)}\, y^{2k+\bar{s}-2}\, dxdy,$$

$$= \int_0^\infty \int_0^1 f(z) e^{-2\pi m y}\, e^{-2\pi i m x}\, y^{2k+\bar{s}-2}\, dxdy$$

$$= \int_0^\infty a_m(y)\, e^{-2\pi m y} y^{2k+\bar{s}-2}\, dxdy$$

Theorem (THE HOLOMORPHIC PROJECTION THEOREM)

Let $k \geq 1$, $f \in N_{2k}(\Gamma_0(N))$ and suppose f has a Fourier expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m z}.$$

For $m \geq 1$ set

$$b_m = \begin{cases} \frac{(4\pi m)^{2k-1}}{(2k-1)!} \int_0^\infty a_m(y)\, e^{-2\pi m y}\, y^{2k-2}\, dy & \text{if } k > 1 \\ 4\pi m \lim_{s\to 0^+} \int_0^\infty a_m(y)\, e^{-2\pi m y}\, y^{s}\, dy & \text{if } k = 1. \end{cases}$$

Then $\pi_{hol}(f)$ defined by

$$\pi_{hol}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi i m z}$$

is an element of $S_{2k}(\Gamma_0(N))$ and satisfies

$$(f, g)_{2k} = (\pi_{\text{hol}}(f), g)_{2k}$$

for every $g \in S_{2k}(\Gamma_0(N))$.

Proof Let f be given as in the statement. Define

$$F : S_{2k}(\Gamma_0(N)) \longrightarrow \mathbb{C} \quad \text{by } F(g) = (f, g)_{2k}.$$

is an element of $S_{2k}(\Gamma_0(N))$ and satisfies

$$(f, g)_{2k} = (\pi_{\text{hol}}(f), g)_{2k}$$

for every $g \in S_{2k}(\Gamma_0(N))$.

Proof Let f be given as in the statement. Define

$$F : S_{2k}(\Gamma_0(N)) \longrightarrow \mathbb{C} \quad \text{by } F(g) = (f, g)_{2k}.$$

By The Riesz Representation Theorem there is a unique function $\pi_{\text{hol}}(f) \in S_{2k}(\Gamma_0(N))$ such that

$$(f, g)_{2k} = (\pi_{\text{hol}}(f), g)_{2k}$$

for all $g \in S_{2k}(\Gamma_0(N))$.

is an element of $S_{2k}(\Gamma_0(N))$ and satisfies

$$(f, g)_{2k} = (\pi_{\text{hol}}(f), g)_{2k}$$

for every $g \in S_{2k}(\Gamma_0(N))$.

Proof Let f be given as in the statement. Define

$$F : S_{2k}(\Gamma_0(N)) \longrightarrow \mathbb{C} \quad \text{by } F(g) = (f, g)_{2k}.$$

By The Riesz Representation Theorem there is a unique function $\pi_{\text{hol}}(f) \in S_{2k}(\Gamma_0(N))$ such that

$$(f, g)_{2k} = (\pi_{\text{hol}}(f), g)_{2k}$$

for all $g \in S_{2k}(\Gamma_0(N))$.

Since $\pi_{\text{hol}}(f) \in S_{2k}(\Gamma_0(N))$,

$$\pi_{\text{hol}}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi i m z}$$

for some constants b_m. Assume $m \geq 1$ and $k > 1$. Then

$$(f, P_{m,0}^{(2k)})_{2k} = \int_0^{\infty} a_m(y)\, e^{-4m\pi y}\, y^{2k-2}\, dxdy$$

and

Since $\pi_{\text{hol}}(f) \in S_{2k}(\Gamma_0(N))$,

$$\pi_{\text{hol}}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi imz}$$

for some constants b_m. Assume $m \geq 1$ and $k > 1$. Then

$$(f, P_{m,0}^{(2k)})_{2k} = \int_0^{\infty} a_m(y)\, e^{-4m\pi y}\, y^{2k-2}\, dxdy$$

and

$$\begin{aligned}(\pi_{\text{hol}}(f), P_{m,0}^{(2k)})_{2k} &= \int_0^{\infty} b_m\, e^{-4m\pi y}\, y^{2k-2}\, dxdy \\ &= b_m \int_0^{\infty} e^{-4m\pi y}\, y^{2k-2} \\ &= \frac{(2k-2)!}{(4\pi m)^{2k-1}}\, b_m\end{aligned}$$

Since $\pi_{\text{hol}}(f) \in S_{2k}(\Gamma_0(N))$,

$$\pi_{\text{hol}}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi i m z}$$

for some constants b_m. Assume $m \geq 1$ and $k > 1$. Then

$$(f, P_{m,0}^{(2k)})_{2k} = \int_0^{\infty} a_m(y)\, e^{-4m\pi y}\, y^{2k-2}\, dxdy$$

and

$$\begin{aligned}(\pi_{\text{hol}}(f), P_{m,0}^{(2k)})_{2k} &= \int_0^{\infty} b_m\, e^{-4m\pi y}\, y^{2k-2}\, dxdy \\ &= b_m \int_0^{\infty} e^{-4m\pi y}\, y^{2k-2} \\ &= \frac{(2k-2)!}{(4\pi m)^{2k-1}}\, b_m\end{aligned}$$

Since $P_{m,0}^{(2k)} \in S_{2k}(\Gamma_0(N))$ these two inner-products are equal

$$b_m = \frac{(4\pi m)^{2k-1}}{(2k-1)!} \int_0^\infty a_m(y)\, e^{-2\pi m y}\, y^{2k-2}\, dy.$$

EXTENDING THE DEFINITION

Since $P_{m,0}^{(2k)} \in S_{2k}(\Gamma_0(N))$ these two inner-products are equal

$$b_m = \frac{(4\pi m)^{2k-1}}{(2k-1)!} \int_0^\infty a_m(y)\, e^{-2\pi my}\, y^{2k-2}\, dy.$$

EXTENDING THE DEFINITION

Assume a function $f(z)$ has an expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi imz}.$$

Extend the definition to such functions by defining

$$\pi_{\text{hol}}^{(k)}(f) = \pi_{\text{hol}}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi mz}$$

where $b_m = \dfrac{(4\pi m)^{2k-1}}{(2k-1)!} \displaystyle\int_0^\infty a_m(y)\, e^{-2\pi my}\, y^{2k-2}\, dy.$

Since $P_{m,0}^{(2k)} \in S_{2k}(\Gamma_0(N))$ these two inner-products are equal

$$b_m = \frac{(4\pi m)^{2k-1}}{(2k-1)!} \int_0^\infty a_m(y)\, e^{-2\pi my}\, y^{2k-2}\, dy.$$

EXTENDING THE DEFINITION

Assume a function $f(z)$ has an expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi imz}.$$

Extend the definition to such functions by defining

$$\pi_{\text{hol}}^{(k)}(f) = \pi_{\text{hol}}(f) = \sum_{m=1}^{\infty} b_m e^{2\pi mz}$$

where $b_m = \dfrac{(4\pi m)^{2k-1}}{(2k-1)!} \displaystyle\int_0^\infty a_m(y)\, e^{-2\pi my}\, y^{2k-2}\, dy.$

Proposition

If $f(z)$ is holomorphic and has an expansion

$$f(z) = \sum_{m=1}^{\infty} a_m e^{2\pi imz},$$

then

$$\pi_{hol}(f) = f$$

Proof Since

$$\int_0^{\infty} e^{-2\pi my}\, y^{2k-2}\, dy. = \frac{(2k-1)!}{(4\pi m)^{2k-1}},$$

for $k \geq 1$ the result follows. □

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$E_2 = 1 - 24\Phi_1$$
$$E_4 = 1 + 240\Phi_3$$
$$E_2^* = E_2 - \frac{3}{\pi y}.$$

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$E_2 = 1 - 24\Phi_1$$
$$E_4 = 1 + 240\Phi_3$$
$$E_2^* = E_2 - \frac{3}{\pi y}.$$

E_2^* is a non-holomorphic modular form of weight 2,

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$E_2 = 1 - 24\Phi_1$$
$$E_4 = 1 + 240\Phi_3$$
$$E_2^* = E_2 - \frac{3}{\pi y}.$$

E_2^* is a non-holomorphic modular form of weight 2, and E_4 is a holomorphic modular form of weight 4.

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$\begin{aligned} E_2 &= 1 - 24\Phi_1 \\ E_4 &= 1 + 240\Phi_3 \\ E_2^* &= E_2 - \frac{3}{\pi y}. \end{aligned}$$

E_2^* is a non-holomorphic modular form of weight 2, and E_4 is a holomorphic modular form of weight 4.

$$(E_2^*)^2 = E_2^2 - \frac{6}{\pi y} E_2 + \frac{9}{\pi y^2}$$

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$E_2 = 1 - 24\Phi_1$$
$$E_4 = 1 + 240\Phi_3$$
$$E_2^* = E_2 - \frac{3}{\pi y}.$$

E_2^* is a non-holomorphic modular form of weight 2, and E_4 is a holomorphic modular form of weight 4.

$$(E_2^*)^2 = E_2^2 - \frac{6}{\pi y} E_2 + \frac{9}{\pi y^2}$$

Then

$$f = (E_2^*)^2 - E_4 \in N_4(\Gamma_0(1))$$

EXAMPLE. Let

$$\Phi_1 = \sum_{n=1} \sigma_1(n) q^n, \qquad \Phi_3 = \sum_{n=1} \sigma_3(n) q^n.$$

Then

$$E_2 = 1 - 24\Phi_1$$
$$E_4 = 1 + 240\Phi_3$$
$$E_2^* = E_2 - \frac{3}{\pi y}.$$

E_2^* is a non-holomorphic modular form of weight 2, and E_4 is a holomorphic modular form of weight 4.

$$(E_2^*)^2 = E_2^2 - \frac{6}{\pi y} E_2 + \frac{9}{\pi y^2}$$

Then

$$f = (E_2^*)^2 - E_4 \in N_4(\Gamma_0(1))$$

$$\pi_{\text{hol}}^{(4)}(f) = E_2^2 - E_4 - \pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right)$$

Since

$$\frac{(4m\pi)^3}{3!}\int_0^\infty e^{-4\pi m y}\, y\, dy = 2\pi m$$

$$\pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right) = 12\, q\, \frac{d}{dq} E_2$$

$$\pi_{\text{hol}}^{(4)}(f) = E_2^2 - E_4 - \pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right)$$

Since

$$\frac{(4m\pi)^3}{3!}\int_0^\infty e^{-4\pi m y} y\, dy = 2\pi m$$

$$\pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right) = 12\, q\, \frac{d}{dq} E_2$$

Hence

$$\pi_{\text{hol}}^{(4)}(f) = E2^2 - E_4 - 12\, q\, \frac{d}{dq} E_2 \in S_4(\Gamma_0(1))$$

$$\pi_{\text{hol}}^{(4)}(f) = E_2^2 - E_4 - \pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right)$$

Since

$$\frac{(4m\pi)^3}{3!}\int_0^\infty e^{-4\pi m y} y\, dy = 2\pi m$$

$$\pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right) = 12\, q\, \frac{d}{dq}E_2$$

Hence

$$\pi_{\text{hol}}^{(4)}(f) = E2^2 - E_4 - 12\, q\, \frac{d}{dq}E_2 \in S_4(\Gamma_0(1))$$

But $\dim S_4(\Gamma_0(1)) = 0$ which implies that

$$q\, \frac{d}{dq}E_2 = \frac{1}{12}\left(E_2^2 - E_4\right)$$

$$\pi_{\text{hol}}^{(4)}(f) = E_2^2 - E_4 - \pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right)$$

Since

$$\frac{(4m\pi)^3}{3!}\int_0^\infty e^{-4\pi m y} y\, dy = 2\pi m$$

$$\pi_{\text{hol}}^{(4)}\left(\frac{6}{\pi y}(E_2 - 1)\right) = 12\, q\, \frac{d}{dq} E_2$$

Hence

$$\pi_{\text{hol}}^{(4)}(f) = E2^2 - E_4 - 12\, q\, \frac{d}{dq} E_2 \in S_4(\Gamma_0(1))$$

But $\dim S_4(\Gamma_0(1)) = 0$ which implies that

$$q\, \frac{d}{dq} E_2 = \frac{1}{12}\left(E_2^2 - E_4\right)$$

EXTENSION TO $\Gamma_1(N)$
BECKWITH, RAUM and RICHTER:
Consider N-periodic continuous functions

$$f(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} c(f;n;y)\exp(2\pi i n z).$$

Assume the are coefficients $\widetilde{c}(f|_k\gamma;0) \in \mathbb{C}$ and $a > 0$ such that

$$(f|_k\gamma)(z) = \widetilde{c}(f|_k\gamma;0) + O(y^{-a})$$

as $y \to \infty$ for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and for all $n \in \frac{1}{N}\mathbb{Z}$, $n \geq 1$,

$$c(f;n;y) = O(y^{2-k}), \qquad \text{as } y \to 0.$$

Then we define

EXTENSION TO $\Gamma_1(N)$

BECKWITH, RAUM and RICHTER:

Consider N-periodic continuous functions

$$f(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} c(f; n; y) \exp(2\pi i n z).$$

Assume the are coefficients $\widetilde{c}(f|_k\gamma; 0) \in \mathbb{C}$ and $a > 0$ such that

$$(f|_k\gamma)(z) = \widetilde{c}(f|_k\gamma; 0) + O(y^{-a})$$

as $y \to \infty$ for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and for all $n \in \frac{1}{N}\mathbb{Z}$, $n \geq 1$,

$$c(f; n; y) = O(y^{2-k}), \qquad \text{as } y \to 0.$$

Then we define

$$\pi_{\text{hol}}^{(k)}(f) := \widetilde{c}(f;0) + \sum_{n \in \frac{1}{N}\mathbb{Z}} c(\pi_{\text{hol}}^{(k)}(f); n) \exp(2\pi i n z).$$

where

$$c(\pi_{\text{hol}}^{(k)}(f); n) := \frac{(4\pi m)^{k-1}}{(k-1)!} \lim_{s \to 0} \int_0^\infty c(f; n, y)\, e^{-2\pi n y}\, y^{s+k-2}\, dy$$

Theorem (BECKWITH, RAUM and RICHTER)

Let f be as above.

- *If f is holomorphic then $\pi_{hol}^{(k)}(f) = f$.*
- *If $(f|_2\gamma)(z) = f(z)$ for all $\gamma \in \Gamma_1(N)$ then $\pi_{hol}^{(2)}(f)$ is a quasi-modular form of weight 2 for $\Gamma_1(N)$.*

$$\pi_{\text{hol}}^{(k)}(f) := \widetilde{c}(f;0) + \sum_{n\in\frac{1}{N}\mathbb{Z}} c(\pi_{\text{hol}}^{(k)}(f);n)\exp(2\pi i n z).$$

where

$$c(\pi_{\text{hol}}^{(k)}(f);n) := \frac{(4\pi m)^{k-1}}{(k-1)!}\lim_{s\to 0}\int_0^\infty c(f;n,y)\,e^{-2\pi n y}\,y^{s+k-2}\,dy$$

Theorem (BECKWITH, RAUM and RICHTER)

Let f be as above.

- *If f is holomorphic then* $\pi_{hol}^{(k)}(f) = f$.
- *If* $(f|_2\gamma)(z) = f(z)$ *for all* $\gamma \in \Gamma_1(N)$ *then* $\pi_{hol}^{(2)}(f)$ *is a quasi-modular form of weight 2 for* $\Gamma_1(N)$.

HOLOMORPHIC PROJECTION WITH RADICALS

Lemma

Suppose

$$f(\tau) = -i \left(\sum_{m=1}^{\infty} \chi_1(m) e^{2\pi i a m^2 \tau} \right) \left(\sum_{n=1}^{\infty} \chi_2(n) \int_{-\overline{\tau}}^{i\infty} \frac{e^{2\pi i b n^2 z}}{\sqrt{-i(z+\tau)}}\, dz \right)$$

Then the weight k holomorphic projection of f is given by

$$\pi_{hol}^{(k)}(f)(\tau) = \frac{2^k}{\pi} \sum_{\substack{m=1 \\ am^2 - bn^2 > 0}}^{\infty} \sum_{n=1}^{\infty} \chi_1(m)\chi_2(n)(am^2 - bn^2)^{k-1} e^{2\pi i \tau (am^2 - bn^2)}$$

$$\times \int_0^{\infty} \frac{dx}{(x^2 + 2am^2)^{k-1}(x^2 + 2bn^2)}.$$

When $k = 2$, this becomes

$$\pi_{\text{hol}}^{(2)}(f)(\tau) = \frac{1}{\sqrt{2ab}} \sum_{\substack{m=1 \\ am^2-bn^2>0}}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)}{mn} \left(m\sqrt{a} - n\sqrt{b}\right) \times \exp(2\pi i\tau(am^2 - bn^2)).$$

HOLOMORPHIC PROJECTION WITHOUT RADICALS

Let a, b, c be positive rationals then the number of integer solutions (m, n) to

$$am^2 - bn^2 = c$$

is usually infinite. Thus the series in the previous theorem does NOT converge as a formal power series.

Lemma (ANDREWS, DYSON and HICKERSON)

Let (x_1, y_1) be the fundamental solution to

$$x^2 - Dy^2 = 1$$

ie the solution in which x_1, y_1 are minimal positive. If $m > 0$ then each equivalence class of solutions of

$$u^2 - Dv^2 = m$$

HOLOMORPHIC PROJECTION WITHOUT RADICALS

Let a, b, c be positive rationals then the number of integer solutions (m, n) to

$$am^2 - bn^2 = c$$

is usually infinite. Thus the series in the previous theorem does NOT converge as a formal power series.

Lemma (ANDREWS, DYSON and HICKERSON)

Let (x_1, y_1) be the fundamental solution to

$$x^2 - Dy^2 = 1$$

ie the solution in which x_1, y_1 are minimal positive. If $m > 0$ then each equivalence class of solutions of

$$u^2 - Dv^2 = m$$

contains a unique (u, v) with $u > 0$ and

$$-\frac{y_1}{x_1+1}u < v \leq \frac{y_1}{x_1+1}u.$$

If m ¡ 0 the corresponding conditions are $v > 0$ and

$$-\frac{Dy_1}{x_1+1}v < u \leq \frac{Dy_1}{x_1+1}v.$$

NOTE: Two solutions (u, v), (u', v') of $u^2 - Dv^2 = m$ are equivalent if and only if

$$u' + v'\sqrt{D} = \pm(x_1 + y_1\sqrt{D})^r(u + v\sqrt{D})$$

for some integer r.

Using this Lemma it is often possible to write the weight two holomorphic projection of f given in the previous theorem as Hecke-Rogers type series

$$\pi_{\text{hol}}^{(2)}(f)(\tau) = c \sum_{m=1}^{\infty} \sum_{n=1}^{\lfloor \alpha m \rfloor} \chi_1(m)\chi_2(n)(m - \beta n)q^{am^2 - bn^2}$$

for certain rationals c, α, β.

JONATHAN BRADLEY-THRUSH

Theorem 3.2.1. *Let a, b, c and d be integers such that $abcd$ is not a square number. Let χ_1 and χ_2 be Dirichlet characters with periods p_1 and p_2 respectively. Suppose that p_1 divides bc and p_2 divides ad. Let $(m,n)=(\alpha,\beta)$ be the fundamental solution of the equation*

$$m^2 - abcdn^2 = 1$$

(ie. minimal such that m and n are both positive) and assume that $(\alpha-1)/p_1$ and $(\alpha+1)/p_2$ are not both integers. Let

$$f(\tau) = -\mathrm{i}\left(\sum_{m=1}^{\infty}\chi_1(m)m\mathrm{e}^{\frac{2\pi\mathrm{i}am^2\tau}{b}}\right)\sum_{n=1}^{\infty}\chi_2(n)n\int_{-\bar{\tau}}^{\mathrm{i}\infty}\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}cn^2}{d}}}{\sqrt{-\mathrm{i}(z+\tau)}}\,\mathrm{d}z.$$

Suppose first that χ_2 is an even character (ie. $\chi_2(-1)=1$). Then the holomorphic projection of f (of weight 2) is given by

$$\pi_{hol}(f)(\tau) = \sqrt{\frac{b}{2a}}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor\frac{ad\beta}{\alpha+1}m\rfloor}\chi_1(m)\chi_2(n)\left(\frac{\alpha+\chi_1(\alpha)\chi_2(\alpha)}{bc\beta}m-n\right)\mathrm{e}^{2\pi\mathrm{i}\tau(\frac{a}{b}m^2-\frac{c}{d}n^2)}.$$

If instead χ_2 is an odd character (ie. $\chi_2(-1)=-1$), then

$$\pi_{hol}(f)(\tau) = \sqrt{\frac{d}{2c}}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor\frac{ad\beta}{\alpha+1}m\rfloor}\chi_1(m)\chi_2(n)\left(m-\frac{\alpha+\chi_1(\alpha)\chi_2(\alpha)}{ad\beta}n\right)\mathrm{e}^{2\pi\mathrm{i}\tau(\frac{a}{b}m^2-\frac{c}{d}n^2)}.$$

Many authors have been able to express Ramanujan's mock theta functions as the holomorphic part of the component of a vector-valued harmonic Maass form of weight $1/2$.

Let N be a positive integer, $0 \leq a < 2N$, and define

$$\theta_{N,a}(\tau) := \sum_{n \equiv a \pmod{2N}} n \exp\left(\frac{2\pi i n^2 \tau}{4N}\right)$$

Many authors have been able to express Ramanujan's mock theta functions as the holomorphic part of the component of a vector-valued harmonic Maass form of weight $1/2$.

Let N be a positive integer, $0 \leq a < 2N$, and define

$$\theta_{N,a}(\tau) := \sum_{n \equiv a \pmod{2N}} n \exp\left(\frac{2\pi i n^2 \tau}{4N}\right)$$

EXAMPLE We use KLEIN and KUPKA's results for the third order mock theta functions:

$$\phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n},\ \psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n},\ \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

$$F_{(3)}(\tau):=\begin{pmatrix} q^{-\frac{1}{48}}\ \phi(q^{\frac{1}{2}}) \\ q^{-\frac{1}{48}}\ \phi(-q^{\frac{1}{2}}) \\ 2\ q^{-\frac{1}{48}}\ \psi(q^{\frac{1}{2}}) \\ 2\ q^{-\frac{1}{48}}\ \psi(-q^{\frac{1}{2}}) \\ \sqrt{2}\ q^{\frac{1}{6}}\ \nu(q^{\frac{1}{2}}) \\ \sqrt{2}\ q^{\frac{1}{6}}\ \nu(-q^{\frac{1}{2}}) \end{pmatrix},$$

where $q=e^{2\pi i\tau}$,

<u>EXAMPLE</u> We use KLEIN and KUPKA's results for the third order mock theta functions:

$$\phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n},\ \psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n},\ \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

$$F_{(3)}(\tau):=\begin{pmatrix} q^{-\frac{1}{48}}\ \phi(q^{\frac{1}{2}}) \\ q^{-\frac{1}{48}}\ \phi(-q^{\frac{1}{2}}) \\ 2\ q^{-\frac{1}{48}}\ \psi(q^{\frac{1}{2}}) \\ 2\ q^{-\frac{1}{48}}\ \psi(-q^{\frac{1}{2}}) \\ \sqrt{2}\ q^{\frac{1}{6}}\ \nu(q^{\frac{1}{2}}) \\ \sqrt{2}\ q^{\frac{1}{6}}\ \nu(-q^{\frac{1}{2}}) \end{pmatrix},$$

where $q=e^{2\pi i\tau}$,

and

$$G_{(3)}(\tau) := \frac{i}{\sqrt{24}} \int_{-\overline{\tau}}^{i\infty} \frac{g_{(3)}(z)}{\sqrt{-i(z+\tau)}}\, dz,$$

where $g_{(3)}$ is the vector $(g_{(3),0}, \ldots, g_{(3),5})^T$ with components

$$g_{(3),0}(z) := -(\theta_{12,1}(z) + \theta_{12,5}(z) + \theta_{12,7}(z) + \theta_{12,11}(z)),$$
$$g_{(3),1}(z) := -(\theta_{12,1}(z) - \theta_{12,5}(z) + \theta_{12,7}(z) - \theta_{12,11}(z)),$$
$$g_{(3),2}(z) := \theta_{12,1}(z) + \theta_{12,5}(z) + \theta_{12,7}(z) + \theta_{12,11}(z),$$
$$g_{(3),3}(z) := \theta_{12,1}(z) - \theta_{12,5}(z) + \theta_{12,7}(z) - \theta_{12,11}(z),$$
$$g_{(3),4}(z) := -\sqrt{2}\,(\theta_{12,4}(z) + \theta_{12,8}(z)),$$
$$g_{(3),5}(z) := \sqrt{2}\,(\theta_{12,4}(z) + \theta_{12,8}(z)).$$

Theorem (KLEIN and KUPKA)

The function $H_{(3)}$, defined for $\tau \in \mathbb{H}$ by

$$H_{(3)}(\tau) := F_{(3)}(\tau) - G_{(3)}(\tau),$$

is a vector valued harmonic weak Maass form of weight $1/2$ satisfying

$$H_{(3)}(\tau+1) = \begin{pmatrix} 0 & \zeta_{48}^{-1} & 0 & 0 & 0 & 0 \\ \zeta_{48}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{48}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_{48}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6 \\ 0 & 0 & 0 & 0 & \zeta_6 & 0 \end{pmatrix} H_{(3)}(\tau)$$

and

$$H_{(3)}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} H_{(3)}(\tau).$$

It can be shown that each component of $H_{(3)}(\tau)$ transforms as a modular form with multiplier on $\Gamma(2)$.

and

$$H_{(3)}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} H_{(3)}(\tau).$$

It can be shown that each component of $H_{(3)}(\tau)$ transforms as a modular form with multiplier on $\Gamma(2)$.

We examine the third component of $H_{(3)}(\tau)$

$$H_{(3),2}(\tau) = 2q^{-1/48}\psi(q^{1/2})$$

$$\frac{-i}{2\sqrt{6}}\int_{-\overline{\tau}}^{i\infty} \frac{(\theta_{12,1}(z)+\theta_{12,5}(z)+\theta_{12,7}(z)+\theta_{12,11}(z))}{\sqrt{-i(z+\tau)}}\,dz$$

$$= 2q^{-1/48}\psi(q^{1/2}) - \frac{i}{2\sqrt{6}}\sum_{n=1}^{\infty}\left(\frac{n}{6}\right) n \int_{-\overline{\tau}}^{i\infty} \frac{\exp\left(\frac{\pi i n^2 z}{24}\right)}{\sqrt{-i(z+\tau)}}\,dz$$

RECALL

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \eta^3(\tau) = q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3$$

$$= q^{1/8}\left(1-3q+5q^3-7q^6+\cdots\right)$$

$$= \sum_{m=1}^{\infty} m\left(\frac{-4}{m}\right) q^{m^2/8}$$

We examine the third component of $H_{(3)}(\tau)$

$$H_{(3),2}(\tau) = 2q^{-1/48}\psi(q^{1/2})$$

$$\frac{-i}{2\sqrt{6}}\int_{-\overline{\tau}}^{i\infty} \frac{(\theta_{12,1}(z)+\theta_{12,5}(z)+\theta_{12,7}(z)+\theta_{12,11}(z))}{\sqrt{-i(z+\tau)}}\,dz$$

$$= 2q^{-1/48}\psi(q^{1/2}) - \frac{i}{2\sqrt{6}}\sum_{n=1}^{\infty}\left(\frac{n}{6}\right) n \int_{-\overline{\tau}}^{i\infty} \frac{\exp\left(\frac{\pi i n^2 z}{24}\right)}{\sqrt{-i(z+\tau)}}\,dz$$

RECALL

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \eta^3(\tau) = q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3$$

$$= q^{1/8}\left(1-3q+5q^3-7q^6+\cdots\right)$$

$$= \sum_{m=1}^{\infty} m\left(\frac{-4}{m}\right)q^{m^2/8}$$

Thus

$$\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau) = q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2}) - \frac{i}{4\sqrt{6}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\exp\left(\frac{i\pi m^2\tau}{8}\right) \times \int_{-i\overline{\tau}} \frac{\exp\left(\frac{i\pi n^2 z}{24}\right)}{\sqrt{-i(z+\tau)}}\,dz$$

$$\pi_{\mathrm{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right) = q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2}) + \frac{1}{6}\sum_{\substack{m=1\\ 3m^2-n^2>0}}\sum_{n=1}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m-n\sqrt{3})q^{m^2/16-n^2/48}$$

Thus

$$\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau) = q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2}) - \frac{i}{4\sqrt{6}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\exp\left(\frac{i\pi m^2\tau}{8}\right) \times \int_{-i\overline{\tau}} \frac{\exp\left(\frac{i\pi n^2 z}{24}\right)}{\sqrt{-i(z+\tau)}}\,dz$$

$$\pi_{\mathrm{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right) = q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2}) + \frac{1}{6}\sum_{\substack{m=1\ n=1 \\ 3m^2-n^2>0}}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m - n\sqrt{3})q^{m^2/16-n^2/48}$$

AFTER SOME WORK we find that

$$\frac{1}{6}\sum_{\substack{m=1 \\ 3m^2-n^2>0}}\sum_{n=1}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m-n\sqrt{3})$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\sqrt{3}\right)q^{m^2/16-n^2/48}$$

THUS

$$\pi_{\mathrm{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

$$=q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2})$$

$$+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{m^2/16-n^2/48}$$

AFTER SOME WORK we find that

$$\frac{1}{6}\sum_{\substack{m=1 \\ 3m^2-n^2>0}}\sum_{n=1}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m-n\sqrt{3})$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\sqrt{3}\right)q^{m^2/16-n^2/48}$$

THUS

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

$$=q^{-1/48}\eta^3(\tau/2)\psi(q^{1/2})$$

$$+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{m^2/16-n^2/48}$$

The function $\eta^3(\tau/2)$ is a modular form of weight $3/2$ on $\Gamma(2)$ with multiplier. This would seem to indicate that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

is holomorphic modular form of weight 2 on $\Gamma(2)$ with multiplier.

The function $\eta^3(\tau/2)$ is a modular form of weight 3/2 on $\Gamma(2)$ with multiplier. This would seem to indicate that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

is holomorphic modular form of weight 2 on $\Gamma(2)$ with multiplier.
We find that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right) = \frac{1}{6}\frac{\eta^7(\tau)}{\eta^3(2\tau)}$$

The function $\eta^3(\tau/2)$ is a modular form of weight 3/2 on $\Gamma(2)$ with multiplier. This would seem to indicate that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

is holomorphic modular form of weight 2 on $\Gamma(2)$ with multiplier. We find that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right) = \frac{1}{6}\frac{\eta^7(\tau)}{\eta^3(2\tau)}$$

which implies that

$$(q;q)_\infty^3\,\psi(q) = \frac{1}{6}\frac{(q^2;q^2)_\infty^7}{(q^4;q^4)_\infty^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{m^2/16-n^2/48-1/12}$$

The function $\eta^3(\tau/2)$ is a modular form of weight 3/2 on $\Gamma(2)$ with multiplier. This would seem to indicate that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right)$$

is holomorphic modular form of weight 2 on $\Gamma(2)$ with multiplier. We find that

$$\pi_{\text{hol}}^{(2)}\left(\frac{1}{2}\eta(\tau/2)H_{(3),2}(\tau)\right) = \frac{1}{6}\frac{\eta^7(\tau)}{\eta^3(2\tau)}$$

which implies that

$$(q;q)_\infty^3\,\psi(q) = \frac{1}{6}\frac{(q^2;q^2)_\infty^7}{(q^4;q^4)_\infty^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{m^2/16-n^2/48-1/12}$$

ANOTHER EXAMPLE

$$(q^4;q^4)_\infty^3\,\psi(-q) = \frac{1}{3}\frac{(q^4;q^4)_\infty^3(q;q)_\infty^3}{(q^2;q^2)_\infty^2} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{3m}\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right)\left(m-\frac{n}{3}\right)q^{m^2/2-n^2/24-11/24}$$

MORE EXAMPLES

Let

$$A(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2}$$

THEN

$$\frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}A(-q)$$
$$=-\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 2m/3\rfloor}\left(\frac{-12}{m}\right)\left(\frac{-4}{n}\right)(m-n)q^{m^2/2-n^2/8+1/24}$$

$$
\begin{aligned}
&(q)_\infty^3 A(-q) \\
&= -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\left(\frac{-4}{mn}\right)(m-n)q^{m^2/8-n^2/8} \\
&- \frac{1}{6}\theta_3(q)^4 + \frac{5}{24}\theta_4(q)^4 - \frac{1}{24}E_2(q)
\end{aligned}
$$