Lecture 3: The Bernstein Center

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Notation

- G: a p-adic group
- L Levi factor of a parabolic subgroup P of G
- $\mathfrak{R}(G)$: the category of all smooth representations of G
- $\mathfrak{X}_{\mathrm{nr}}(L)$: the group of unramified characters of L
- σ an irreducible supercuspidal smooth representation of L.

A few recollection from Lecture 1:

Parabolic induction

The inflation of a representation σ of L to P is the unique representation of P which restricts to σ on L and trivial on U; we denote it as $\mathrm{infl}_L^P(\sigma)$. The parabolic induction functor is the following composition

$$\mathrm{i}_{L,P}^G \colon \mathfrak{R}(L) \xrightarrow{\mathrm{inf}_L^P} \mathfrak{R}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}(G).$$

Parabolic restriction

Let U be the unipotent radical of P. The parabolic restriction functor, or Jacquet (restriction) functor, is the following composition

$$r^G_{L,P} \colon \mathfrak{R}(G) \xrightarrow{\mathrm{Res}^P_L} \mathfrak{R}(P) \xrightarrow{U\text{-coinvariants}} \mathfrak{R}(L).$$

Definition

A smooth representation π of G is supercuspidal if $r^G_{L,P}(\pi) = 0$, for any proper parabolic subgroup P of G.

Proposition

Let π be a smooth representation of G. The following conditions are equivalent

(1) the representation π is supercuspidal

(2) the representation π not a subquotient of any proper parabolically induced representation.

(3) every matrix coefficient of π has compact modulo center support.

Theorem [Harish-Chandra]

Every smooth irreducible representation π of G occurs as an irreducible component of a parabolically induced representation $\mathrm{i}_{L,P}^{G}(\sigma)$, where P is a parabolic subgroup of G with Levi factor L and σ is supercuspidal irreducible smooth representation of L.

The G-conjugacy class $(L,\sigma)_G$ of (L,σ) is uniquely determined and is called the supercuspidal support of π, it is denoted by $\mathrm{Sc}(\pi)$.

Bernstein's idea: grouping together "close" supercuspidal suppports

Definition

Let $\sigma, \sigma' \in \mathrm{Irr}(L)$. We say that σ and σ' are close if $\sigma' = \chi \cdot \sigma$ for some $\chi \in \mathfrak{X}_{\mathrm{nr}}(L)$.

Notation

We denote by

- $\mathfrak{s}$ the G-conjugacy class of the pair $(L, \mathfrak{X}_{\mathrm{nr}}(L)\cdot\sigma)$. We write $\mathfrak{s} = [L,\sigma]_G$ (We set $\mathfrak{s}_L := (L, \mathfrak{X}_{\mathrm{nr}}(L)\cdot\sigma)_L$.)
- $\mathfrak{B}(G)$ the set of such classes $\mathfrak{s}$. It is countable infinite.

Definition

Let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every G-subquotient of π has its supercuspidal support contained in $\mathfrak{s}$.

Theorem [Bernstein]

The category $\mathfrak{R}(G)$ is a direct product:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G)$$

of the full subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$

Questions that we will be interested in

1. What do the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$ look like? Is always $\mathfrak{R}^{\mathfrak{s}}(G)$ the module category of an explicit algebra (as it is for $\mathfrak{s} = [T, \mathrm{triv}]_G$) ?
2. Can we classify $\mathrm{Irr}^{\mathfrak{s}}(G)$?
3. How can we describe unitary/tempered/square-integrable representations in $\mathfrak{R}^{\mathfrak{s}}(G)$?
4. How reflects the Bernstein decomposition in other settings (e.g., noncommutative geometry, Langlands correspondence)?

The group $W^{\mathfrak{s}}$

Let $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$. The group $\mathrm{N}_G(L)$ acts on $\mathfrak{R}(L)$ by

$$(n \cdot \sigma)(l) := \sigma(n^{-1} l n).$$

Set $\mathcal{O} := \mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma$. We define

$$W^{\mathfrak{s}} := \mathrm{N}_G([L, \sigma]_L)/L = \{n \in \mathrm{N}_G(L) \ : \ n \text{ stabilzes } \mathcal{O}\}/L.$$

Proposition

We have

$$W^{\mathfrak{s}} = W_0^{\mathfrak{s}} \rtimes \Gamma^{\mathfrak{s}}$$

where $W_0^{\mathfrak{s}}$ is a finite Weyl group (with roots system denoted by $R^{\mathfrak{s}}$) and $\Gamma^{\mathfrak{s}}$ is a finite group (sending positive roots in $R^{\mathfrak{s}}$ to positive roots).

Center of $\mathfrak{R}^{\mathfrak{s}}(G)$

For $\mathfrak{s} = [L, \sigma]_G$, let $\mathbb{C}[\mathcal{O}]$ be the ring of regular functions on $\mathcal{O}$ (viewed as a complex torus).
The center of $\mathfrak{R}^{\mathfrak{s}}(G)$ is

$$\mathfrak{Z}(\mathfrak{R}^{\mathfrak{s}}(G)) = \mathbb{C}[\mathcal{O}]^{W^{\mathfrak{s}}}.$$

The Hecke algebra of G

We fix a Haar measure on G, and recall that $\mathcal{H}(G)$ denotes the space of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$, that we view as a $\mathbb{C}$-algebra via convolution relative to the Haar measure. We refer to $\mathcal{H}(G)$ as the Hecke algebra of G.

Idempotented algebras

Let A be an associative algebra. Let $E(A)$ denotes the set of idempotents in A (i.e. the elements $e \in A$ such that $e^2 = e$). We define a partial order on $E(A)$ by setting

$$e_1 \leq e_2 \quad \text{if } e_1Ae_1 \subset e_2Ae_2.$$

For every pair of idempotents $e_1 \leq e_2$, we have an inclusion of algebras $e_1Ae_1 \subset e_2Ae_2$.
The algebra A is said to be idempotented if A is the inductive limit of algebras eAe for $e \in E(A)$ ranging over the ordered set of idempotents of A.

Remark

If A is unital, the unit of A is obviously the maximal element of $E(A)$, and therefore A is idempotented.

Definition

Let A be an idempotented algebra. A A-module M is said to be nondegenerate if

$$M = \bigcup_{e \in E(A)} eM.$$

Notation

For every compact open subgroup K of G, let $e_K \in \mathcal{H}(G)$ be the function defined

$$e_K(x) := \begin{cases} \frac{1}{\mathrm{vol}(K)} & \text{if } x \in K \\ 0 & \text{if } x \in G,\ x \notin K. \end{cases}$$

Remark

A $\mathcal{H}(G)$-module M is nondegenerate if and only if

$$M = \bigcup_K e_K M$$

the union ranging over all compact open subgroups K of G, for every idempotent $e \in E(\mathcal{H}(G))$ is dominated by some e_K.

Proposition

The category $\mathfrak{R}(G)$ of smooth representations of G is equivalent to the category of of nondegenerate $\mathcal{H}(G)$-modules.

The Bernstein decomposition of $\mathcal{H}(G)$

By letting G act on $\mathcal{H}(G)$ by left translation, we obtain a decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}(G)$, that are Morita equivalent to unital, Noetherian algebras of finite global dimension, and $\mathfrak{R}(\mathcal{H}(G)^{\mathfrak{s}}) \cong \mathfrak{R}^{\mathfrak{s}}(G)$.

Proposition [Bushnell-Kutzko]

For every $\mathfrak{s} \in \mathfrak{B}(G)$, there exists an idempotent $e^{\mathfrak{s}} \in \mathcal{H}(G)$ sucth that

- $\mathcal{H}(G)^{\mathfrak{s}} = \mathcal{H}(G)e^{\mathfrak{s}}\mathcal{H}(G)$
- $\mathfrak{R}^{\mathfrak{s}}(G) \cong \mathfrak{R}(e^{\mathfrak{s}}\mathcal{H}(G)e^{\mathfrak{s}})$.

Theorem [Crisp]

We have

$$\mathrm{H}(\mathfrak{R}_{\mathrm{f}}(G)) \simeq \mathrm{H}(\mathcal{H}(G))$$

where the right-hand side is the Hochschild and cyclic homology of the Hecke algebra of G.

Proof

For A a Noetherian algebra of finite global dimension, Keller proved in ["On the cyclic homology of exact categories", J. Pure Appl. Algebra **136**: 1, (1999), 1–56] that

$$\mathrm{H}(A) = \mathrm{H}(\mathfrak{R}_{\mathrm{f}}(A)).$$

The functor H commutes with direct sums and is Morita invariant, so the result follows. □

Intertwining algebras

Let K be an open compact subgroup of G, and (λ, V) a smooth irred. rep. of K. Let $(\lambda^\vee, V^\vee)$ be the contragredient of (λ, V). We define $\mathcal{H}(G, \lambda)$ to be the space of compactly supported functions $f\colon G \to \mathrm{End}_G(V^\vee)$ such that

$$f(kgk') = \lambda^\vee(k)f(g)\lambda^\vee(k'), \quad \text{where } k, k' \in K \text{ and } g \in G.$$

The convolution product gives $\mathcal{H}(G, \lambda)$ the structure of a unitary associative $\mathbb{C}$-algebra.

Notation

Let $e_\lambda \in \mathcal{H}(G)$ be the function defined by

$$e_\lambda(x) := \begin{cases} \frac{\dim \lambda}{\mathrm{vol}(K)} \mathrm{tr}(\lambda(x^{-1})) & \text{if } x \in K \\ 0 & \text{if } x \in G,\ x \notin K. \end{cases}$$

Then e_λ is idempotent, and $e_\lambda * \mathcal{H}(G) * e_\lambda$ is a sub-algebra of $\mathcal{H}(G)$ with unit e_λ. **Note:** If $\lambda = \mathrm{triv}$, then $e_\lambda = e_K$ and $\mathcal{H}(G, \lambda) = \mathcal{H}(G//K)$.

Remark

There is a canonical isomorphism of unital algebras:

$$\mathcal{H}(G,\lambda) \otimes_{\mathbb{C}} \mathrm{End}(V) \simeq e_\lambda * \mathcal{H}(G) * e_\lambda.$$

Definition [Bushnell-Kutzko]

The pair (K,λ) is an $\mathfrak{s}$-type for G if the following property is satisfied

$$\lambda \text{ occurs in the restriction of } \pi \in \mathrm{Irr}(G) \text{ if and only if } \pi \in \mathfrak{s}.$$

Example

For I an Iwahori subgroup of G, the pair (I, triv) is an $\mathfrak{s}$-type for $\mathfrak{s} = [T, \mathrm{triv}]_G$, where T a maximal torus of G, and we have $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}_I(G)$.

Theorem [Bushnell-Kutzko]

If (K,λ) is an $\mathfrak{s}$-type, then we can take $e^{\mathfrak{s}} = e_\lambda$.

The GL_n-case

Let $G = \mathrm{GL}_n(F)$, $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ and $\mathfrak{s}_L := [L, \sigma] \in \mathfrak{B}(L)$. We have

$$L = \mathrm{GL}_{n_1}(F)^{e_1} \times \cdots \times \mathrm{GL}_{n_t}(F)^{e_r}$$

$$\sigma = \sigma_1^{\otimes e_1} \otimes \sigma_2^{\otimes e_2} \otimes \cdots \otimes \sigma_t^{\otimes e_t}$$

with σ_1, ..., σ_t pairwise distinct (after unramified twist).
Each representation σ_i is irreducible supercuspidal, and has a torsion number r_i: the order of the cyclic group of all those unramified characters $\chi \in \mathfrak{X}(\mathrm{GL}_{n_i}(F))$ for which $\sigma_i \otimes \chi \simeq \sigma_i$.
The representation σ_i contains an $\mathfrak{s}_i$-type (K_i, λ_i) in $\mathrm{GL}_{n_i}(F)$ with $\mathfrak{s}_i = [\mathrm{GL}_{n_i}, \sigma_i]_{\mathrm{GL}_{n_i}(F)}$. Let

$$K_L := K_1^{e_1} \times K_2^{e_2} \times \cdots \times K_t^{e_t} \quad \text{and} \quad \lambda = \lambda_1^{\otimes e_1} \otimes \lambda_2^{\otimes e_2} \otimes \cdots \otimes \lambda_t^{\otimes e_t}.$$

Then (K_L, λ_L) is an $\mathfrak{s}_L$-type.

The GL_n-case (continued)

We have

$$W^{\mathfrak{s}} = W_0^{\mathfrak{s}} = \prod_{i=1}^{t} S_{e_i}.$$

Theorem [Bushnell-Kutzko]

Let $G := \mathrm{GL}_n(F)$. For every $\mathfrak{s} \in \mathfrak{B}(G)$, there exists an $\mathfrak{s}$-type (K, λ) and an isomorphism of unital $\mathbb{C}$-algebras

$$\mathcal{H}(G, \lambda) \simeq \bigotimes_{i=1}^{t} \mathcal{H}(S_{e_i}, q_F^{r_i}),$$

where $\mathcal{H}(S_{e_i}, q_F^{r_i})$ is the Iwahori-Hecke algebra of $\mathrm{GL}_{e_i}(E_i)$ with E_i/F an unramified extension of degree r_i.
Moreover, these isomorphisms are compatible with twisting, parabolic induction, and preserving tempered representations.

Corollary

For $G = \mathrm{GL}_n(F)$, and every $\mathfrak{s} \in \mathfrak{B}(G)$, we have

$$\mathfrak{R}^{\mathfrak{s}}(G) \overset{\text{Morita}}{\sim} \mathrm{Mod}\left(\bigotimes_{i=1}^{t} \mathcal{H}(S_{e_i}, q_F^{r_i})\right).$$

Conclusion

The study of the category $\mathfrak{R}(\mathrm{GL}_n(F))$ reduces to that of categories of modules of affine Hecke algebras.

More generally, the study of the category of smooth representations of an arbitrary p-adic groups reduces to that of categories of modules of generalized affine Hecke algebras.

We will see more examples later.