CHAPTER 1

School Mathematics and Teachers' Mathematics

A critical pillar of a strong PreK–12 education is a well-qualified teacher in every classroom. This report offers recommendations for the mathematical preparation and professional development of such teachers.

A second pillar is a challenging, world-class curriculum. In mathematics, the substance for this pillar is supplied by the Common Core State Standards (CCSS). These standards are created from progressions: sequences of topics and performances designed to respect the structure of mathematics and cognitive aspects of learning mathematics. This report focuses on teachers' knowledge of the mathematical aspects of these progressions: the sequences of topics and the mathematical structures that underlie these sequences.[1]

The CCSS also include standards for mathematical practice.[2] Their formulation was influenced by the National Council of Teachers of Mathematics process standards, the elements of mathematical proficiency described in the National Research Council report *Adding It Up*, and the discussions of the Park City Mathematics Standards Study Group.[3] Like their students, teachers need to have the varieties of expertise described in these standards—monitoring their own progress as they solve problems, attending to precision, constructing viable arguments, seeking and using mathematical structure, and making strategic use of appropriate tools, e.g., notations, diagrams, graphs, or procedures (whether implemented by hand or electronically). These abilities are supported by the mathematical "habits of mind" described in the original MET report.

At every grade level—elementary, middle, and high school—there is important mathematics that is both intellectually demanding to learn and widely used, such as reasoning strategies that rely on base-ten algorithms in elementary school; ratio, proportion, and exploratory statistics in middle school; algebra, geometry, and data analysis in high school. Teachers need to have more than a student's understanding of the mathematics in these grades. To support curricular coherence, teachers need to know how the mathematics they teach is connected with that of prior and later grades.[4] For example, an elementary teacher needs to know how the associative, commutative, and distributive properties are used together with place value in algorithms for addition and multiplication of whole numbers, and the

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[1] An overview of the CCSS structure appears as Appendix B of this report.

[2] The full text of these standards appears as Appendix C.

[3] Between 2004 and 2008, the Park City Mathematics Study Group (a group of research mathematicians) conducted discussions of school mathematics, including extended discussions with NCTM representatives. *Principles and Standards* and *Adding It Up* (published in 2000 and 2001) summarize findings from previous decades of research in mathematics education.

[4] Such connections are outlined in the Progressions for the CCSS (see the web resources for this report).

significance of these algorithms for decimal arithmetic in later grades. In the middle grades, a teacher needs to know how to build on this foundation; for instance, how to help students to extend these algorithms correctly to decimals and to use the distributive and other properties when adding and subtracting linear expressions. A high school teacher builds on the same ideas in teaching students about calculations with polynomials and other symbolic expressions.

Moreover, to appropriately create, select, or modify tasks, teachers need to understand the mathematical consequences of different choices of numbers, manipulative tools, or problem contexts.[5] They need to recognize the need for definitions (e.g., "What is a fraction?," "What does it mean to add two fractions?") and their consequences ("How do we know that the sum is unambiguously determined?"). Concepts may be defined differently in different resources being used, whether text-based or online (e.g., a trapezoid has at least one pair of parallel sides versus exactly one pair), and have different consequences (e.g., parallelograms are trapezoids—or not). Different assumptions also have different consequences. For example, in discussing properties of numbers ("Does 'number' mean 'whole number' or 'fraction'?"), in geometry ("Does this depend on the parallel postulate?"), or in modeling ("Is the flow uniform or not?").

Software, manipulatives, and many other tools exist to support teaching and learning. In order to use these strategically in teaching, and to help students use them strategically in doing mathematics, teachers need to understand the mathematical aspects of these tools and their uses. Teachers need the ability to find flaws in students' arguments, and to help their students understand the nature of the errors. Teachers need to know the structures that occur in school mathematics, and to help students perceive them.

The technical knowledge inherent in these examples implies that the profession of mathematics teaching requires a high level of expertise.[6] International and domestic studies suggest that an important factor in student success is a highly skilled teaching corps,[7] and that teachers' expertise is developed or hindered by institutional arrangements and professional practices.[8] Widespread expertise is aided by high standards for entry into the profession, and continual improvement of mathematical knowledge and teaching skills. Continual improvement can be promoted by regular interactions among teachers, mathematicians, and mathematics education faculty in creating and analyzing lessons, textbooks, and curriculum

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[5]Examples are given by Ma, *Knowing and Teaching Elementary Mathematics*, Erlbaum, 1999: changes in number, p. 74; change in manipulative and problem context, p. 5.

[6]For a summary (p. 400) and further examples of teaching tasks, see Ball et al., "Content Knowledge for Teaching," *Journal of Teacher Education*, 2008; also Senk et al., "Knowledge of Future Primary Teachers for Teaching Mathematics: An International Comparative Study," *ZDM*, 2012, p. 310.

[7]See, e.g., the findings of the Teacher Education and Development Study in Mathematics (TEDS-M).

[8]These are intertwined and occur on a variety of levels. For example, the institutional arrangement of having teachers share a room affords the professional practice of discussing mathematics. An institutionalized career hierarchy based on teaching shapes the professional activities of Chinese master teachers and "super rank" teachers described in *The Teacher Development Continuum in the United States and China*, National Academies Press, 2010. In Japan, institutional arrangements afford the practice of "lesson study," allowing teachers to communicate with other teachers in their school or district, and with policy-makers (see Lewis, *Lesson Study*, Research for Better Schools, 2002, pp. 20–22).

documents; and examining the underlying mathematics.[9] To support the spread of expertise in PreK–12 mathematics teaching, the mathematical education of teachers should become a central concern of the mathematics community. In particular, the mathematical education of teachers will need to become a central concern of more mathematicians and collegiate mathematics departments.

Current efforts to improve PreK–12 mathematics education in the United States recognize that school systems, communities, families, and teachers, as well as students themselves, all share responsibility for high-quality mathematics learning.[10] In a similar fashion, high-quality mathematical education of teachers is the responsibility of institutions of higher education, professional societies, accrediting organizations, and school districts, as well as PreK–12 teachers themselves. Their collective goal needs to be continual improvement in the preparation and further education of mathematics teachers.

This report describes the mathematical knowledge that teachers at different levels need. It puts special emphasis on professional development, because mathematical knowledge for teaching can and should continue to grow throughout a teacher's career. The report discusses the kinds of experiences that can create, extend, and deepen knowledge at each stage of a teacher's career:

    i. opportunities for beginning teachers;

    ii. increasing expertise for teachers with several years experience;

    iii. enhancing the skills of very experienced teachers.

Collegiate mathematics faculty members have vital roles to play in these experiences, and this report describes how they can contribute in productive ways.

Professional development should include self-directed study as well as activities that involve school-district mathematics supervisors and faculty in mathematics education and mathematics. To assist mathematics faculty with little experience in offering professional development opportunities for teachers, this report draws on the experiences of a range of professional development programs funded by the National Science Foundation and United States Department of Education's Math Science Partnerships, and other foundation- and public-sector-based initiatives. Interested readers are invited to learn more about these programs and contact program leaders for assistance in adopting and adapting the programs to their locations (see the web resources associated with this report).

Each different level of teacher education presents particular challenges for the education of mathematics teachers. Perhaps the most publicized challenges involve

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[9]Chapter 2 discusses this claim further, but note the findings of *Effects of Teacher Professional Development on Gains in Student Achievement*, Council of Chief State School Officers, 2009. Most successful professional development programs continued for 6 months or more, and the mean contact time with teachers was 91 hours.

[10]For example, the Mathematics Common Core Coalition (comprised of professional societies and assessment consortia) addresses educators, teachers, teacher leaders, supervisors, administrators, governors and their staffs, other policy-makers, and parents.

the education of elementary teachers. Like many undergraduates,[11] future elementary teachers may enter college with only a superficial knowledge of K–12 mathematics, including the mathematics that they intend to teach. For example, they may not know rationales for computations with fractions or the role of place value in base-ten algorithms, and may not have the opportunity to learn them as undergraduates.[12] Moreover, much that is useful to teachers is known about teaching–learning paths for early mathematics,[13] but, too often mathematicians who are new to this area lack the knowledge or resources to help future teachers develop an understanding of these paths and their mathematical stepping-stones.[14] After elementary teachers begin teaching, it is rare for them to have any sustained professional development centered on mathematics.[15] This report's recommendations for elementary teachers call for comprehensive professional development programs in mathematics coupled with more in-depth pre-service study of school mathematics. To do this, the recommended number of semester-hours of mathematics courses specifically designed for teachers is raised to 12 from the original MET Report's 9.

Far too frequently, middle grades teachers have the same preparation as elementary generalists.[16] This must stop. This report repeats the original MET Report's recommendation that grades 5–8 mathematics be taught by teachers who specialize in this subject and raises the recommended number of semester-hours in

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[11]The CBMS surveys (conducted every five years) consistently document large proportions of undergraduates enrolled in remedial mathematics courses (see, e.g., Table S.2 of the 2005 report).

[12]The 2005 CBMS survey suggests that many mathematics departments do not have courses especially designed for elementary teachers (see Table SP.6). In 2010, Masingila et al. surveyed 1,926 U.S. higher education institutions that prepared elementary teachers. Of those who responded (43%), about half (54%) reported that requirements included two mathematics courses designed for teachers. See "Who Teaches Mathematics Content Courses for Prospective Elementary Teachers in the United States? Results of a National Survey," *Journal of Mathematics Teacher Education*, 2012, Table 2. A more detailed picture for three states is presented by McCrory & Cannata, "Mathematics Classes for Future Elementary Teachers: Data from Mathematics Departments," *Notices of the American Mathematical Society*, 2011.

[13]Chapter 2 gives an overview of teaching–learning paths.

[14]In Masingila et al.'s survey less than half of respondents reported giving training or support to instructors of mathematics courses for elementary teachers.

[15]For example, when surveyed in 2000, 86% of K–4 teachers reported studying mathematics for less than 35 hours over a period of three years, an average of less than 12 hours per year. See Horizon Research's 2000 National Survey of Science and Mathematics Education. More recent studies show large increases in elementary student mathematics achievement when their teachers receive content-based professional development. Student scores of teachers who do not receive such professional development do not show these gains (see the sections on curriculum-specific professional development in Chapter 2 and on mathematics specialists in Chapter 4). Thus, unsatisfactory student performance may suggest a greater need for content-based professional development.

[16]The Association for Middle Level Education (AMLE) position statement notes, "in some states, virtually anyone with any kind of degree or licensure is permitted to teach young adolescents." According to the AMLE web site, 28 states and the District of Columbia offer separate licenses for middle grades generalists. Separate licenses, however, do not necessarily imply the existence of separate preparation programs or different mathematics requirements. The 2005 CBMS survey found that 56% of mathematics departments at four-year institutions had the same mathematics requirements for K–8 certification in early and later grades (see Table SP.5). See also the discussion of opportunity to learn for U.S. prospective lower secondary teachers in Tatto & Senk, "The Mathematics Education of Future Primary and Secondary Teachers: Methods and Findings from the Teacher Education and Development Study in Mathematics," *Journal of Mathematics Teacher Education*, 2011, p. 127.

mathematics to 24. All states need to institute certification programs for middle grades mathematics teachers.

Because many practicing middle grades mathematics teachers received certification by meeting expectations that were more appropriate for elementary teachers, opportunities for content-based professional development are needed that address their situation. This need is even more critical in the context of the increased expectations indicated by the CCSS.

Although high school mathematics teachers frequently major in mathematics, too often the mathematics courses they take emphasize preparation for graduate study or careers in business rather than advanced perspectives on the mathematics that is taught in high school. This report offers suggestions for rethinking courses in the mathematics major in order to provide opportunities for future teachers to learn the mathematics they need to know to be well-prepared beginning teachers who will continue to learn new mathematical content and deepen their understanding of familiar topics. As stated in MET I, "college mathematics courses should be designed to prepare prospective teachers for the life-long learning of mathematics, rather than to teach them all they will need to know." This viewpoint is especially important in the context of the greater sophistication and breadth of the mathematical expectations for high school students described by the CCSS.