Lecture 4: Extended quotients

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Geometric extended quotient

Let X be a space and Γ a (finite) group acting on X. For $x \in X$, let $\Gamma_x \subset \Gamma$ be the fixator of x:

$$\Gamma_x := \{\gamma \in \Gamma \,:\, \gamma \cdot x = x\}.$$

The quotient X/Γ of X by (the action of) Γ is the set Γ-orbits in X. We replace X by a bigger space $\widetilde{X}$ on which Γ is still acting, and take the quotient of $\widetilde{X}$ by Γ: the geometric extended quotient of X by Γ is the quotient

$$(X//\Gamma)_{\rm geo} := \{(x, \gamma) \,:\, x \in X, \gamma \in \Gamma_x\}/\Gamma.$$

Notation

For $\gamma \in \Gamma$, we set

$$X^\gamma := \{x \in X \,:\, \gamma \cdot x = x\},$$

and denote by $\mathrm{Z}_\Gamma(\gamma)$ the centralizer of γ in Γ.

Remark

The geometric extended quotient is the disjoint union

$$(X//\Gamma)_{\text{geo}} = \bigsqcup X^\gamma / \mathrm{Z}_\Gamma(\gamma)$$

where one γ is chosen in each Γ-conjugacy class.

Spectral extended quotient

Instead of elements of Γ_x, we can consider irred. repres. of Γ_x.
The (spectral) extended quotient of X by Γ is the quotient

$$X//\Gamma := \{(x, \tau) \,:\, x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\}\,/\Gamma.$$

The extended quotients $(X//\Gamma)_{\text{geo}}$ and $X//\Gamma$ are in bijection but not in a canonical way in general.

Example 1

Let $X := (\mathbb{C}^\times)^n$. The symmetric group S_n acts on X by permuting the coordinates. First, we form the ordinary quotient:

$$X/S_n = (\mathbb{C}^\times)^n/S_n =: \mathrm{Sym}^n(\mathbb{C}^\times),$$

which is called the *n*-fold symmetric product of $\mathbb{C}^\times$.
Next, we form the geometric extended quotient $(X/\!/S_n)_{\mathrm{geo}}$.
The conjugacy class of $\gamma \in S_n$ determines a partition of n. Let the distinct parts of the partition be $n_1, \ldots, n_l$ with n_i repeated d_i times so that $d_1 n_1 + d_2 n_2 + \cdots + d_l n_l = n$. We get

$$X^\gamma \simeq (\mathbb{C}^\times)^{n_1} \times \cdots \times (\mathbb{C}^\times)^{n_l}$$

$$\mathrm{Z}_{S_n}(\gamma) \simeq (\mathbb{Z}/d_1\mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/d_l\mathbb{Z}) \wr S_{n_l}$$

The cyclic group $\mathbb{Z}/d_i\mathbb{Z}$ acting trivially, we have

$$X^\gamma/\mathrm{Z}_{S_n}(\gamma) \simeq \mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_l}\mathbb{C}^\times.$$

Example 2

Let $\mathcal{T}$ be a torus in a complex reductive group $\mathcal{G}$ and $W := \mathrm{N}_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$ the corresponding Weyl group, acting on $\mathcal{T}$ by conjugation, then we can consider the extended quotients $(\mathcal{T}//W)_{\mathrm{geo}}$ and $\mathcal{T}//W$.

Example 3

Let G be a p-adic group, let L be a Levi subgroup of G and let $\mathrm{Irr}_{\mathrm{cusp}}(L)$ denote the set of isomorphism classes of supercuspidal irreducible smooth representations of L.
The group $W(L) := \mathrm{N}_G(L)/L$ acts on $\mathrm{Irr}_{\mathrm{cusp}}(L)$ and we can form the panoramic (spectral) p-adic extended quotient:

$$\mathrm{Irr}_{\mathrm{cusp}}(L)//W(L).$$

The ABPS Conjecture (coarse form) [A.-Baum-Plymen-Solleveld]

When $\mathbb{F} = F$ is a non-archimedean local field, for every $\mathfrak{s} \in \mathfrak{B}(G)$, the set $\mathrm{Irr}^{\mathfrak{s}}(G)$ of irreducible objects of the category $\mathfrak{R}^{\mathfrak{s}}(G)$ has a very simple geometric structure given by a (possibly twisted by a 2-cocycle) extended quotient $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_{\natural}$, where $T^{\mathfrak{s}}$ is a complex torus and $W^{\mathfrak{s}}$ is the finite group

$$W^{\mathfrak{s}} := \{w \in \mathrm{N}_G(L)/L \,:\, {}^w\mathfrak{s} = \mathfrak{s}\}.$$

Remarks

- If G is a quasi-split classical group [A.-Moussaoui-Solleveld] or the exceptional group G_2 [A.-Xu], no twisting is needed.
- However, for $G = \mathrm{SL}_n(D)$, with D/F a division algebra, there are cases which require a twisting [A.-Baum-Plymen-Solleveld].

Notation

Let Γ be a group acting on a topological space X and let Γ_x denote the stabilizer in Γ of $x \in X$. Let $\natural$ be a collection of 2-cocycles

$$\natural_x \colon \Gamma_x \times \Gamma_x \to \mathbb{C}^\times,$$

such that $\natural_{\gamma x}$ and $\gamma_* \natural_x$ define the same class in $H^2(\Gamma_{\gamma x}, \mathbb{C}^\times)$, where $\gamma_* \colon \Gamma_x \to \Gamma_{\gamma x}$ sends γ' to $\gamma\gamma'\gamma^{-1}$.

Let $\mathbb{C}[\Gamma_x, \natural_x]$ be the group algebra of Γ_x twisted by $\natural_x$, which is defined to be the $\mathbb{C}$-vector space $\mathbb{C}[\Gamma_x, \natural_x]$ with basis $\{t_\gamma \, : \, \gamma \in \Gamma_x\}$ and multiplication rules $t_\gamma t_{\gamma'} := \natural_x(\gamma, \gamma') t_{\gamma\gamma'}$, for any $\gamma, \gamma' \in \Gamma_x$. We set

$$\widetilde{X}_\natural := \{(x, \tau) \, : \, x \in X, \, \tau \in \mathrm{Irr}\, \mathbb{C}[\Gamma_x, \natural_x]\}$$

and topologize $\widetilde{X}_\natural$ by decreeing that a subset of $\widetilde{X}_\natural$ open if its projection to the first coordinate is open in X.

Definition

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$f_{\gamma,x} \colon \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

satisfying the conditions

(a) if $\gamma x = x$, then $f_{\gamma,x}$ is conjugation by an element of $\mathbb{C}[\Gamma_x, \natural_x]^\times$;

(b) $f_{\gamma',\gamma x} \circ f_{\gamma,x} = f_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma$ and $x \in X$.

Define a Γ-action on $\widetilde{X}_\natural$ by $\gamma \cdot (x, \tau) := (\gamma x, \tau \circ f_{\gamma,x}^{-1})$.

The twisted extended quotient of X by Γ with respect to $\natural$ is defined to be

$$(X/\!/\Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

A key example

Let $G = \mathrm{SL}_5(D)$ with D a central division algebra of dimension 4 over F. Let V_4 denote the non-cyclic group of order 4. Let W_F denote the Weil group of F. There exists a classical Langlands parameter $\phi\colon W_F \to \mathrm{PGL}_2(\mathbb{C})$ for $\mathrm{SL}_1(D)$ which factors through V_4:

$$\phi\colon W_F \to V_4 \to \mathrm{PGL}_2(\mathbb{C}).$$

Let τ be the supercuspidal representation of $\mathrm{GL}_1(D) = D^\times$ which has, as its Langlands parameter, a lift φ of ϕ to $\mathrm{GL}_2(\mathbb{C})$:

$$\phi\colon W_F \xrightarrow{\varphi} \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}).$$

The group of characters χ for which $\chi\tau \simeq \tau$ is isomorphic to V_4 and comprises the four characters $\{1, \gamma, \eta, \gamma\eta\}$, where γ, η are quadratic characters.

A key example (continued)

Let $\mathfrak{s} := [L, \sigma]_G$ with $L := (D^\times)^5 \cap \mathrm{SL}_5(D)$ and $\sigma = \tau \otimes 1 \otimes \gamma \otimes \eta \otimes \gamma\eta$. We have $W^{\mathfrak{s}} \simeq V_4$ and

$$\mathcal{H}(G)^{\mathfrak{s}} \underset{\text{Morita}}{\sim} \mathcal{O}(T^{\mathfrak{s}}) \rtimes_\natural V_4$$

where $\natural$ is the 2-cocycle associated to the above projective representation of V_4. We have

$$\mathrm{Irr}(\mathcal{O}(T^{\mathfrak{s}}) \rtimes_\natural V_4) = (T^{\mathfrak{s}} /\!/ V_4)_\natural.$$

This example shows that for inner forms of $\mathrm{SL}_5(F)$ there are Bernstein components where the twisting is non-trivial.

Definition/Notation

An irreducible smooth representation of G is said to be tempered if it is unitary and its matrix coefficients lie in $L^{2+\epsilon}(G/Z)$ for all $\epsilon > 0$.
Let $\mathrm{Irr}^{\mathrm{t}}(G)$ denote the tempered dual of G, i.e., the set of isomorphism classes of irreducible tempered representations of G.

Conjecture ABPS (strong form) [A.-Baum-Plymen-Solleveld]

Let G be a p-adic group, and $\mathfrak{s}$ a point in the Bernstein spectrum of G. There is a bijection

$$\nu^{\mathfrak{s}} \colon (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_{\natural} \to \mathrm{Irr}^{\mathfrak{s}}(G)$$

such that the following properties are satisfied

1. $\nu^{\mathfrak{s}}$ maps $T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}}$ onto $\mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}^{\mathrm{t}}(G)$, where $T^{\mathfrak{s}}_{\mathrm{cpt}}$ is the maximal compact subgroup of $T^{\mathfrak{s}}$
2. there is an algebraic family $\vartheta_z \colon T^{\mathfrak{s}}_{\mathrm{cpt}}/\!/W^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$ of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^{\times}$, such that ϑ_1 is the natural projection and

   $$\vartheta_{\sqrt{q}} = \mathrm{Sc}^{\mathfrak{s}} \circ \nu^{\mathfrak{s}}$$

   where $\mathrm{Sc}^{\mathfrak{s}}(\pi) := \mathfrak{s}$ and q is the order of the residue field of F.
3. $\nu^{\mathfrak{s}}$ comes from a canonical stratified equivalence of the two unital finite-type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$–algebras $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$ and $\mathcal{H}(G)^{\mathfrak{s}}$
4. $\nu^{\mathfrak{s}}$ is compatible with the local Langlands correspondence.