Lecture 2: Twisted extended affine and graded Hecke algebras

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Coxeter matrices

Let S be a finite set and let $(m_{s,s'})_{s,s'\in S^2}$ be a matrix (called a Coxeter matrix) with entries in $\mathbb{Z}_{\geq 1} \cup \{+\infty\}$ such that

- $m_{s,s} = 1$ for every $s \in S$
- $m_{s,s'} = m_{s',s} \geq 2$ for all $s, s' \in S$ such that $s \neq s'$.

Remark

The matrix $(m_{s,s'})_{s,s'\in S^2}$ is completely described by a graph (called a Coxeter graph) with set of vertices in bijection with S, where the vertices corresponding to $s \neq s'$ are joined by

- an edge if $m_{s,s'} = 3$
- a double edge if $m_{s,s'} = 4$
- a triple edge if $m_{s,s'} = 6$
- a quadruple edge if $m_{s,s'} = +\infty$

Definition

Let W be the group defined by the generators s ($s \in S$) and relations

$$(ss')^{m_{s,s'}} = 1 \text{ for any } s, s' \in S \text{ such that } m_{s,s'} < +\infty.$$

The group W is called a Coxeter group and the pair (W, S) a Coxeter system.

Remark

- In W we have $s^2 = 1$ (for every $s \in S$). These relations are called the quadratic relations.
- The equalities

  $$(ss')^{m_{s,s'}} = 1$$

  or equivalently

  $$\underbrace{ss'ss'\cdots}_{m_{s,s'} \text{ terms}} = \underbrace{s'ss's\cdots}_{m_{s,s'} \text{ terms}}$$

  are called the braid relations.

Example

$W = S_n$ the symmetric group acting by permutation on the set $\{1, 2, \ldots, n\}$, and $S = \{(12), (23), \ldots, (n-1n)\}$. The group W is said to be of type A_{n-1}.

Example

$W = \{\pm 1\} \rtimes S_n$, and $S = \{(12), (23), \ldots, (n-1n), (\mathrm{id}, (1, \ldots, 1, -1))\}$. The group W is said to be of type B_n or C_n.

Definition

Let (W, S) be a Coxeter system. For $w \in W$ let $\ell(w)$ be the smallest integer m such that

$$w = s_1 s_2 \cdots s_m \quad \text{with } s_1, s_2, \ldots, s_m \in S.$$

We then say that $w = s_1 s_2 \cdots s_m$ is a reduced expression and call $\ell(w)$ the length of w.

Remark

By definition, we have $\ell(1) = 0$, and $\ell(s) = 1$ for every $s \in S$. One can check that for every $w \in S$ and $s \in S$, we have $\ell(sw) = \ell(w) \pm 1$ and similarly $\ell(ws) = \ell(w) \pm 1$.

Deformation of the quadratic relations

An affine Hecke algebra is a deformation of the group algebra of an affine Coxeter group (W, S): we keep the braid relations of W, but replace the quadratic relations

$$s^2 - 1 = (s-1)(s+1) = 0 \quad (s \in S)$$

by relations of the form

$$(s - q_s)(s+1) = 0 \quad (s \in S)$$

where $q_s \in \mathbb{C}^\times$. That gives rise to an associative algebra.

Definition

Let (W, S) be a Coxeter group, equipped with a function $\mathbf{q}\colon s \mapsto q_s$ from S to $\mathbb{C}$ satisfying

$$q_s = q_{s'} \quad \text{if } s \text{ and } s' \text{ are conjugate under } W.$$

There is a unique algebra structure $\mathcal{H}(W, \mathbf{q})$ on the $\mathbb{C}$-vector space spanned by elements T_w $(w \in W)$ such that

(1) $T_1 = 1$,

(2) $(T_s - q_s)(T_s + 1) = 0$ for any $s \in S$ (quadratic relations),

(3) $T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots$ where both sides have $m_{s,s'}$ elements (braid relations),

(4) $T_{ww'} = T_w T_{w'}$ if $\ell(ww') = \ell(w) + \ell(w')$.

The algebra $\mathcal{H}(W, \mathbf{q})$ is called the Hecke-Iwahori algebra of $(W, \mathbf{q})$. .

Another basis

Let

$$C_w := \mathbf{v}^{\ell(w)} \sum_{y \leq w} P_{y,w}(\mathbf{v}^2)\, T_y$$

where $P_{y,w}$ are the Kazhdan–Lusztig polynomials.
Then the elements C_w for $w \in \widetilde{W}$ form an $\mathcal{A}$-basis of $\mathcal{H}$, called the Kazhdan-Lusztig basis of $\mathcal{H}$.

Definition

A root datum is a 4-tuple $\mathcal{R} = (X, Y, R, R^\vee)$ such that X, Y are free abelian groups of finite rank together with a perfect pairing

$$\langle\ ,\ \rangle \colon X \times Y \to \mathbb{Z},$$

$R \subset X \backslash \{0\}$, $R^\vee \subset Y \backslash \{0\}$ are the finite sets of roots and coroots respectively, in bijection $\alpha \leftrightarrow \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$,

- for every $\alpha \in R$, the reflection

  $$s_\alpha \colon X \to X, \quad s_\alpha(x) := x - \langle x, \alpha^\vee \rangle,$$

  stabilizes R,
- for every $\alpha \in R^\vee$, the reflection

  $$s_{\alpha^\vee} \colon Y \to Y, \quad s_{\alpha^\vee}(y) := y - \langle \alpha, y \rangle,$$

  stabilizes $R^\vee$.

Examples

1. SL_2: $X = \mathbb{Z} \supset R = \{\pm 2\}$, $Y = \mathbb{Z} \supset R^\vee = \{\pm 1\}$
2. PGL_2: $X = \mathbb{Z} \supset R = \{\pm 1\}$, $Y = \mathbb{Z} \supset R = \{\pm 2\}$.

We see that SL_2 and PGL_2 are interchanged by swapping roots and coroots.

Definitions

The finite Weyl group associated to the root datum $\mathcal{R}$ is

$$W_{\mathrm{f}} := \langle s_\alpha \; : \; \alpha \in R \rangle.$$

Then W_{f} acts on both X and Y. The extended affine Weyl group associated to $\mathcal{R}$ is

$$\widetilde{W} := W_{\mathrm{f}} \ltimes \mathbb{Z}X.$$

Example

Let $X = Y = \mathbb{Z}$, $R = \{\pm 1\}$ and $R^\vee = \{\pm 2\}$. Then $\widetilde{W} = S_2 \ltimes \mathbb{Z}$, an infinite dihedral group.

Fix $q \in \mathbb{R}_{>1}$ and let $\lambda, \lambda^* \colon R \to \mathbb{C}$ be functions such that

- if $\alpha, \beta \in R$ are W-associate, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,
- if $\alpha^\vee \notin 2Y$, then $\lambda^*(\alpha) = \lambda(\alpha)$.

Remark

$\alpha^\vee \in 2Y$ is only possible for short roots α in a type B component of R.

Notation

For $\alpha \in R$ we set

$$q_{s_\alpha} := q^{\lambda(\alpha)} \quad \text{and, if } \alpha^\vee \in R^\vee_{\max},\ q_{s'_\alpha} := q^{\lambda^*(\alpha)}, \tag{1}$$

where $R^\vee_{\max}$ is a certain set of "maximal elements" of $R^\vee$.

Definition

The affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$ is the vector space $\mathcal{H}(W_{\mathrm{f}}, q) \otimes_{\mathbb{C}} \mathbb{C}[X]$ with the with the multiplication rules:

- $\mathcal{H}(W_{\mathrm{f}}, q)$ and $\mathbb{C}[X]$ are embedded as subalgebras;
- for $\alpha \in \Pi$ (a basis of R) and $x \in X$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} =$$

$$\Big((q^{\lambda(\alpha)} - 1) + \theta_{-\alpha}(q^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - q^{(\lambda(\alpha)-\lambda^*(\alpha))/2})\Big)\frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}},$$

  where $\{\theta_x \,:\, x \in X\}$ is a basis of $\mathbb{C}[X]$.

It is an associative algebra with unit element $T_1 \otimes \theta_0$.

We say that $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$ has equal parameters if

$$\lambda(\alpha) = \lambda(\beta) = \lambda^*(\alpha) = \lambda^*(\beta) \quad \text{for all } \alpha, \beta \in R,$$

and then we denote it simply by $\mathcal{H}(\mathcal{R}, q)$.

Motivation

The geometric information contained in the affine Hecke algebra can be recovered from the corresponding graded Hecke algebra. Graded Hecke algebras are simplified versions of affine Hecke algebras, more or less in the same way that a Lie algebra is a simplification of a Lie group.

The actors

- $\mathfrak{a}$ finite dimensional real inner product space
- $\mathfrak{a}^*$ linear dual of $\mathfrak{a}$
- $\mathfrak{t}$ and $\mathfrak{t}^*$ complexifications of $\mathfrak{a}$ and $\mathfrak{a}^*$
- $S(\mathfrak{t}^*)$ symmetric algebra of $\mathfrak{t}^*$
- R (reduced) root system in $\mathfrak{a}^*$, and $W_{\mathfrak{f}} := W(R)$ Weyl group of R
- $R^\vee$ dual root system in $\mathfrak{a}$
- Π basis of R, and $S := \{s_\alpha \,:\, \alpha \in \Pi\}$ set of simple reflections in W
- for every $\alpha \in \Pi$, let $k_\alpha \in \mathbb{C}$ such that $k_\alpha = k_\beta$ if α and β are W-conjugate.

Definition

Let $\widetilde{R} := (\mathfrak{a}^*, R, \mathfrak{a}, R^\vee, \Pi)$. Let $\mathbb{H}(\widetilde{R})$ be the complex vector

$$\mathbb{H}(\widetilde{R}) := \mathbb{C}[W_{\mathrm{f}}] \otimes S(\mathfrak{t}^*).$$

Define multiplication in $\mathbb{H}(\widetilde{R})$ by the following rules

- $\mathbb{C}[W_{\mathrm{f}}]$ and $S(\mathfrak{t}^*)$ are canonically embedded as subalgebras
- for every $\lambda \in \mathfrak{t}^*$ and $\alpha \in \Pi$:

$$\lambda \cdot s_\alpha = s_\alpha \cdot s_\alpha(\lambda) + k_\alpha \langle \lambda, \alpha^\vee \rangle.$$

Then $\mathbb{H}(\widetilde{R})$ is an algebra, which is called a graded Hecke algebra.

Remark

- We define a grading on $\mathbb{H}(\widetilde{R})$ by requiring that W_{f} has degree zero, and $\mathfrak{t}^*$ has degree one.
- Graded Hecke algebras have more diverse applications than affine Hecke algebras. They appear in the representation theory of reductive groups over local fields, both in the p-adic case and in the real case. Further, they can be realized with Dunkl operators, which enables them to act on many interesting function spaces.