Lecture 6: The strong form of the ABPS Conjecture

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NSF-CBMS Conference: Representations of p-adic groups and noncommutative geometry

St John's University, Queens, NY

June 9-13, 2025

Notation/Definition

- Let X be an affine algebraic variety over the complex numbers $\mathbb{C}$. We denote by $k := \mathcal{O}(X)$ the coordinate algebra of X.
- Let A be a k-algebra and let $\mathrm{Prim}(A)$ denote set of primitive ideals of A.
- An ideal I in a k-algebra A is a k-ideal if $\lambda a \in I$ for all $(\lambda, a) \in k \times I$.

Definition

A representation of A is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V) \quad \text{and} \quad k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that

1. $k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital
2. $(\omega a)v = \omega(av) = a(\omega v)$ for all $(\omega, a, v) \in k \times A \times V$.

Central character

The central character of a finite type k-algebra A is a map $\mathrm{Irr}(A) \longrightarrow X$ defined as follows. Let φ be an irreducible representation of A. For $\omega \in k$, define $T_\omega\colon V \to V$ by $T_\omega(v) = \omega v$ for all $v \in V$. Then T_ω is an intertwining operator for $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. It is a scalar multiple of I_V (the identity operator of V).

$$T_\omega = \lambda_\omega I_V \qquad \text{for some } \lambda_\omega \in \mathbb{C}.$$

The map $\omega \mapsto \lambda_\omega$ is a unital morphism of $\mathbb{C}$-algebras $\mathcal{O}(X) \to \mathbb{C}$ and thus is given by evaluation at a unique ($\mathbb{C}$ rational) point p_φ of X:

$$\lambda_\omega = \omega(p_\varphi) \qquad \text{for all } \omega \in \mathcal{O}(X).$$

The central character $\mathrm{cc}\colon \mathrm{Irr}(A) \longrightarrow X$ is $\varphi \mapsto p_\varphi$.

Proposition

A Morita equivalence between two finite type k-algebras A, B preserves the central character, i.e. there is commutativity in the diagram

$$\begin{array}{ccc} \mathrm{Irr}(A) & \longrightarrow & \mathrm{Irr}(B) \\ {\scriptstyle \mathrm{cc}}\downarrow & & \downarrow{\scriptstyle \mathrm{cc}} \\ X & \underset{\mathrm{I}_X}{\longrightarrow} & X \end{array}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters.

Remark

In general, if $u \neq 1$, the k-algebras $\mathcal{H}(W_{\mathrm{f}} \ltimes X, u)$ and $\mathbb{C}[W_{\mathrm{f}} \ltimes X])$ are not isomorphic as k-algebras, or even not Morita equivalent.

Definition/Proposition

The exceptional set $\mathfrak{E}(A)$ is the set of all $x \in X$ such that the fibre of cc over x has cardinality greater than 1:

$$\mathfrak{E}(A) := \{x \in X \,:\, \sharp \mathrm{cc}^{-1}(x) > 1\}.$$

If A and B are Morita equivalent as k-algebras, then we will have $\mathfrak{E}(A) = \mathfrak{E}(B)$.

Remark

In general, if $u \neq 1$, the k-algebras $\mathcal{H}(W_{\mathrm{f}} \ltimes X, u)$ and $\mathbb{C}[W_{\mathrm{f}} \ltimes X])$ have different exceptional sets.

Consequence

In some situations, Morita equivalence are too strong and we are led to use a weakening of this concept, which we call stratified equivalence. The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A.

Definition

Let A, B two finite type k-algebras. A morphism of k-algebras $f\colon A \to B$ is spectrum preserving if

1. Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $f^{-1}(J) \subset I$.
2. The resulting map $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is a bijection.

Definition

A morphism of k-algebras $f\colon A \to B$ is spectrum preserving with respect to filtrations if there are k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

in A, and k-ideals $0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B$ in B, such that for every $j \in \{1, 2, \ldots, r\}$, we have $f(I_j) \subset J_j$, and

$$I_j/I_{j-1} \to J_j/J_{j-1} \text{ is spectrum preserving.}$$

Remark

The primitive ideal spaces of the subquotients I_j/I_{j-1} and J_j/J_{j-1} are called the strata for stratifications of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$.

- Each stratum of $\mathrm{Prim}(A)$ is mapped homeomorphically onto the corresponding stratum of $\mathrm{Prim}(B)$.
- However, the map $\mathrm{Prim}(A) \to \mathrm{Prim}(B)$ might not be a homeomorphism.

Support of a k-module

If $\mathfrak{P} \subset k$ is a prime ideal and M a k-module, then we denote by $M_{\mathfrak{P}}$ the localization of M at $\mathfrak{P}$, that is,

$$M_{\mathfrak{P}} := S^{-1}M$$

where $S = k \backslash \mathfrak{P}$. The support of M is defined to be the set of maximal ideals $\mathfrak{P} \subset k$ such that the $M_{\mathfrak{P}} \neq 0$. It is a closed subset of $\mathrm{Max}(k)$ in the Zariski topology, where $\mathrm{Max}(k)$ denotes the maximal ideal spectrum of k.

Lemma

Let $f\colon A \to B$ be a spectrum preserving with respect to filtrations morphism of finite type k-algebras. Then the k-modules A and B have the same support.

Theorem [Baum-Nistor]

Let $f\colon A \to B$ be a spectrum preserving with respect to filtrations morphism of finite type k-algebras. Then the induced map

$$f_*\colon \mathrm{HP}(A) \longrightarrow \mathrm{HP}(B)$$

is an isomorphism.

Proof

See Theorem 7 in [P. Baum and V. Nistor, "Periodic cyclic homology of Iwahori-Hecke algebras", K-Theory **27** (2002), 329–357].

Algebraic variation of k-structure

Let A be a unital $\mathbb{C}$-algebra, and $\psi\colon k \to \mathrm{Z}(A[t, t^{-1}])$ a unital morphism of $\mathbb{C}$-algebras. For $\zeta \in \mathbb{C}^\times$, consider the composition

$$k \xrightarrow{\psi} \mathrm{Z}(A[t, t^{-1}]) \xrightarrow{\mathrm{ev}(\zeta)} \mathrm{Z}(A).$$

and denote by A_ζ the unital k-algebra so obtained. We call such a family $\{A_\zeta\}_{\zeta \in \mathbb{C}^\times}$ an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$.

Definition

With k fixed, we consider the collection of all finite type k-algebras. On this collection, a stratified equivalence is the equivalence relation generated by the two elementary steps:

- ES1 If there is a morphism of k-algebras $f\colon A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.
- ES2 If there is $\{A_\eta\}_{\zeta \in \mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, such that each A_ζ is a unital finite type k-algebra, then for any $\zeta, \zeta' \in \mathbb{C}^\times$, $A_\zeta \sim A_{\zeta'}$.

Equivalent description

Two finite type k-algebras A, B are stratified equivalent if and only if there is a finite sequence A_0, A_1, A_2, ..., A_r of finite type k-algebras with $A_0 = A$, $A_r = B$, and for each $j \in \{0, 1, \ldots, r-1\}$ one of the following three possibilities holds:

- a morphism of k-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k-algebras $A_{j+1} \to A_j$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\eta\}_{\zeta \in \mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, is given such that each A_ζ is a unital finite type k-algebra, and ζ', ζ" in $\mathbb{C}^\times$ have been chosen such that $A_j = A_{\zeta'}$ and $A_{j+1} = A_{\zeta"}$.

Note

To define a stratified equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology are determined:

$$\mathrm{Prim}(A) \leftrightarrow \mathrm{Prim}(B) \quad \text{and} \quad \mathrm{HP}_*(A) \simeq \mathrm{HP}_*(B).$$

Proposition [A-Baum-Plymen-Solleveld]

If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are stratified equivalent:

$$A \underset{\text{Morita}}{\sim} B \Longrightarrow A \sim B.$$

In contrast, there exist k-algebras that are stratified equivalent but not Morita equivalent, e.g., the affine Hecke algebra $\mathcal{H}_q(W^{\mathrm{aff}})$ (with $q \neq 1$) associated to an affine Weyl group W^{aff} and the group algebra $\mathbb{C}[W^{\mathrm{aff}}]$ are stratified equivalent for almost all q, but they are not Morita equivalent en general.

Key example

The Hecke-Iwahori algebra $\mathcal{H}(W, u)$, with $u \neq 1$ not a root of unity, is stratified equivalent to $\mathcal{H}(W, 1)$ by the three elementary steps

$$\mathcal{H}(W, u) \rightsquigarrow \mathcal{J}_u \rightsquigarrow \mathcal{J}_1 \rightsquigarrow \mathcal{H}(W, 1).$$

The second elementary step (i.e. passing from $\mathcal{J}_u$ to $\mathcal{J}_1$) is an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$. The first elementary step uses Lusztig's map ϕ_u, and the third elementary step uses Lusztig's map ϕ_1.

Hence

$$\mathcal{H}(W, u) \text{ is stratified equivalent to } \mathcal{H}(W, 1) = \mathbb{C}[W].$$

Notation

- Let $\mathcal{R} = (X, Y, R, R^\vee, \Pi)$ be a based reduced root datum. Here Π is a basis of simple roots of R. The set Π determines a subset R_+ of positive roots.
- Let X^+ denote the cone of dominant elements in X:

$$X^+ := \{x \in X \ : \ \langle x, \alpha^\vee \rangle \geq 0 \ \text{ for all } \alpha \in R_+\}.$$

  We put $X^- := -X^+$.
- The extended affine Weyl group $\widetilde{W} = W_{\mathrm{f}} \ltimes X$, which acts as a group of affine transformations on $\mathbb{Q} \otimes_{\mathbb{Z}} X$.
- Let $\Omega := \left\{ w \in \widetilde{W} \ : \ \ell(w) = 0 \right\}$.
- $Z_X := X^+ \cap X^-$ is a sublattice of X which is central in $\widetilde{W}$. We have $Z_X \subset \Omega$.
- We fix a basis (z_i) of Z_X and define a norm $\| \ \|$ on $\mathbb{Q} \otimes_{\mathbb{Z}} Z_X$ by $\| \sum_i l_i z_i \| = \sum_i |l_i|$.

Definition of a norm on the extended affine Weyl group

We define a norm $\mathcal{N}$ on $\widetilde{W}$ by setting

$$\mathcal{N}(w) := \ell(w) + \|\overline{w(0)}\|$$

with $w(0)$ the image of $0 \in \mathbb{Q} \otimes_{\mathbb{Z}} X$ under the affine transformation w, and with $\overline{w(0)}$ the projection of $w(0)$ onto $\mathbb{Q} \otimes_{\mathbb{Z}} Z_X$.

Remark

We have

- $\mathcal{N}(ww') \leq \mathcal{N}(w) + \mathcal{N}(w')$ for all $w, w' \in \widetilde{W}$
- $\mathcal{N}(w) = 0$ if and only if w is an element of Ω of finite order.

The Hecke algebra of the root datum

Given a root datum $\mathcal{R}$ and a (positive real) label function $\overline{q}\colon w \mapsto q_w$ on $\widetilde{W}$, there exists a unique associative complex Hecke algebra $\mathcal{H} := \mathcal{H}(\widetilde{W}, \overline{q})$ with $\mathbb{C}$-basis N_w indexed by $w \in \widetilde{W}$, satisfying the relations:

- $(N_s + q_s^{1/2})(N_s - q_s^{-1/2}) = 0$ for every affine simple reflection s
- $N_{ww'} = N_w N_{w'}$ if $\ell(ww') = \ell(w) + \ell(w')$.

Hilbert algebra structure on $\mathcal{H}$

The anti-linear map $h \mapsto h^*$ defined by

$$\left(\sum_{w\in\widetilde{W}} c_w N_w\right)^* := \sum_{w\in\widetilde{W}} \overline{c_{w^{-1}}} N_w$$

is an anti-involution of $\mathcal{H}$. Thus it gives $\mathcal{H}$ the structure of an involutive algebra.

Definition

The linear functional $\tau \colon \mathcal{H} \to \mathbb{C}$ defined by

$$\tau\left(\sum_{w\in\widetilde{W}} c_w N_w\right)^{:} = c_e$$

is a positive trace for the involutive algebra $(\mathcal{H}, *)$.

The Hilbert completion of $\mathcal{H}$

The basis N_w of $\mathcal{H}$ is orthonormal with respect to the preHilbert structure on $\mathcal{H}$

$$(x, y) := \tau(x^* y).$$

We denote the Hilbert completion of $\mathcal{H}$ with respect to $(\cdot, \cdot)$ by $L^2(\mathcal{H})$. This is a separable Hilbert space with Hilbert basis $(N_w)_{w\in\widetilde{W}}$.

The Hilbert structure of $\mathcal{H}$

Let $x \in \mathcal{H}$. The operators $\lambda(x)\colon \mathcal{H} \to \mathcal{H}$ (given by $\lambda(x)(y) := xy$) and $\rho(x)\colon \mathcal{H} \to \mathcal{H}$ (given by $\rho(x)(y) := yx$) extend to $B(L^2(\mathcal{H}))$, the algebra of bounded operators on $L^2(\mathcal{H})$. This gives $\mathcal{H}$ the structure of a Hilbert algebra.

Definition

The operator norm completion of $\lambda(\mathcal{H}) \subset B(L^2(\mathcal{H}))$ is a C^*-algebra which is called the reduced C^*-algebra $C^*_{\mathrm{r}}(\mathcal{H})$ of $\mathcal{H}$.

Notation

We define norms p_n for $n = 1, 2, \ldots$ on $\mathcal{H}$ by

$$p_n(h) := \sup_{w \in \widetilde{W}} |(N_w, h)| \cdot (1 + \mathcal{N}(w))^n.$$

Definition [Delorme-Opdam]

The Schwartz completion $\mathcal{S}$ of $\mathcal{H}$ by

$$\mathcal{S} := \left\{ x = \sum_{w \in \widetilde{W}} x_w N_w \ : \ p_n(x) < \infty \text{ for all } n \in \mathbb{Z}_+ \right\}.$$

Remark

The multiplication operation of $\mathcal{H}$ is continuous with respect to the family p_n of norms. The completion $\mathcal{S}$ is a (nuclear, unital) Fréchet algebra, and we have a continuous embedding

$$\mathcal{S} \subset C_{\mathrm{r}}^*(\mathcal{H}).$$

The subalgebra $\mathcal{S}$ is dense and symmetric (i.e., $\mathcal{S}^* = \mathcal{S}$).