Lecture 7: Tempered unipotent representations

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Buildings

- The notion of (spherical) building was first introduced by Jacques Tits as a mean of understanding algebraic groups over an arbitrary field.
- The general idea is to construct a space upon which the group acts in a nice manner, and to use information about this space and the action to learn about the group itself.
- The Riemannian symmetric space associated to a Lie group $\mathcal{G}$ is the quotient space $\mathcal{G}/K$, where K is a maximal compact subgroup of $\mathcal{G}$.
- One motivation to introduce Bruhat-Tits buildings is because they provide a very useful non-archimedean replacement of Riemannian symmetric spaces.

The Bruhat-Tits building $\beta(G)$ of a p-adic group G

It is a set equipped with the following structures:

- it is a complete metric space, with an affine structure
- it is the product of a polysimplicial complex and a real vector space
- it has a collection of distinguished subsets, known as apartments which are indexed by the maximal split tori
- the group G acts isometrically on $\beta(G)$ as simplicial automorphisms.

For every $x \in \beta(G)$, the stabilizer $\mathrm{Stab}_G(x)$ of x in G is a compact open subgroup of G.

Notation/Definition

- Let $\mathfrak{o}_F$ denote the ring of valuation of F, let ϖ_F be a uniformizer of F, and let $k_F = \mathfrak{o}_F/\varpi_F\mathfrak{o}_F$ be the residue field of F.
- Let $V := F^n$ and $G := \mathrm{GL}_n(F) = \mathrm{GL}(V)$. An $\mathfrak{o}_F$-lattice is a free $\mathfrak{o}_F$-submodule of V of rank n. Let Latt denote the set of lattices.
- We say that two lattices $\mathcal{L}$ and $\mathcal{L}'$ are equivalent (denoted as $\mathcal{L} \simeq \mathcal{L}'$) if $\mathcal{L} = c\mathcal{L}'$ for some $c \in F^\times$. We denote by $\Lambda := [\mathcal{L}]$ the equivalence class of a lattice $\mathcal{L}$.
- The vertices in $\beta(\mathrm{GL}_n(F))$ correspond to the equivalence classes of lattices in V.
- A $(k-1)$-simplex corresponds to k vertices $\Lambda_1, \ldots, \Lambda_k$ such that:

$$\varpi_F\mathcal{L}_k \subsetneq \mathcal{L}_1 \subsetneq \cdots \subsetneq \mathcal{L}_k.$$

Iwahori subgroups of $\mathrm{GL}_n(F)$

- A $(n-1)$-simplex is called a chamber.
- The $\mathrm{GL}_n(F)$-stabilizer of a chamber is called an Iwahori subgroup.
- An Iwahori subgroup is equal to the preimage of a Borel subgroup $B \subset \mathrm{GL}_n(k_F)$ under the reduction map $\mathrm{GL}_n(\mathfrak{o}_F) \to \mathrm{GL}_n(k_F)$.

Example

For $G = \mathrm{GL}_2(F)$, a Borel subgroup is G-conjugate to

$$\left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} \; : \; \bar{a}, \bar{d} \in k_F^\times, \, \bar{b} \in k_F \right\}.$$

and Iwahori subgroup is G-conjugate to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; : \; a, d \in \mathfrak{o}_F^\times, \, b \in \mathfrak{o}_F, \, c \in \mathfrak{p}_F \right\}.$$

Iwahori subgroups of a p-adic group G

The group G is the group of F-points of a quasi-split connected reductive algebraic group $\mathbf{G}$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, and let denote by $\overline{g}$ the reduction of $g \in \mathbf{G}(\mathfrak{o}_F)$. The subgroup

$$I := \{g \in \mathbf{G}(\mathfrak{o}_F) \ : \ \overline{g} \in \mathbf{B}(k_F)\}$$

is an Iwahori subgroup of G.

Definition

Let I be an Iwahori subgroup of G. An irreducible representation $(\pi, \mathcal{V})$ of G is called I-spherical if there exists $0 \neq v \in \mathcal{V}$ such that $\pi(g)(v) = v$ for all $g \in I$. The representation $(\pi, \mathcal{V})$ is called Iwahori–spherical if it is I-spherical for some Iwahori subgroup of G.

Parahoric subgroups

- The stabilizer of a lattice chain is called a parahoric subgroup $\mathrm{GL}_n(F)$. It is the preimage of a parabolic subgroup in $\mathrm{GL}_n(k_F)$.
- The parahoric subgroups of G are precisely the compact open subgroups of G containing an Iwahori subgroup, i.e., the normalizer of the maximal pro-p subgroups of G.
- A parahoric subgroup $G_{x,0} \subset G_x$ of G belongs to a short exact sequence

$$1 \longrightarrow G_{x,0+} \longrightarrow G_{x,0} \longrightarrow \underline{\mathbb{G}}_{x,0}(k_F) \longrightarrow 1$$

where $G_{x,0+}$ is called the pro-unipotent radical of $G_{x,0}$ and $\underline{\mathbb{G}}_{x,0}(k_F)$ is the group of k_F-points of a connected reductive algebraic group $\underline{\mathbb{G}}_{x,0}$.

Representations of finite reductive group [Deligne-Lusztig]

Let $\underline{\mathbb{G}}$ be a connected reductive algebraic group defined over a finite field $k := \mathbb{F}_q$. For any irreducible representation τ of $\mathbb{G} := \underline{\mathbb{G}}(k)$, there exists a k-rational maximal torus $\underline{\mathbb{T}}$ of $\underline{\mathbb{G}}$ and a character of $\mathbb{T} := \underline{\mathbb{T}}(k)$ such that τ occurs in the Deligne-Lusztig (virtual) character $R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta)$, i.e., such that $\langle \tau, R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta) \rangle_{\mathbb{G}} \neq 0$, where the character of τ is also denoted by τ, and $\langle\ ,\ \rangle_{\mathbb{G}}$ is the usual scalar product on the space of class functions on $\mathbb{G}$:

$$\langle f_1, f_2 \rangle_{\mathbb{G}} = |\mathbb{G}|^{-1} \sum_{g \in \mathbb{G}} f_1(g)\, \overline{f_2(g)}.$$

If $\theta = 1$ (the trivial character of $\mathbb{T}$) then the representation τ is called unipotent.

Definition

An irreducible smooth representation (π, V) of G is said to have unipotent parahoric restriction (or, for short, to be unipotent) if there is $x \in \beta(G)$ such the $G_{x,0+}$-invariants in V contain an irreducible cuspidal unipotent representation π_x in the sense of Deligne-Lusztig theory, i.e., such that

$$\langle \pi_x, R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1) \rangle \neq 0,$$

for a maximal torus $\mathbb{T}$ of $\mathbb{G}_{x,0}$, with $R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1)$ a Deligne-Lusztig character.

Example

Every Iwahori-spherical representation of G is unipotent.
Iwahori-spherical representations belongs to $\mathfrak{R}^{\mathfrak{s}}(G)$, where $\mathfrak{s} = [T, 1]_G$, with T a maximal torus of G.

Supercuspidal unipotent representations

These are the representations π of G such that there exist a vertex $x \in \beta_{\rm red}(G)$ and an irreducible unipotent cuspidal representation π_x of $\mathbb{G}_{x,0}$, such π is compactly induced from $\widetilde{\pi}_x$, an extension to $\mathrm{N}_G(G_{x,0})$ of the inflation of π_x to $G_{x,0}$:

$$\pi = \mathrm{c}{-}\mathrm{Ind}_{\mathrm{N}_G(G_{x,0})}^{G}(\pi_x),$$

where $\mathrm{N}_G(G_{x,0})$ is the normalizer of $G_{x,0}$ in G (a totally disconnected group that is compact mod center). It coincides with the fixator under the action of G on $\beta_{\rm red}(G)$ of the image of x in $\beta_{\rm red}(G)$.

The category $\mathfrak{R}^{\mathrm{u}}(G)$ of unipotent representations of G

We have

$$\mathfrak{R}^{\mathrm{u}}(G) := \prod_{\substack{\mathfrak{s}=[L,\sigma] \\ \sigma \text{ unipotent}}} \mathfrak{R}^{\mathfrak{s}}(G).$$

Parametrization of $\mathrm{Irr}^{\mathrm{u}}(G)$ [Lusztig, Feng-Opdam-Solleveld]

$$\mathrm{Irr}^{\mathrm{u}}(G) \xleftrightarrow{1-1} (s, u, \rho)_{G^\vee}$$

where $G^\vee$ is the complex reductive group dual to G, $s, u \in G^\vee$, with s semisimple, u unipotent, such that $sus^{-1} = u^q$, and $\rho \in \mathrm{Irr}(A(s, u))$, with

$$A(s, u) := \mathrm{Z}_{G^\vee}(s, u)/\mathrm{Z}_{G^\vee} \cdot \mathrm{Z}_{G^\vee}(s, u)^\circ$$

where $\mathrm{Z}_{G^\vee}$ is the center of $G^\vee$.

Remark

We decompose s into its compact and hyperbolic parts:

$$s = s_{\mathrm{c}}\, s_{\mathrm{h}}.$$

The tempered unipotent representations are those such that $s_{\mathrm{h}} = 1$.

Local Langlands correspondence

Extension of nonarchimedean local fields

Let E be a finite extension of F. We set $f := [k_E : k_F]$ and let e be the integer such that $\varpi_F \mathfrak{o}_E = \mathfrak{p}_E^e$. The integers f and e are the residue degree and ramification degree of E over F. We say that the extension E/F is unramified if $e = 1$.

The absolute Galois group of F

- $\overline{F}$ separable algebraic closure of F:

$$\Gamma_F := \mathrm{Gal}(\overline{F}/F) := \varprojlim_{E} \mathrm{Gal}(E/F)$$

where E/F ranges over finite Galois extensions with $E \subset \overline{F}$

Finite unramified extensions

- F_m the unramified extension of F of degree m. It is Galois and $\mathrm{Gal}(F_m/F)$ is cyclic.
- There is a unique element ϕ_m of $\mathrm{Gal}(F_m/F)$ which acts on $k_{F_m} \simeq \mathbb{F}_{q^m}$ as $x \mapsto x^q$. Set $\Phi_m := \phi_m^{-1}$.
- $\Phi_m \mapsto 1$ defines a canonical isomorphism $\mathrm{Gal}(F_m/F) \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$.

Maximal unramified extension

- $F_{\mathrm{nr}} :=$ the composite of all the F_m is the maximal unramified extension of F
- canonical isomorphism of topological groups

$$\mathrm{Gal}(F_{\mathrm{nr}}/F) \simeq \varprojlim_{m\geq 1} \mathbb{Z}/m\mathbb{Z} =: \widehat{\mathbb{Z}}.$$

- Let $\Phi_F \in \mathrm{Gal}(F_{\mathrm{nr}}/F)$ be the unique element that acts on F_m as Φ_m, for all m.

The inertia group of F

The inertia group of F defined to be $I_F := \mathrm{Gal}(\overline{F}/F_{\mathrm{nr}})$. We have an exact sequence of topological groups

$$1 \to I_F \to \Gamma_F \to \widehat{\mathbb{Z}} \to 0.$$

In 1951, Weil introduced a modification of the absolute Galois group of a local or global field.

The abstract Weil group of F

Let ${}_aW_F$ be the inverse image in Γ_F of the cyclic subgroup $\langle \Phi_F \rangle$ of $\mathrm{Gal}(F_{\mathrm{nr}}/F)$ generated by Φ_F. It is a dense subgroup of Γ_F and fits into an exact sequence of abstract groups:

$$1 \to I_F \to {}_aW_F \to \mathbb{Z} \to 0.$$

Definition

The (absolute) Weil group of F is the topological group, with underlying abstract group ${}_aW_F$, so that

1. I_F is an open subgroup of W_F, and
2. the topology on I_F, as subspace of W_F, coincides with its natural topology as $\mathrm{Gal}(\overline{F}/F_{\mathrm{nr}}) \subset \Gamma_F$.

2. Archimedean case.

We have $W_{\mathbb{C}} = \mathbb{C}^\times$ (the group of non-zero complex numbers).

Definition

The absolute Galois group $\Gamma_{\mathbb{R}}$ is cyclic of order two, and the Weil group $W_{\mathbb{R}}$ is the nonsplit extension

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \to 1.$$

Explicitly, $W_{\mathbb{R}}$ is the group obtained from $\mathbb{C}^\times$ by adjoining an element j such that $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for all for $z \in \mathbb{C}^\times$.

The GL_1-case (Class Field Theory)

Let $\mathbb{F}$ be a local field. The local Langlands correspondence for $\mathrm{GL}_1(\mathbb{F}) = \mathbb{F}^\times$ is the canonical bijection between the

- irreducible complex representations of $\mathrm{GL}_1(\mathbb{F})$ (automorphic side)
- one-dimensional representations of $W_\mathbb{F}$ (Galois side)

Theorem

The local Langlands correspondence for $\mathrm{GL}_n(\mathbb{F})$ is a canonical bijection between the (equivalence classes of the) following objects:

- irreducible smooth complex representations of $\mathrm{GL}_n(\mathbb{F})$
- n-dimensional complex representations of $W'_\mathbb{F}$, where

$$W'_\mathbb{F} := \begin{cases} W_\mathbb{F} & \text{if } \mathbb{F} \text{ archimedean} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } \mathbb{F} \text{ nonarchimedean.} \end{cases}$$

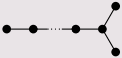
Proof:

- $\mathbb{F} = \mathbb{R}$: [Langlands]
- $\mathbb{F}$ nonarch & $\mathrm{char}(\mathbb{F}) > 0$: [Laumon-Rapoport-Stuhler, 1993]
- $\mathbb{F}$ nonarch & $\mathrm{char}(\mathbb{F}) = 0$: [Harris-Taylor, 1998[; [Henniart, 2000]; [Scholze, 2010] □

Notations

- $\mathbb{F}$: local field
- G: group of $\mathbb{F}$-rational points of a reductive algebraic group defined over $\mathbb{F}$
- $G^\vee$: complex Lie group with root datum dual to that of G (Langlands dual group of G).

Some groups G with their Langlands dual groups

G	Dynkin diagram	$G^\vee$
$\mathrm{GL}_n(\mathbb{F})$		$\mathrm{GL}_n(\mathbb{C})$
$\mathrm{SL}_n(\mathbb{F})$		$\mathrm{PGL}_n(\mathbb{C})$
$\mathrm{PGL}_n(\mathbb{F})$		$\mathrm{SL}_n(\mathbb{C})$
$\mathrm{Sp}_{2n}(\mathbb{F})$		$\mathrm{SO}_{2n+1}(\mathbb{C})$
$\mathrm{SO}_{2n+1}(\mathbb{F})$		$\mathrm{Sp}_{2n}(\mathbb{C})$
$\mathrm{SO}_{2n}(\mathbb{F})$		$\mathrm{SO}_{2n}(\mathbb{C})$
$\mathrm{G}_2(\mathbb{F})$		$\mathrm{G}_2(\mathbb{C})$

Definition

An *L-parameter* is a continuous morphism

$$\varphi\colon W'_{\mathbb{F}} \longrightarrow G^\vee$$

such

- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is morphism of algebraic groups,
- $\varphi(w)$ is a semisimple element of $G^\vee$, for any $w \in W_{\mathbb{F}}$.

The Local Langlands Correspondence (LLC)

predicts a surjective map, satisfying several properties,

$$\left\{\begin{matrix}\text{irred. smooth} \\ \text{repres. } \pi \text{ of } G\end{matrix}\right\}/\text{iso.} \xrightarrow{\mathcal{L}} \{L\text{-parameters}\}/G^\vee\text{-conjugacy}$$

with finite fibers, called *L*-packets.

Remarks

- In the nonarchimedean case, in order to obtain a bijection LLC between the group side and the Galois side, the conjectural map $\mathcal{L}$ was later enhanced: on the Galois side, one considers enhanced *L*-parameters: (φ_π, ρ_π), where the enhancement ρ_π is a representation of a certain component group.
- It may be useful to consider simultaneously inner twists of a given group G. This leads to "compound" *L*-packets.

Examples of unipotent L-packets with "very different" colors

The unipotent discrete series of $\mathrm{G}_2(F)$ belong to the following L-packets

- $\{\mathrm{St}_{\mathrm{G}_2}\}$
- $\{\pi[1], \pi(1), \pi(1)'\}$
- $\{\pi[-1], \pi(\eta_2)\}$
- $\{\pi[\zeta_3], \pi[\zeta_3^2], \pi(\eta_3)\}$

Remark:

- $\pi[1]$, $\pi[-1]$, $\pi[\zeta_3]$ and $\pi[\zeta_3^2]$ are supercuspidal: they belong to four distinct Bernstein blocks $\mathfrak{R}^{\mathfrak{s}}(G)$ for which $L = G$
- $\mathrm{St}_{\mathrm{G}_2}$, $\pi(1)$, $\pi(1)'$, $\pi(\eta_2)$ and $\pi(\eta_3)$ are in the principal series of G, they are Iwahori-spherical: they all belong to the principal unipotent block $\mathfrak{R}^{\mathfrak{s}}(G)$ with $\mathfrak{s} = [T, 1]_G$, where T maximal torus of G.

Examples of non-unipotent depth-zero L-packets in $\mathrm{G}_2(F)$ [A.-Xu]

- One L-packet, consisting of one singular supercuspidal representation coming from reductive quotient $\mathbb{G}_{x_0} \simeq G_2(\mathbb{F}_q)$ and $\mathrm{Z}_{\underline{\mathbb{G}}_{x_0}^\vee}(s) \simeq \mathrm{SU}_3(\mathbb{F}_q)$, and an intermediate series representation $\pi(\sigma)$ whose cuspidal support lives in $\mathrm{GL}_2^{\mathrm{l.r.}}$. This case only occurs when $q \equiv -1 \mod 3$.
- Two L-packets, each consisting of one singular supercuspidal representation coming from reductive quotient $\mathbb{G}_{x_3} \simeq \mathrm{SO}_4(\mathbb{F}_q)$ and $\mathrm{Z}_{\underline{\mathbb{G}}_{x_2}^\vee}(s) \simeq \mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_2)(\mathbb{F}_q)$ (here we take the non-split form of O_2), and one generic principal series representation $\pi(\eta_2')$, where η_2' is a ramified quadratic character.
- Three L-packets, each consisting of two singular supercuspidal representations coming from reductive quotient $\mathbb{G}_{x_1} \simeq \mathrm{SL}_3(\mathbb{F}_q)$ and $\mathrm{Z}_{\underline{\mathbb{G}}_{x_1}^\vee}(s) = \underline{\mathbb{T}}^\vee \rtimes \mu_3$, and a generic principal series representation $\pi(\eta_3')$, where η_3' is a ramified cubic character.