Lecture 1: Introduction and motivation

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NSF-CBMS Conference: Representations of p-adic groups and noncommutative geometry

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Our objectives

1. to describe the category of smooth representations of p-adic groups, and, in particular, its irreducible objects, from a noncommutative geometry point of view: the ABPS Conjecture
2. to achieve an analogous description of the Galois side of the local Langlands correspondence
3. to match both descriptions in order to realize the explicit and categorical local Langlands correspondence.

Definitions

- A locally compact field is a field F with a locally compact Hausdorff topology, such that $(F, +)$ is a topological group, and that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ on $F^\times$ are both continuous.
- A local field is a locally compact field that is not discrete.

Topology

The topology on a local field F is always induced by an absolute value $|\ |_F \colon F \to \mathbb{R}_{\geq 0}$. Furthermore F is complete with respect to $|\ |_F$.

Let F be a nonarchimedean field The ring

$$\mathfrak{o}_F := \{x \in F \,:\, |x|_F \leq 1\}$$

is called the ring of integers of F,

$$\mathfrak{p}_F := \{x \in F \,:\, |x|_F < 1\}$$

is the unique maximal ideal of $\mathfrak{o}_F$, it is of the form $\mathfrak{p}_F = (\varpi_F)$, and $k_F := \mathfrak{o}_F/\mathfrak{p}_F$ is called the residue field of F. It is a finite field.
The normalized absolute value $|\ |_F$ is defined by $|\varpi_F|_F := q^{-1}$, where $q = |k_F|$.

Classification

Up to continuous isomorphisms, local fields are classified as follows:

- $\mathbb{R}$ and $\mathbb{C}$, equipped with the usual absolute values
- nonarchimedean local fields:
  - the fields $\mathbb{Q}_p := \mathbb{Z}_p[1/p]$ where p is a prime number or their finite extensions
  - the fields $\mathbb{F}_p((t)) := \mathbb{F}_p[[t]][1/t]$ of Laurent series in the variable t or their finite extensions

Notation

- F nonarchimedean local field
- G group of F-points of a connected reductive group (we will call G a p-adic reductive group)
- **Examples of such groups**: $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_n(F)$, with n a positive integer.

The general linear group

Taking the topology on F determined by the absolute value $|\ |_F$, the set $\mathrm{M}_n(F) \simeq F^{n^2}$ of $n \times n$-matrices with entries in F is given the product topology.
The determinant $\det\colon \mathrm{M}_n(F) \to F$ is a polynomial in the matrix entries, hence it is a continuous map.
The general linear group $\mathrm{GL}_n(F)$ is defined to be the inverse image of $F^\times = F \backslash \{0\}$ (an open subset of F) under the map det:

$$\mathrm{GL}_n(F) := \{g \in \mathrm{M}_n(F) \,:\, \det(g) \neq 0\}.$$

It is an open subset of $\mathrm{M}_n(F)$ and we give it the topology it inherits in this way.

Definition

A locally profinite group is a topological group which is Hausdorff, totally disconnected, and locally compact. A compact locally profinite group is called profinite.

Example: $\mathrm{GL}_n(F)$

The subgroup $\mathrm{GL}_n(\mathfrak{o}_F) = \det^{-1}(\mathfrak{o}_F^\times) \cap \mathrm{M}_n(\mathfrak{o}_F)$ is closed in $\mathrm{M}_n(\mathfrak{o}_F)$ hence compact, because $\mathfrak{o}_F^\times \subset \mathfrak{o}_F$ is closed and det is continuous. The p-adic group $\mathrm{GL}_n(F)$ is locally profinite, with a basis of compact, open neighborhoods given by

$$\mathrm{GL}_n(\mathfrak{o}_F) \supset \mathrm{Id} + \mathfrak{p}_F \, \mathrm{M}_n(\mathfrak{o}_F) \supset \mathrm{Id} + \mathfrak{p}_F^2 \, \mathrm{M}_n(\mathfrak{o}_F) \supset \mathrm{Id} + \mathfrak{p}_F^3 \, \mathrm{M}_n(\mathfrak{o}_F) \supset \cdots$$

Remark

Any p-adic group is locally profinite.

Representations of locally profinite groups

Definition

Let G be a locally profite group and let $C_c^\infty(G)$ denote the space of locally constant, compactly supported functions on G. We fix a left Haar measure m on G, i.e. a non-zero linear form on $C_c^\infty(G)$, which is invariant under left translation by G.

We define a product $*$, called convolution, on $C_c^\infty(G)$ by setting

$$f_1 * f_2(x) := \int_G f_1(g) f_2(g^{-1}x) d\mathrm{m}(g), \quad \text{for } f_1, f_2 \in C_c^\infty(G) \text{ and } x \in G.$$

This product gives $C_c^\infty(G)$ the structure of a $\mathbb{C}$-algebra, called the Hecke algebra of G and denoted $\mathcal{H}(G)$. This algebra has no unit element unless G is discrete.

Example

If G is discrete, then $\mathcal{H}(G) = \mathbb{C}[G]$ as $\mathbb{C}$-algebras.

Remark

The algebra $\mathcal{H}(G)$ has a unit if and only if G is discrete.

Definition

A representation of a locally profinite group G on a complex vector space V is called smooth if for every $v \in V$, its stabilizer $\{g \in G \ : \ \pi(g)(v) = v\}$ is open,

Proposition

Let G be a locally profinite group. Let (π, V) be a representation of G. The following conditions are equivalent:

1. (π, V) is smooth.
2. $V = \bigcup_{K \subset G} V^K$, where K runs through all compact open subgroups of G.
3. The action map $G \times V \to V$, $(g, v) \mapsto \pi(g)(v)$ is continuous, when V is endowed with the discrete topology and $G \times V$ with the product topology.

Proof

(1) $\Rightarrow$ (3): Let $(g, v) \in G \times V$. Then $g\mathrm{Stab}_G(v) \times \{v\}$ is an open neighborhood of (g, v) such that $\pi(g\mathrm{Stab}_G(v))(\{v\}) = \{gv\}$. Hence, the action map is continuous.

(3) $\Rightarrow$ (2): Let $v \in V$ and denote the action map by α. As $\alpha^{-1}(\{v\}) \subset G \times V$ is open, there exists an open compact subgroup K of G such that $\alpha(K \times \{v\}) \subset \{v\}$. In other words, $v \in V^K$.

(2) $\Rightarrow$ (1): Let $v \in V$. By assumption, there exists a compact open subgroup K of G such that $K \subset \mathrm{Stab}_G(v)$, and $\mathrm{Stab}_G(v) = \bigcup_{g \in \mathrm{Stab}_G(v)} gK$ is open.

□

Example

A group homomorphism $\chi\colon G \to \mathbb{C}^\times$ is called a *character*. Then $(\chi, \mathbb{C})$ is smooth if and only if $\mathrm{Ker}(\chi)$ is open. The trivial representation is the G-representation $(\mathbf{1}, \mathbb{C})$, where $\mathbf{1}(g) = 1$ for all $g \in G$.

An important tool to construct representations is the induction. There are two kinds of inductions that will play an important role for us: smooth induction and compact induction.

Smooth induction

Let G be a locally profinite group, let H be a closed subgroup of G and let (σ, W) be a smooth representation of H. The (smooth) induced representation $(\mathrm{Ind}_H^G \sigma, \mathrm{Ind}_H^G W)$ is defined as follows: $\mathrm{Ind}_H^G W$ is the space of functions $f\colon G \to W$ satisfying

1. $f(hg) = \sigma(h) f(g)$ for all $h \in H$, $g \in G$, and
2. there exists a compact open subgroup $K_f \subset G$ such that $f(gk) = f(g)$ for all $k \in K_f$.

The action of G on $\mathrm{Ind}_H^G W$ is via right translation:

$$(\mathrm{Ind}_H^G \sigma)(g)(f))(x) := f(xg) \quad \text{for all } g \in G,\ f \in \mathrm{Ind}_H^G W,\ x \in G.$$

Compact induction

The compact induction of (σ, W) from H to G is the subrepresentation $(\mathrm{c-Ind}_H^G\sigma, \mathrm{c-Ind}_H^G(W))$ of $(\mathrm{Ind}_H^G\sigma, \mathrm{Ind}_H^G W)$ consisting of functions $f \in \mathrm{Ind}_H^G W$ whose support has compact image in $H\backslash G$.

Representations of p-adic groups

Parabolic subgroups

Suppose now that P is a parabolic subgroup of a p-adic reductive group G with Levi decomposition $P = L \ltimes U$. For $G = \mathrm{GL}_n(F)$, a parabolic subgroup P is (up to conjugation) a subgroup of upper block-triangular matrices, U is formed by those matrices whose diagonal blocks are identity matrices, and L is the subgroup of block-diagonal matrices, a product of smaller $\mathrm{GL}_{N_i}(F)$.

Parabolic induction

Let P be a parabolic subgroup of G, with Levi factor L . The inflation of a representation σ of L to P is the unique representation of P which restricts to σ on L and trivial on U; we denote it as $\mathrm{infl}_L^P(\sigma)$. The parabolic induction functor is the following composition

$$\mathrm{i}_{L,P}^G \colon \mathfrak{R}(L) \xrightarrow{\inf_L^P} \mathfrak{R}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}(G).$$

Remark

Every locally compact topological group G has a left Haar measure m which is unique up to scalar. For any $x \in G$, $g \mapsto \mathrm{m}(xgx^{-1})$ is another left Haar measure, so there is $\delta_G(x) \in \mathbb{R}_+^\times$ such that $\mathrm{m}g) = \delta_G(x)\mathrm{m}(xgx^{-1})$. The map $\delta_G \colon G \to \mathbb{R}_+^\times$ obtained is a homomorphism and is called the modulus character. It is trivial on any compact subgroup, so that it is smooth if G contains a compact open subgroup. It is trivial iff G has a bi-invariant Haar measure. This happens, for example, if G reductive or if G compact.

Normalized parabolic induction:

If σ is a smooth presentation of L the normalized parabolic induction is $\mathrm{i}_L^G(\delta^{1/2}\sigma)$. It preserves unitarity.

Parabolic restriction

Let U be the unipotent radical of P. The parabolic restriction functor, or Jacquet (restriction) functor, is the following composition

$$\mathrm{r}_{L,P}^G \colon \mathfrak{R}(G) \xrightarrow{\mathrm{Res}_L^P} \mathfrak{R}(P) \xrightarrow{U\text{-coinvariants}} \mathfrak{R}(L).$$

Definition

A representation π of G is supercuspidal if $\mathrm{r}_{L,P}^G\pi = 0$ for every proper parabolic subgroup P of G.

Matrix coefficients

Let (π, V) be a smooth representation of G. For $v \in V$ and $\xi \in \widetilde{V}$, we call a matrix coefficient of π are the locally constant function $G \to G$

$$g \mapsto \langle \xi, \pi(g^{-1}v) \rangle.$$

Proposition

An irreducible smooth representation of G is supercuspidal if and only if its matrix coefficients are compactly supported modulo the center of G.

Theorem [Harish-Chandra]

1. Every smooth irreducible representation π of G embeds into the parabolic induction of an irreducible supercuspidal representation σ of a Levi subgroup L of G.
2. The G-conjugacy class of (L, σ) is uniquely determined and called the supercuspidal support of π.

Proposition

If σ is a smooth representation of L and π a smooth representation of G, then

$$\mathrm{Hom}_L(\mathrm{r}^G_{L,P}(\pi), \sigma) = \mathrm{Hom}_G(\pi, \mathrm{i}^G_{L,P}(\sigma)),$$

i.e., $\mathrm{r}_{L,P}$ is the left adjoint of $\mathrm{i}^G_{L,P}$.

Remark

It has been discovered by Casselman for admissible representations and generalized by Bernstein to arbitrary smooth representations that there is another non-obvious adjointness between the two functors. Namely, the parabolic restriction functor turns out to be also right adjoint to the parabolic functor with respect to the opposite parabolic (this result is known as the second or Bernstein adjointness theorem).

Geometrical Lemma [Bernstein–Zelevinsky; Casselman]

(It is one of the main tools in the classification of irreducible smooth representations in terms of parabolically induced representations.)
Given two parabolic subgroups P, Q of G, with respective Levi factors L, M, the composite functor $\mathrm{r}^G_{M,Q} \circ \mathrm{i}^G_{L,P}$ admits a filtration with subquotients of the form $\mathrm{i}^M_{M_w} \mathrm{Ad}(w) \mathrm{r}^L_{L_w}$, where w ranges over a set $W_{M,L}$ of coset representatives for $P \backslash G / Q$, and L_w, M_w are Levi factors of certain parabolic subgroups of L and M, respectively.

Proposition

The parabolic induction and parabolic restriction functors are both exact, and preserve the property of being finitely generated, so they induce canonical maps in Hochschild and cyclic homology.

Notation

The Hochschild and cyclic homology groups of an object C are related by an exact sequence

$$\ldots \to \mathrm{HC}_{n+1}(C) \to \mathrm{HC}_{n-1}(C) \to \mathrm{HH}_n(C) \to \mathrm{HC}_n(C) \to \mathrm{HC}_{n-2}(C) \to \ldots.$$

We will use the notation H(C) to refer to this sequence.

The Geometrical Lemma becomes the Mackey formula in homology

We consider the Hochschild and cyclic homology groups $\mathrm{HH}_*(\mathfrak{R}_{\mathrm{f}}(G))$ and $\mathrm{HC}_*(\mathfrak{R}_{\mathrm{f}}(G))$ associated to the category $\mathfrak{R}_{\mathrm{f}}(G)$ of finitely generated smooth complex representations of G. These two homology theories are related by the long exact sequence $\mathrm{H}(\mathfrak{R}_{\mathrm{f}}(G))$. The above filtration becomes a sum in homology:

$$\mathrm{r}^G_{M,Q} \circ \mathrm{i}^G_{L,P} = \sum_{w \in W_{M,L}} \mathrm{i}^M_{M_w} \mathrm{Ad}(w) \mathrm{r}^L_{L_w} : \mathrm{H}(\mathcal{H}(L)) \longrightarrow \mathrm{H}(\mathcal{H}(M)).$$

Definition

Let I be an Iwahori subgroup of G. The Iwahori-Hecke algebra of G, denoted by $\mathcal{H}(G//I)$, is the convolution algebra of compactly supported smooth functions on G

$$f\colon G \to \mathbb{C}, \quad \text{such that } f(i_1 g i_2) = f(g) \text{ for all } g \in G, \text{ and } i_1, i_2 \in I.$$

Theorem [Borel (1976)]

Let $\mathfrak{R}_I(G)$ denote the full subcategory of smooth representations of G that are generated by their I-fixed vectors, i.e., the representations (π, V) such that $V = \mathcal{H}(G)V^I$, with

$$V^I := \{v \in V \,:\, \pi(i)(v) = v \;\text{ for any } i \in I\}.$$

The functor $V \mapsto V^I$ is an equivalence between $\mathfrak{R}_I(G)$ and the category of $\mathcal{H}(G//I)$-modules.