Inverse problems in nonlinear acoustic imaging

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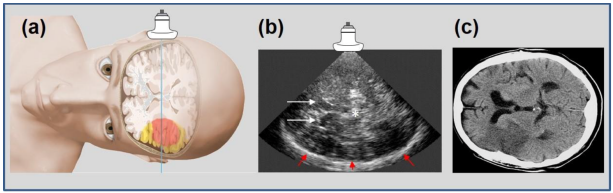
Motivation

- ultrasound imaging: uses high-frequency sound waves to image human tissues
- nonlinear effects, due to the interaction between the ultrasound waves and the medium



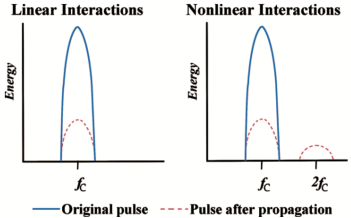


Figure: ultrasound imaging

Not adequately modeled using linear acoustic equations

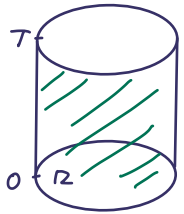
Nonlinear Acoustic Model

The acoustic waves are modeled by the Westervelt-type equation

$$\begin{aligned} \partial_t^2 p - c^2 \Delta p - \beta \partial_t^2 (p^2) &= 0, && \text{in } (0,T)\times\Omega, \\ p &= f, && \text{on } (0,T)\times\partial\Omega, \\ p = \partial_t p &= 0, && \text{on } \{t=0\}, \end{aligned}$$

- $\Omega \subset \mathbb{R}^3$: a bounded domain with smooth boundary
- $p(t,x)$: the pressure field of the ultrasound waves
- $c(x)$ or $c(t,x) > 0$: smooth wave speed
- By [Kaltenbacher-Rundell, 2022] a damping term Du: e.g.,

$$D = -\partial_t, \quad -\partial_t \Delta, \quad \partial_t (-\Delta)^s, \ \ldots$$



Inverse problems

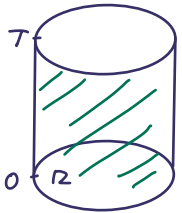
Consider

$$\begin{aligned} \partial_t^2 p - c^2 \Delta p - \beta \partial_t^2 (p^2) = 0, &\quad \text{in } (0,T)\times\Omega, \\ p = f, &\quad \text{on } (0,T)\times\partial\Omega, \\ p = \partial_t p = 0, &\quad \text{on } \{t=0\}, \end{aligned}$$

- well-posedness: for small compatible boundary sources $f \in C^6([0,T]\times\partial\Omega)$
- consider the Dirichlet-to-Neumann map (DN map)

$$\Lambda_\beta : f = p|_{(0,T)\times\Omega} \to \partial_\nu p|_{(0,T)\times\partial\Omega}.$$

Inverse problems: Can we recover β from Λ_β?



Related results

The Westervelt equation with time-independent coefficients:

① 2nd-order linear

② Gaussian beams

- [Kaltenbacher-Rundell, 2021]: constant wave speed
- [Acosta-Uhlmann-Zhai, 2022]: the DN map determines $\beta(x)$, assuming
  - either no conjugate points or under the foliation condition
- [Eptaminitakis-Stefanov, 2023]: constant wave speed, weakly nonlinear geometric optics

Related results

① 4th-order linear

② interaction of distorted plane waves

The Westervelt equation with time-dependent coefficients:

- [Uhlmann-Z, 2023]: with no additional geometric assumptions, the DN map determines $\beta_{m+1}(t,x)$ from $\sum_{m=1}^{+\infty} \beta_{m+1}\partial_t^2(p^{m+1})$ assumption
- [Z, 2023]: $\partial_t^2 p - c^2\Delta p + \langle b, \nabla_{t,x} p\rangle + hp - \beta\partial_t^2(p^2) = 0$ up to gauge $\rightsquigarrow (-\Delta)^s$ strong unique continuation
- [Li-Z, 2023, 2023]: $\partial_t^2 p - \Delta p + Du - \beta\partial_t^2(p^2) = 0$, where $D = -\partial_t\Delta$ or $\partial_t(-\Delta)^s$.

$\partial_t^2 p - \Delta p - \partial_t\Delta p = 0$

① complex geometric optics (CGO)

② fundamental solution

Inverse problems

Consider

$$\begin{aligned} \partial_t^2 p - c^2 \Delta p - \beta \partial_t^2 (p^2) &= 0, && \text{in } (0,T)\times\Omega, \\ p &= f, && \text{on } (0,T)\times\partial\Omega, \\ p = \partial_t p &= 0, && \text{on } \{t=0\}, \end{aligned}$$

- well-posedness: for small compatible boundary sources $f \in C^6([0,T]\times\partial\Omega)$
- consider the Dirichlet-to-Neumann map (DN map)

$$\Lambda_\beta : f = p|_{(0,T)\times\Omega} \to \partial_\nu p|_{(0,T)\times\partial\Omega}.$$

Inverse problems: Can we recover β from Λ_β?

Idea 1: multi-fold linearization

$\Box_c p - \beta \partial_t^2 (p^2) = 0 \quad \cdots$

$p = f \quad (0,T) \times \partial\Omega$

- choose sources depending on several small parameters $\epsilon_1, \epsilon_2, \ldots$
- derive asymptotic expansion of the solution p w.r.t these small parameters
- relate the measurements with a product of solutions v_j to linear problems

For example, 2nd-order linearization: let $f = \epsilon_1 f_1 + \epsilon_2 f_2$, where $f_1, f_2 \in C^6([0,T] \times \partial\Omega)$

$\epsilon_1, \epsilon_2 \ll 1$ Let v_j solve

$$\begin{cases} \Box_c v_j = 0 & (0,T) \times \mathbb{R} \\ v_j = f_j & (0,T) \times \partial\Omega \\ v_j = \partial_t v_j = 0 & t = 0 \end{cases}$$

$v = \epsilon_1 v_1 + \epsilon_2 v_2$

$$\circledast \quad \Box_c(p-v) = \beta \partial_t^2(p^2) \qquad (0,T)\times\Omega$$

$$(p-v) = 0 \qquad (0,T)\times\partial\Omega$$

$$p-v \Rightarrow \partial_t(p-v) = 0 \qquad t=0$$

Formally $p = v + V_2 + R$, with $R \to O(\varepsilon_i\varepsilon_j\varepsilon_k)$

$\varepsilon_i\varepsilon_j$-terms

$(v+V_2+R)^2$

$\varepsilon_i\varepsilon_j$-terms in $\circledast$:

$$\Box_c V_2 = \beta \partial_t^2(v^2) = \beta \partial_t^2(\varepsilon_1^2 v_1^2 + 2\varepsilon_1\varepsilon_2 v_1 v_2 + \varepsilon_2^2 v_2^2)$$

Defme $\quad v_2 = \dfrac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2} p \Big|_{\varepsilon_1=\varepsilon_2=0} = \dfrac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2} V_2 \Big|_{\varepsilon_1=\varepsilon_2=0}$

$$\Box_c v_2 = 2\beta \partial_t^2 v_1 v_2$$

Let p be the solution to the Westervelt equation with $f = \epsilon_1 f_1 + \epsilon_2 f_2$, and denote

$$U_2 = \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} p \mid_{\epsilon_1 = \epsilon_2 = 0} .$$

Then U_2 solves

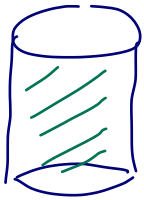
$$\begin{aligned} \Box_c U_2 - 2\beta(x)\partial_t^2(v_1 v_2) &= 0, && \text{in } (0,T)\times\Omega, \\ U_2 &= 0, && \text{on } (0,T)\times\partial\Omega, \\ U_2 = \partial_t U_2 &= 0, && \text{on } \{t=0\}, \end{aligned}$$

where v_j are linear solutions to

$$\begin{aligned} \Box_c v_j &= 0, && \text{in } (0,T)\times\Omega, \\ v_j &= f_j, && \text{on } (0,T)\times\partial\Omega, \\ v_j = \partial_t v_j &= 0, && \text{on } \{t=0\}. \end{aligned}$$

Let v_0 be the solution to the backward wave equation

$$\begin{aligned} \Box_c v_0 &= 0, && \text{in } (0,T)\times\Omega, \\ v_0 &= f_0, && \text{on } (0,T)\times\partial\Omega, \\ v_0 = \partial_t v_0 &= 0, && \text{on } \{t=T\}, \end{aligned}$$



Using integration by parts, we have

meas.

$$\int_0^T \int_{\partial\Omega} \partial_\nu U_2 f_0 \, dS \, dt = 2\int_0^T \int_\Omega \beta(x)\partial_t(v_1 v_2)\partial_t v_0 \, dx \, dt.$$

Note

$$\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Lambda_\beta(\epsilon_1 f_1 + \epsilon_2 f_2)\,|_{\epsilon_1=\epsilon_2=0} = \partial_\nu U_2|_{(0,T)\times\partial\Omega}.$$

.

$\Lambda_\beta(f) = \partial_\nu p|_{(0,T)\times\partial\Omega}$

Idea 2: construct linear waves

We would like to construct linear solutions

$$\begin{aligned} \Box_c v_j &= 0, && \text{in } (0,T)\times\Omega,\\ v_j &= f_j, && \text{on } (0,T)\times\partial\Omega,\\ v_j = \partial_t v_j &= 0, && \text{on } \{t=0\}, \end{aligned} \qquad \begin{aligned} \Box_c v_0 &= 0, && \text{in } (0,T)\times\Omega,\\ v_0 &= f_0, && \text{on } (0,T)\times\partial\Omega,\\ v_0 = \partial_t v_0 &= 0, && \text{on } \{t=0\}, \end{aligned}$$

such that we could recover $\beta(x)$ from

$$\int_0^T \int_\Omega \beta(x)\partial_t(v_1 v_2)\partial_t v_0 \, dx \, dt.$$

For nonlinear wave equations with time-dependent coefficients, to construct linear solutions, we consider

$\Box_g u = 0$

- distorted plane waves: choose $f \in I^{\mu}(\Lambda)$ and then construct $u \in I^{\mu-\frac{3}{2},-\frac{1}{2}}(\Lambda, \Lambda^g)$
- Gaussian beams: $u(t,x) = \underbrace{e^{i\rho\Theta(t,x)}a(t,x)}_{\text{principal}} + \underbrace{R_\rho(t,x)}_{\text{remainder}}$ with a large parameter ρ.

$\Box_g u = 0$

  - the principal part of u is supported near a null geodesic γ

$P = \partial_t^2 - \Delta$

Hörmander's theorem on propagation of singularities

Let $P \in \Psi^m(\mathbb{R}^n)$ with real principal symbol p_m homogeneous of degree m in ξ. Suppose

$$Pu = f,$$

WF(u)

then $\mathrm{WF}(u) \setminus \mathrm{WF}(f)$ is contained in the characteristic set $\mathrm{char}(\sigma(P))$ and is invariant under null bicharacteristics.

Here the integral curves $(x(t), \xi(t))$ of the Hamiltonian vector fields solve

$$\dot{x}(t) = \partial_\xi p_m(x(t), \xi(t)), \qquad \dot{\xi}(t) = -\partial_x p_m(x(t), \xi(t)).$$

These integral curves on which $p_m = 0$ are called the null bicharacteristics.

$(x(0), \xi(0)) \in \mathrm{WF}(u) / \mathrm{WF}(f)$

$\to (x(t), \xi(t)) \in \mathrm{WF}(u)$ if away from $\mathrm{WF}(f)$

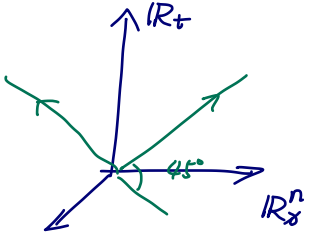
Example: $P = \partial_t^2 - \Delta$ on $\mathbb{R} \times \mathbb{R}^n$

The principal symbol is

$$\sigma(P)(t, x, \tau, \xi) = -\tau^2 + |\xi|^2.$$

The characteristic set is

$$\mathrm{char}(P) = \{(t, x, \tau, \xi) : \ \tau = \pm|\xi|\}.$$



Consider the Hamiltonian $H(t, x, \tau, \xi) = \frac{1}{2}(-\tau^2 + |\xi|^2)$.

Then the bicharacteristics are solutions to

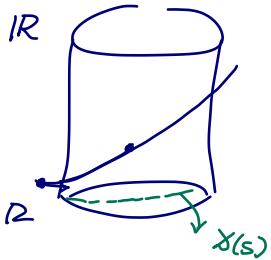
$$\dot{t}(s) = -\tau(s), \quad \dot{x}(s) = \xi(s), \quad \dot{\tau}(s) = 0, \quad \dot{\xi}(s) = 0,$$

and therefore when $|\xi_0| = 1$ we have

$$\rightsquigarrow (t(s), x(s), \tau(s), \xi(s)) = (t_0 \mp s, x_0 + s\xi_0, \pm 1, \xi_0).$$

Example: $P = \partial_t^2 - \Delta_{g_0}$ on $\mathbb{R} \times \Omega$

$Pu = f$



Here (Ω, g_0) is a 3-dimensional Riemannian manifold.
The principal symbol is

$$\sigma(P)(t, x, \tau, \xi) = -\tau^2 + |\xi|^2_{g_0}.$$

The characteristic set is

$$\mathrm{char}(P) = \{(t, x, \tau, \xi) : \ \tau = \pm|\xi|_{g_0}\}.$$

Consider the Hamiltonian $H(t, x, \tau, \xi) = \frac{1}{2}(-\tau^2 + |\xi|^2_{g_0})$.

The bicharacteristics are solutions to

$$\dot{t}(s) = -\tau(s), \quad \dot{x}(s) = g_0^{-1}\xi(s), \quad \dot{\tau}(s) = 0, \quad \dot{\xi}(s) = -\nabla_x(|\xi|^2_{g_0}).$$

Note that $x(s)$ satisfies the geodesic equation and when $|\xi_0|_{g_0} = 1$ they are unit speed geodesics on Ω.

Construction of Gaussian beams

Gaussian beams:

$$u(t,x) = e^{i\rho\Theta(t,x)}\underbrace{a(t,x)}_{\text{principal}} + \underbrace{R_\rho(t,x)}_{\text{remainder}}$$

with a large parameter ρ.

- the principal term of u is supported near a null geodesic γ
- Θ: phase function, complex-valued
- $R_\rho \to 0$ as $\rho \to +\infty$

Lorentzian case

Let (M, g) be an $1+3$-dimensional globally hyperbolic Lorentzian manifold, in the sense that $M = \mathbb{R} \times N$ and the metric tensor is of the form

$$g = -\alpha(t,x)dt^2 + \kappa(t,x),$$

where $\alpha > 0$ is a smooth function and $\kappa(t, \cdot)$ is a smooth family of Riemannian metrics on the 3-dimensional manifold N.
We consider the Laplace-Beltrami operator

$$\Box_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x^j} \right).$$

The null bicharacteristics are corresponding to the null geodesic flow in M.

Example

- $\mathbb{R}\times\mathbb{R}^3$: $\quad \Box = \partial_t^2 - \Delta = \Box_{g_e}$

  $g_e = -dt^2 + (dx)^2$

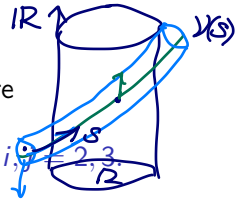
- $\mathbb{R}\times\Omega \qquad (\Omega, g_0) \qquad \Box_g = \Box = \partial_t^2 - \Delta_{g_0}$

  $g = -dt^2 + g_0$

$\rightsquigarrow \mathbb{R}\times\Omega \qquad \Box_c = \partial_t^2 - c^2\Delta \qquad g_0 = \frac{1}{c^2}(dx)^2$ on Ω

$g = -dt^2 + \frac{1}{c^2}(dx)^2 \qquad \Box_g \neq \Box_c \qquad$ but $\quad \sigma_p(\Box_g) = \sigma_p(\Box_c)$

For a null geodesic $\gamma : [0, s_0] \to M$,



- Pseudo-orthonormal frame: Let $\{e_0, e_1, e_2, e_3\}$ be a frame of $T_{\gamma(0)}$, where

$$e_0 = \dot{\gamma}(0), \quad g(e_1, e_1) = 0, \quad g(e_0, e_1) = -2, \quad g(e_i, e_j) = \delta_{ij}, \quad i, j = 2, 3.$$

  We consider its parallel transport $\{e_0(s), e_1(s), e_2(s), e_3(s)\}$ along $\gamma(s)$

- Fermi Coordinates: given by inverting the map

$$(s, y) \mapsto \exp_{\gamma(s)} \left(\sum_{k=1}^{n} y^k e_k(s) \right) \in M.$$

  Note that in this coordinates (s, y), we have $\gamma(s) = (s, 0)$ and

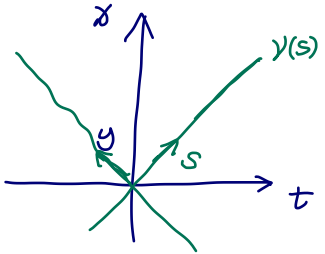
$$g|_\gamma = -2ds\, dy_1 + \sum_{k=2}^{n} dy_k\, dy_k.$$

Example $\mathbb{R}_t \times \mathbb{R}_x$ $\quad g_0 = -dt^2 + dx^2$

$= -(dt - dx)(dt + dx)$

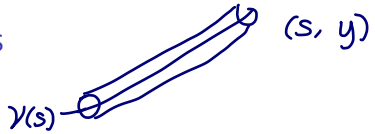
Set $s = \frac{1}{\sqrt{2}}(t+x)$, $\quad y = \frac{1}{\sqrt{2}}(t-x)$ $\qquad (t, x)$

$g_0 = -2 ds dy$



(s, y)

Gaussian Beams



Gaussian beam solutions are constructed as an WKB ansatz

$$u_\rho(s,y) = e^{i\rho\Theta(s,y)}a(s,y)$$

in a neighborhood of γ to approximately solve the equation $\Box_g u = 0$.
Here Θ is the so-called phase function and a is the so-called amplitude.
Note we have

$$\Box_g\left(e^{i\rho\Theta}a\right) = e^{i\rho\Theta}\left(\rho^2 g(d\Theta,d\Theta)a - 2i\rho g(d\Theta,da) + i\rho(\Box_g\Theta)a + \Box_g a\right).$$

Thus, we would like to choose Θ and a, such that the remainder is $\mathcal{O}(\rho^{-K})$ in $H^k([0,T]\times N)$ for a given $K \in \mathbb{N}$.

$(s, y) \quad \to g = -2dsdy + \sum_{k=2}^{n} dy_k dy_k$

We first construct approximate solution Θ to the Eikonal equation

$$\to g(d\Theta, d\Theta) = 0.$$

Then we plug in Θ to appropriately solve the amplitude a from some transport equation derived from

$$-2i\rho g(d\Theta, da) + i\rho(\Box_g \Theta)a + \Box_g a = 0.$$

Construction of Θ

To solve the Eikonal equation approximately, we consider

$$\Theta = \sum_{j=0}^{N} \Theta_j(s, y),$$

where $\Theta_j(s, y)$ is homogeneous of order j in $y \in \mathbb{R}^n$.
We would like to solve the Eikonal equation up to order N along γ, i.e.,

$$\partial_y^\alpha g(d\Theta, d\Theta)(s, 0) = 0, \quad \text{for any } |\alpha| \leq N$$

By choosing

$$\Theta_0 = 0 \quad \text{and} \quad \Theta_1 = y_1,$$

we have the Eikonal equation is satisfied up to order 1.

Construction of Θ

To solve the Eikonal equation up to order 2, we choose

$$\Theta_2(s,y) = \sum_{1\le i,j\le 3} H_{ij}(s) y^i y^j,$$

where H is a $n \times n$ symmetric matrix with $\mathrm{Im} H(s) > 0$ and

$$\frac{d}{ds}H + HCH + D = 0, \quad H(0) = H_0, \quad \text{with } \mathrm{Im} H_0 > 0,$$

where C, D are matrices with

$$C_{11} = 0,\ C_{ii} = 2,\ i = 2,3,\ C_{ij} = 0,\ i \neq j, \qquad \text{and} \qquad D_{ij} = \frac{1}{4}(\partial_i^2 g^{11}).$$

This Riccati equation can be uniquely solved and therefore one can solve the Eikonal equation up to order 2.

Construction of Θ

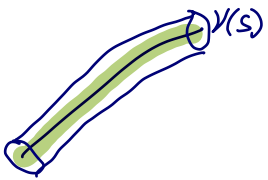
$N = 2$

More generally, we can solve the Eikonal equation up to order N, by solving more ODEs. Here N is an integer depending on our choice of k and K.

For such Θ that $g(d\Theta, d\Theta)$ vanishes to order N along γ, we can show it satisfies

$$g(d\Theta, d\Theta) \leq C|y|^{N+1}.$$

Constriction of a



Next, we plug the approximate solution Θ in the equation

$$-2i\rho g(d\Theta, da) + i\rho(\Box_g \Theta)a + \Box_g a = 0.$$

If the amplitude has an expansion

$$a = \chi(\frac{|y|}{\delta})(a_0 + \rho^{-1}a_1 + \rho^{-2}a_2 + \cdots + \rho^{-N}a_N),$$

then we have

$$\rightarrow \quad -2ig(d\Theta, da_0) + i(\Box_g \Theta)a_0 = 0,$$

$$\rightarrow \quad -2ig(d\Theta, da_j) + i(\Box_g \Theta)a_j - \Box_g a_{j-1} = 0,$$

for $j = 1, \ldots, N$.
We would like to solve these transport equations recursively in j starting from a_0.
Note such a is compactly supported near $\gamma(s)$.

Gaussian beams

$\Theta = \theta + y_1 + y \cdot H(s) y$ $N=2$

For $u_\rho(s,y) = e^{i\rho\Theta} a$ constructed above,

$$\Box_g \left(e^{i\rho\Theta} a\right) = e^{i\rho\Theta} \left(\rho^2 g(d\Theta, d\Theta)a - 2i\rho g(d\Theta, da) + i\rho(\Box_g \Theta)a + \Box_g a\right)$$

$$= e^{i\rho(y_1 + y\cdot H(s)y)} O(|y|^3)$$

$\mathrm{supp}(a)$ $|y| < \delta$

$$\leq C e^{-c\rho|y|^2} |y|^3$$

For sufficiently large N and k, we have

$$\|\Box_g(u_\rho)\|_{H^k([0,T]\times N)} \leq C\rho^{-K}, \quad K = \frac{N+1-k}{2} - 1.$$

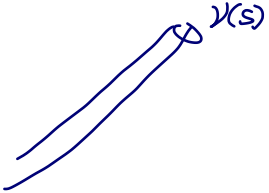
Then we construct a solution to $\Box_g u = 0$ by

$$u = e^{i\rho\Theta(s,y)} a(s,y) + R_\rho, \quad \|R_\rho\|_{H^{k+1}([0,T]\times N)} \leq C\rho^{-K}.$$

Choose v_1, v_2, v_0

$$v_2 = v_1 = e^{i\rho\theta} a_\rho + R_\rho$$

$$\Rightarrow v_0 = e^{i2\rho\bar{\theta}} a_{2\rho} + R_{2\rho}$$



$$v_1 v_2 v_0 = e^{-4\rho\, y\cdot \mathrm{Im}H(s)\, y}\, a_\rho^2\, a_{2\rho}$$

Then with $g = -\mathrm{d}t^2 + c^{-2}(\mathrm{d}x)^2$,

$$\int_0^T \int_\Omega \beta(x)\partial_t(v_1 v_2)\partial_t v_0 \, dx\, dt \approx v_1 v_2 v_0 = \text{weighted geodesic X-ray transform of } \beta(x) + O(\rho^{-1}).$$

Main Result I

Consider the Riemannian metric

$$g_0 = c^{-2}(\mathrm{d}x)^2.$$

We say (Ω, g_0) is non-trapping, if there exists $T > 0$ such that

$$\mathrm{diam}_{g_0}(\Omega) = \sup\{\text{lengths of all geodesics in } (\Omega, g_0)\} < T.$$

Westervelt type (Acosta-Uhlmann-Zhai, 2022)

Let (Ω, g_0) be non-trapping and $\partial\Omega$ be strictly convex w.r.t. g_0. Assume either no conjugate points or satisfying the foliation condition. Then the DN map Λ_β determines $\beta(x)$ for each $x \in \Omega$.

- Can we remove these geometric assumptions?
- Can we consider a more general nonlinearity?

General Nonlinear Acoustic Model

By [Kaltenbacher-Rundell, 2021], one can consider

$$\partial_t^2 p - c^2 \Delta p - \partial_t (f_w(x,p)\partial_t p) = 0.$$

If f_W is analytic in p, we can assume a general nonlinear term F given by

$$F(x,p,\partial_t p,\partial_t^2 p) = \sum_{m=1}^{+\infty} \beta_{m+1}(x)\partial_t^2(p^{m+1}),$$

where $\beta_{m+1}(x), m \geq 1$ are smooth.

- well-posedness: for small boundary sources $f \in C^6([0,T]\times\partial N)$
- consider the Dirichlet-to-Neumann map (DN map)

$$\Lambda_F : f = p|_{(0,T)\times\partial\Omega} \to \partial_\nu p|_{(0,T)\times\partial\Omega}$$

Main result II

Can we recover $\beta_{m+1}(x),\ m \geq 1$ from the DN map Λ_F?

Theorem (Uhlmann-Z, 2022)

Let (Ω, g_0) be nontrapping with strictly convex boundary. If the nonlinear term $F = \sum_{m=1}^{+\infty} \beta_{m+1}(x)\partial_t^2(p^{m+1})$ satisfies some nonvanishing condition, then the DN map Λ_F determines F.

Remark:

- allow conjugate points in Ω
- consider the Lorentzian m. $M = \mathbb{R} \times \Omega$ with $g = -\mathrm{d}t^2 + c^{-2}(\mathrm{d}x)^2$
- if the wave speed $c(x)$ is unknown, one can recover it from the first order linearization of the DN map

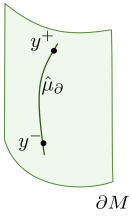
Main result II'

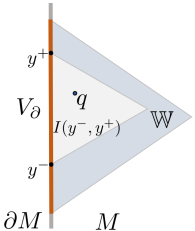
(M, g) globally hyperbolic, with timelike and strictly null-convex boundary

Theorem (Uhlmann-Z, 2022)

If the nonlinear term $F = \sum_{m=1}^{+\infty} \beta_{m+1}(t, x')\partial_t^2(p^{m+1})$ *satisfies some nonvanishing condition, then the DN map* Λ_F *determines* F*, in the set*

$$\mathbb{W} = \bigcup_{y^-, y^+ \in (0,T)\times\partial\Omega} I(y^-, y^+) \cap M^{int}.$$



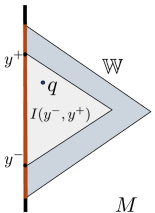


Main result III

$\Box_g p + \langle b, \nabla_{t,x} p\rangle + hp - \beta \partial_t^2(p^2) = 0$ in a global hyperbolic Lorentzian m. (M, g)

Theorem (Z, 2023)

With some assumptions on h, the DN map $\Lambda_{b,h,\beta}$ determines $b(t,x)$ and $\beta(t,x)$ up to a natural obstruction in the set $\mathbb{W}$.



- gauge invariance: for nonvanishing $\rho(x) \in C^\infty(\Omega)$ with $\rho|_{\partial\Omega} = 1$, set

$$b^\rho = b + 2\rho^{-1}\,\mathrm{d}\rho, \quad h^\rho = h + \langle b(x), \rho^{-1}\,\mathrm{d}\rho\rangle + \rho^{-1}c^2\Delta\rho, \quad \beta^\rho = \rho\beta,$$

  then $\Lambda_{b,h,\beta} = \Lambda_{b^\rho, h^\rho, \beta^\rho}$.

- without knowing h, one can recover b and β almost up to the gauge
- allows caustics and works for nonlinearity $= \sum_{m=1}^{+\infty} \beta_{m+1}(t,x)\partial_t^2(p^{m+1})$

Nonlinear Acoustic Model with strong damping

$$\underbrace{\partial_t^2 p - \Delta p - \partial_t \Delta p}_{\text{strongly damped wave op.}} + q(t,x)p - \beta(t,x)\partial_t^2(p^2) = 0, \quad \text{in } (0,T)\times\Omega,$$

$$p = f, \quad \text{on } (0,T)\times\partial\Omega,$$

$$p = \partial_t p = 0, \quad \text{on } \{t=0\},$$

- $\Omega \subset \mathbb{R}^n$: a bounded domain with smooth boundary, for $n \geq 2$
- $p(t,x)$: the pressure field of the ultrasound waves
- $q(t,x), \beta(t,x)$ are smooth

Forward problem: studied in many works [Pata, Squassina, Larsson, Thomee, Chidaglia, Burq, Christianson, Wang, ...]

Inverse problem:

- nonlinear model with memory: [Colombo 2007], [Colombo-Guidetti 2008]
- linear model: [Wu, 2011], [Kim-Moon-Seo, 2023], [Li-Z,2023]

Inverse problem

$$\begin{aligned} \partial_t^2 p - \Delta p + \partial_t(-\Delta)p + q(t,x)p - \beta(t,x)\partial_t^2(p^2) = 0, &\quad \text{in } (0,T)\times\Omega, \\ p = f, &\quad \text{on } (0,T)\times\partial\Omega, \\ p = \partial_t p = 0, &\quad \text{on } \{t=0\}, \end{aligned}$$

- well-posedness: for small compatible boundary sources f
- consider the Dirichlet-to-Neumann map (DN map)

$$\Lambda_{q,\beta} : f = p|_{(0,T)\times\Omega} \to \partial_\nu p|_{(0,T)\times\partial\Omega}.$$

Theorem (Li-Z, 2023)

The DN map $\Lambda_{q,\beta}$ determines $q(t,x)$ and $\beta(t,x)$, for any $(t,x) \in (0,T)\times\Omega$.

Use complex geometrical optics and fundamental solutions constructed by Hörmander.

Thank you very much!

Solvability of the Riccati equation

Lemma

The Riccati equation above has a unique solution. Moreover the solution H is symmetric and $Im(H(s)) > 0$. One has $H(\tau) = Z(\tau)Y(\tau)^{-1}$, where $Y(\tau)$ and $Z(\tau)$ solve the ODEs

$$\frac{d}{ds}Y(s) = CZ(s), \quad Y(s_-) = Y_0,$$
$$\frac{d}{ds}Z(s) = -D(s)Y(s), \quad Z(s_-) = H_0 Y_0.$$

In addition, $Y(\tau)$ is nondegenerate.