Lecture 10: A Galois version of the Bernstein Center

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The cuspidality conjecture

The local Langlands correspondence restricts to a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(G).$$

State of art

The cuspidality conjecture is known to hold for all the Levi subgroups (including the groups themselves) of

- general linear groups and split classical p-adic groups [Moussaoui, 2017],
- inner forms of linear groups and of special linear groups, and quasi-split unitary p-adic groups [A-Moussaoui-Solleveld, 2018],
- the p-adic group G_2 [A-Xu, 2022],
- pure inner forms of quasi-split classical p-adic groups [A-Moussaoui-Solleveld, 2022].

Another example of extended quotient

$G^\vee$ and $L^\vee$ complex dual groups of G, L and $\Phi_{\mathrm{e,cusp}}(L)$ the set of $L^\vee$-conjugacy classes of cuspidal enhanced Langlands parameters for L. The group $W(L^\vee) := \mathrm{N}_{G^\vee}(L^\vee)/L^\vee$ acts on $\Phi_{\mathrm{e,cusp}}(L)$ and we can form the panoramic (spectral) Galois extended quotient:

$$\Phi_{\mathrm{e,cusp}}(L)/\!/W(L^\vee).$$

Conjecture

The local Langlands correspondence induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(L) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)/\!/W(L^\vee)$$

for any Levi subgroup L of G.

Slogan: The local Langlands correspondence matches the panoramic extended quotients.

Remark

The groups $W(L)$ and $W(L^\vee)$ are canonically isomorphic.

Property $C(L)$

There is a bijection $\mathfrak{L}_L \colon \mathrm{Irr}_{\mathrm{cusp}}(L) \xrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)$ such that

$$\mathfrak{L}_L \circ \mathrm{Ad}(w) = \mathrm{Ad}(w^\vee) \circ \mathfrak{L}_L \quad \text{for any } w \in W_G(L)$$

where $w \mapsto w^\vee$ denotes the canonical bijection $W(L) \to W(L^\vee)$.

Remark

(Cuspidality conjecture) $\Leftrightarrow$ (Property $C(G)$ is satisfied with $\mathfrak{L}_G = \mathrm{LLC}$).

A Langlands correspondence between panoramic extended quotients:

If Property $C(L)$ is satisfied for L a Levi subgroup of G, then $\mathfrak{L}_L$ matches the panoramic extended quotients, i.e., induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W_G(L) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)/\!/W_{G^\vee}(L^\vee).$$

Theorem [A-Moussaoui-Solleveld, 2018]

For any G, there is a bijection:

$$\Phi_{\mathrm{e}}(G) \xleftrightarrow{1-1} \bigsqcup_{L\in\mathfrak{L}(G)} (\Phi_{\mathrm{e,cusp}}(L)/\!/W_{G^\vee}(L^\vee))_{L\natural}.$$

Theorem [Solleveld, 2020]

For any G, there is a bijection:

$$\mathrm{Irr}(G) \xleftrightarrow{1-1} \bigsqcup_{L\in\mathfrak{L}(G)} (\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W_G(L))_{\natural}.$$

Corollary

If Property $C(L)$ is satisfied for all the Levi subgroups L of G (including $L = G$), and if the bijection $\mathfrak{L}_L$ is "compatible with the twists", then we get a bijection

$$\mathrm{Irr}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}(G).$$

Notation

- $\mathfrak{X}_{\mathrm{nr}}(L^\vee) := \{\zeta\colon W_F/I_F \to \mathrm{Z}^\circ_{L^\vee}\}$, which acts on the set of cuspidal enhanced L-parameters for L.
- $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})]_{G^\vee}$ the $G^\vee$-conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{\mathrm{nr}}(L^\vee) \cdot (\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}}))$, where $(\varphi_{\mathrm{cusp}}, \rho_{\mathrm{cusp}})$ is a cuspidal enhanced L-parameter for L Let $\mathfrak{B}^\vee(G)$ denote the set of such $\mathfrak{s}^\vee$.

Theorem [A-Moussaoui-Solleveld]

Let G be an arbitrary p-adic reductive group.
The set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of enhanced L-parameters for G is partitioned into series à la Bernstein as

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \tag{1}$$

where $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$: the fiber of $\mathfrak{s}^\vee$ under the map Sc.
Moreover, for every $\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)$, we have

$$\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \overset{1-1}{\longleftrightarrow} (\Phi_{\mathrm{e}}^{\mathfrak{s}_L^\vee}(L)/\!/W^{\mathfrak{s}^\vee})_{L^\natural}. \tag{2}$$

Theorem

If G is

- an inner form of $\mathrm{GL}_n(F)$ [A-Baum-Plymen-Solleveld, 2019],
- the exceptional group of type G_2 [A-Xu, 2022],
- a pure inner form of a quasi-split classical p-adic group [A-Moussaoui-Solleveld, 2022],

then, for every $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ and for $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, \mathrm{LLC}^L(\sigma)]_{G^\vee}$,

$$\mathrm{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} \mathrm{Irr}^{\mathfrak{s}_L}(L)/\!/W^{\mathfrak{s}} \xleftrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}_L^\vee}(L)/\!/W^{\mathfrak{s}^\vee} \xleftrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \qquad (3)$$

coincides with the LLC, and the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Irr}^{\mathfrak{s}}(G) & \xrightarrow[1-1]{\mathrm{LLC}} & \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \\ \downarrow{\scriptstyle \mathrm{Sc}} & & \downarrow{\scriptstyle \mathrm{Sc}} \\ \mathrm{Irr}^{\mathfrak{s}_L}(L) & \xrightarrow[\mathrm{LLC}]{1-1} & \Phi_{\mathrm{e}}^{\mathfrak{s}_L^\vee}(L) \end{array}$$

Theorem [A-Moussaoui-Solleveld, 2022]

If G a classical group (including the group $\mathrm{GSpin}_n(F)$), then the bijection

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G) \stackrel{1-1}{\longleftrightarrow} \bigsqcup_{\mathfrak{s}^\vee\in\mathfrak{B}^\vee(G)} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) = \Phi_{\mathrm{e}}(G)$$

induced by (3) coincides with the local Langlands correspondence that was constructed by Arthur.

Theorem [A-Moussaoui-Solleveld, 2018]

For every $\mathfrak{s} \in \mathfrak{B}^\vee(G)$, there exist a (twisted) extended affine Hecke algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ such that

$$\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \stackrel{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)).$$

Theorem: a categorical local Langlands correspondence

1. If G an inner form of $\mathrm{GL}_n(F)$ or a pure inner form of a quasi-split classical p-adic group [A.-Moussaoui-Solleveld], or
2. if G is the group G_2 with $p \neq 2, 3$ [A.-Xu]

then, for every $\mathfrak{s} \in \mathfrak{B}(G)$, we have

$$\mathfrak{R}^{\mathfrak{s}}(G) \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee))$$

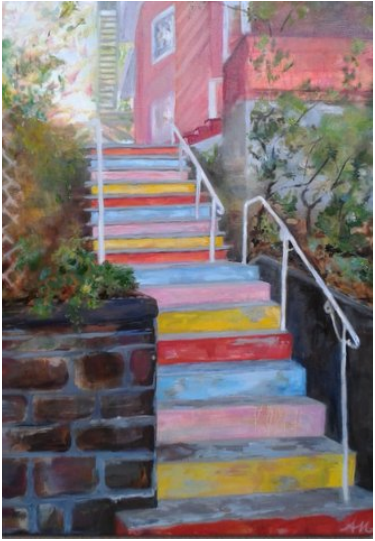
where

$$\mathfrak{s} \overset{\mathrm{LLC}^L}{\longleftrightarrow} \mathfrak{s}^\vee.$$

Hence, we have a categorical local Langlands correspondence:

$$\mathfrak{R}(G) \overset{\text{Morita}}{\sim} \prod_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \mathrm{Mod}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)).$$

Thank you very much for your attention!



From Staten Island (Oil painting 2016).