Lecture 5: Two sided cells and asymptotic Hecke algebras

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Notation

Let $\mathcal{R} = (X, Y, R, R^\vee)$ be a root datum. It delivers the following items:

- $W_{\mathrm{f}} := W(R)$ finite Weyl group
- $\widetilde{W} := W_{\mathrm{f}} \ltimes X$ extended affine Weyl group
- $\mathcal{T} := \mathrm{Hom}(X, \mathbb{C})$ a complex torus
- for each $u \in \mathbb{C}^\times$, the affine Hecke algebra $\mathcal{H} := \mathcal{H}(\widetilde{W}, u)$

Remarks

- Let $\mathcal{G}$ be the reductive group over $\mathbb{C}$ corresponding to $\mathcal{R}$. Then $\mathcal{T}$ is a maximal torus of $\mathcal{G}$.
- The algebra $\mathcal{H}$ has equal parameters. Partly conjectural versions of the following results exist also for unequal parameters affine Hecke algebras. See: G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monogr. Ser., **18**, Amer. Math. Soc., Providence, RI, 2003. vi+136 pp.

Recollection

The algebra $\mathcal{H}$ has two $\mathcal{A}$-bases, where $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$ with $u := v^2$.

- the basis $(T_w)_{w \in \widetilde{W}}$, with multiplication is defined by the relations

$$(T_s - u)(T_s + 1) = 0 \quad \text{if } s \text{ is a simple reflection}$$

$$T_w T_{w'} = T_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

- the Kazhdan-Lusztig basis $(C_w)_{w \in \widetilde{W}}$.

Notation

Let $w, w' \in \widetilde{W}$. We write $w \leq_{\mathrm{LR}} w'$ if $\lambda_w \neq 0$ in the expression

$$hC_{w'}h' = \sum_{z \in \widetilde{W}} \lambda_z C_z \quad (a_z \in \mathcal{A})$$

for some $h, h' \in \mathcal{H}$.

Definition

The relation $\leq_{\mathrm{LR}}$ defines a preorder on $\widetilde{W}$. The corresponding equivalence classes are called two-sided cells and the preorder gives rise to a partial order $\leq_{\mathrm{LR}}$ on the set of two-sided cells of $\widetilde{W}$.

Definition

Let $\mathcal{H}_{\leq w}$ be the minimal based (i.e., spanned over $\mathbb{Z}$ by a subset of the Kazhdan-Lusztig basis) two-sided ideal of $\mathcal{H}$ that contains C_w.

Characterization

Two elements $w, w' \in \widetilde{W}$ lie in the same two-sided cell of $\widetilde{W}$ iff

$$\mathcal{H}_{\leq w} = \mathcal{H}_{\leq w'}.$$

Let $\mathcal{C}(\widetilde{W})$ denote the set of two-sided cells of $\widetilde{W}$.

Theorem [Lusztig]

There is a natural bijective correspondence $\mathbf{c} \mapsto \mathcal{O}_{\mathbf{c}}$ between the set of two-sided cells in $\widetilde{W}$ and the set of $\mathcal{G}$-conjugacy classes of unipotent elements of $\mathcal{G}$.

Remark

We define a partial order $\leq$ on unipotent $\mathcal{G}$-conjugacy classes by setting

$$\mathcal{O}' \leq \mathcal{O} \quad \text{if } \mathcal{O}' \subset \overline{\mathcal{O}},$$

where $\overline{\mathcal{O}}$ is the closure of $\mathcal{O}$.
The above theorem gives an order on two-sided cells. This order coincides with the natural one. (This was conjectured by Lusztig and proved by Bezruknavnikov.)

Definitions [Lusztig]

- Let $h_{w,w',z}$ denote the structure constants of $\mathcal{H}$ with respect to the basis $(C_w)_{w\in W}$, i.e.,

$$C_w C_{w'} = \sum_{z\in\widetilde{W}} h_{w,w',z} C_z.$$

- For $\mathbf{c}$ a given two-sided cell of $\widetilde{W}$, let $-a(\mathbf{c})$ be the lowest possible degree of non-zero term in $h_{w,w',z}$, $w, w' \in \mathbf{c}$, $z \in \widetilde{W}$.

Notation

We will denote by $\mathcal{B}_{\mathbf{c}}$ the Springer fiber:

$$\mathcal{B}_{\mathbf{c}} := \{B \text{ Borel subgroup } : \ u \in B\}$$

where $u \in \mathcal{O}_{\mathbf{c}}$.

Proposition [Lusztig]

1. The a-function sends every two-sided cell $\mathbf{c}$ of $\widetilde{W}$ to the dimension of $\mathcal{B}_{\mathbf{c}}$.
2. For an affine Weyl group the a-function is bounded by the length of the longest element of the corresponding Weyl group.

Example

The exceptional group $\mathrm{G}_2(\mathbb{C})$ has five unipotent classes

$$1 \leq \mathrm{A}_1 \leq \widetilde{\mathrm{A}}_1 \leq \mathrm{G}_2(a_1) \leq \mathrm{G}_2.$$

We will refer to 1, A_1, $\widetilde{\mathrm{A}}_1$, $\mathrm{G}_2(a_1)$ and G_2 as the trivial, minimal, subminimal, subregular and regular class, respectively.

Example (continued)

Thus, the group $\widetilde{W} := W_{\mathrm{f}} \ltimes X(\mathcal{T})$ of $\mathrm{G}_2(\mathbb{C})$ has five two-sided cells:

$$\mathbf{c}_0 \leq \mathbf{c}_3 \leq \mathbf{c}_2 \leq \mathbf{c}_1 \leq \mathbf{c}_e.$$

The Lusztig bijection is described as follows:

$$\mathbf{c}_e \leftrightarrow \mathrm{G}_2 \quad \mathbf{c}_1 \leftrightarrow \mathrm{G}_2(a_1) \quad \mathbf{c}_2 \leftrightarrow \widetilde{\mathrm{A}}_1 \quad \mathbf{c}_3 \leftrightarrow \mathrm{A}_1 \quad \mathbf{c}_0 \leftrightarrow 1.$$

Explicit description of the cells:

$$\mathbf{c}_e = \left\{ w \in \widetilde{W} \,:\, a(w) = 0 \right\} = \{e\}$$

$$\mathbf{c}_1 = \left\{ w \in \widetilde{W} \,:\, a(w) = 1 \right\}$$

$$\mathbf{c}_2 = \left\{ w \in \widetilde{W} \,:\, a(w) = 2 \right\}$$

$$\mathbf{c}_3 = \left\{ w \in \widetilde{W} \,:\, a(w) = 3 \right\}$$

$$\mathbf{c}_0 = \left\{ w \in \widetilde{W} \,:\, a(w) = 6 \right\} \text{ (the lowest two-sided cell).}$$

Definition [Lusztig]

The asymptotic affine Hecke algebra J is has basis

$$\left\{ t_w \,:\, w \in \widetilde{W} \right\},$$

and multiplication defined by

$$t_w t_{w'} := \sum_{z \in \widetilde{W}} \gamma_{w,w',z}\, t_{z^{-1}}$$

where the structure constant $\gamma_{w,w',z}$ is the constant term of the polynomial $v^{a(z)}\, h_{w,w',z^{-1}}$. It is an associative algebra.

Distinguished involutions

We set

$$\mathcal{D} := \left\{ d \in \widetilde{W} \,:\, a(w) = \ell(w) + 2\deg(P_{1,w}) \right\}.$$

The elements of $\mathcal{D}$ are involutions (i.e., $d^2 = 1$) and each two-sided cell contains exactly one of them.
The element $\sum_{d\in\mathcal{D}} t_d$ is the unit element in the algebra J.

Definition

For $\mathbf{c}$ a two-sided cell of $\widetilde{W}$, the $J_{\mathbf{c}}$ spanned by the t_w, $w \in \mathbf{c}$, is a two-sided ideal of J. The ideal $J_{\mathbf{c}}$ is in fact an associative ring with unit $\sum_{d\in\mathcal{D}\cap\mathbf{c}} t_d$, which is called the based ring of the two-sided cell $\mathbf{c}$.

Theorem [Lusztig]

We have

$$J = \bigoplus_{\mathbf{c}\in\mathcal{C}(\widetilde{W})} J_{\mathbf{c}}.$$

Definition [Lusztig]

Let $\phi_u \colon \mathcal{H} \to \mathcal{J} := J \otimes_{\mathbb{Z}} \mathcal{A}$ the homomorphism

$$C_w \mapsto \sum_{\substack{z \in \widetilde{W}, d \in \mathbf{c} \cap \mathcal{D} \\ a(d)=a(z)}} h_{w,d,z}\, t_z.$$

Proposition

The map ϕ_u is injective, and we have

$$\phi_u(\mathrm{Z}(\mathcal{H})) \subset \mathrm{Z}(\mathcal{J}).$$

Key point for us

The Proposition above provides $\mathcal{J}$ (and also each $\mathcal{J}_{\mathbf{c}}$) with a structure of $\mathrm{Z}(\mathcal{H})$-algebra. This $\mathrm{Z}(\mathcal{H})$-algebra structure is not canonical: it depends on u. There is a canonical isomorphism $\mathrm{Z}(\mathcal{H}) \simeq \mathcal{O}(X) =: k$. We will denote this k-algebra by $\mathcal{J}_u$.

Definition

Let $\mathcal{H}^{\geq i}$ be the $\mathbb{C}$-subspace of $\mathcal{H}$ spanned by all the C_w with $w \in \widetilde{W}$ such that $a(w) \geq i$. This a two-sided ideal of $\mathcal{H}$. We set

$$\mathcal{H}^i := \mathcal{H}^{\geq i}/\mathcal{H}^{\geq i+1}.$$

This is an $\mathcal{H}$-bimodule. It has as $\mathbb{C}$-basis the images $[C_w]$ of the C_w such that $a(w) = i$.
We may regard $\mathcal{H}^i$ as a J-bimodule with multiplication defined by the rule:

$$t_x * [C_w] = \sum_{\substack{z \in \widetilde{W} \\ a(z)=i}} \gamma_{x,w,z^{-1}} C_z$$

$$[C_w] * t_x = \sum_{\substack{z \in \widetilde{W} \\ a(z)=i}} \gamma_{w,x,z^{-1}} C_z$$

for all $w, x \in \widetilde{W}$ such that $a(w) = i$.

Proof of the Proposition

It is enough to show that $\phi_u(z)\cdot t_w = t_w\cdot\phi_u(z)$ for any $z\in k$, $w\in\widetilde{W}$. Let $i := a(w)$, and $z\in k$. We set

$$f_i := \sum_{\substack{d\in\mathcal{D}\\ a(d)=i}} [C_d] \in \mathcal{H}^i.$$

We have $t_w * f_i = f_i * t_w = [C_w]$. Hence we obtain

$$(\phi_u(z)\cdot t_w) * f_i = \phi_u(z) * t_w * f_i = \phi_u(z) * [C_w] = z[C_w].$$

We observe that $hf = \phi_u(h) * f$ for all $f\in\mathcal{H}^i$, $h\in\mathcal{H}$. Hence we get

$$(t_w\phi_u(z)) * f_i = t_w * (\phi_u(z) * f_i) = t_w * (zf_i)$$

and, hence, since $zf_i = f_iz$,

$$(t_w\cdot\phi_u(z)) * f_i = t_w * (f_iz)$$

Observing that $(j * f)h = j * (fh)$, for all $f \in \mathcal{H}^i$, $h \in \mathcal{H}$, and $j \in J$, we obtain

$$(t_w \cdot \phi_u(z)) * f_i = (t_w * f_i)z = [C_w]z. \tag{1}$$

Now, since $z \in k$, we have $z[C_w] = [C_w]z$. Hence

$$(\phi_u(z) \cdot t_w) * f_i = (t_w * f_i)w = [C_x] \cdot \phi_u(z)) * f_i.$$

On can check that $\gamma_{w,w',z} = 0$ implies $a(w) = a(w') = a(z)$. Hence

$$\phi_u(z) \cdot t_w = \sum_{\substack{w' \in \widetilde{W} \\ a(w')=i}} \alpha_{w'} t_{w'} \quad \text{and} \quad t_w \cdot \phi_u(z) = \sum_{\substack{w' \in \widetilde{W} \\ a(w')=i}} \beta_{w'} t_{w'}$$

with $\alpha_{w'}, \beta_{w'} \in \mathbb{C}$. Then (1) implies that

$$\sum_{\substack{w' \in \widetilde{W} \\ a(w')=i}} \alpha_{w'} C_{w'} = \sum_{\substack{w' \in \widetilde{W} \\ a(w')=i}} \beta_{w'} C_{w'}.$$

Hence $\alpha_{w'}, = \beta_{w'}$ for all $w' \in \widetilde{W}$ such that $a(w) = i$. It gives $\phi_u(z) \cdot t_w = t_w \cdot \phi_u(z)$, as required.

Remark

Since $\mathcal{H}(\widetilde{W}, 1) = \mathbb{C}[\widetilde{W}]$ we obtain

$$\mathcal{H}(\widetilde{W}, u) \xrightarrow{\phi_u} \mathcal{J} \xleftarrow{\phi_1} \mathbb{C}[\widetilde{W}].$$

Definition [Lusztig]

Let E be a simple $\mathcal{H}$-module (resp. $\mathcal{J}$-module, with $\mathcal{J}_{\mathbf{c}} := J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathcal{A}$). We attach to E an integer a_E by the following two requirements:

1. $C_w E = 0$ (resp. $t_w E = 0$) for any $w \in \widetilde{W}$ with $a(w) > a_E$;
2. $C_w E \neq 0$ (resp. $t_w E \neq 0$) for some $w \in \widetilde{W}$ such $a(w) = a_E$.

The integer A_E is called the weight of E.
Let $\mathrm{Irr}(\mathcal{H}(\widetilde{W}, u))_a$ (resp. $\mathrm{Irr}(\mathcal{J})_a$) denote the set of simple $\mathcal{H}(\widetilde{W}, u)$-modules (resp. simple $\mathcal{J}$-modules) of weight a.

Notation

Let a be a positive integer. We define

$$J_a := \text{ideal generated by } \{t_w \in J \,:\, a(w) \leq a\}$$

$$I_a := \phi_u^{-1}(J_a).$$

Filtrations

We have the filtrations

$$0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = \mathcal{H}(\widetilde{W}, u)$$

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_r = \mathcal{J}.$$

We have

$$\mathrm{Irr}(\mathcal{H}(\widetilde{W}, u)) = \bigcup_a \mathrm{Irr}(I_a/I_{a-1}) \quad \text{and} \quad \mathrm{Irr}(\mathcal{J}) = \bigcup_a \mathrm{Irr}(J_a/J_{a-1})$$

with $\mathrm{Irr}(I_a/I_{a-1}) = \mathrm{Irr}(\mathcal{H}(\widetilde{W}, u))_a$ and $\mathrm{Irr}(J_a/J_{a-1}) = \mathrm{Irr}(\mathcal{J})_a$.

Theorem [Lusztig]

We assume that u is 1 or is not a root of unity. Then there is a unique bijection

$$\mathcal{L}_u: \quad \mathrm{Irr}(\mathcal{H}(W,u) \to \mathrm{Irr}(\mathcal{J}) \\ E \mapsto E'$$

between the set of isomorphism classes of simple $\mathcal{H}(W,u)$-modules and the set of isomorphism classes of simple $\mathcal{J}$-modules such that

- $a_E = a_{E'}$ and
- the restriction of E to $\mathcal{H}(W,u)$ via ϕ_u is an $\mathcal{H}(W,u)$-module with exactly one composition factor isomorphic to E and all other composition factors of the form $\bar{E}$ with $a_{\bar{E}} < a_E$.

Thus the map $\mathcal{L}_u$, for every a, the map $\mathcal{L}_u$ is a bijection

$$\mathrm{Irr}(\mathcal{J}) \xrightarrow{1-1} \mathrm{Irr}(\mathcal{H}(\widetilde{W},u))$$

and induces a bijection

$$\mathrm{Irr}(\mathcal{J})_a \xrightarrow{1-1} \mathrm{Irr}(\mathcal{H}(\widetilde{W},u))_a.$$