Lecture 8: Tempered representations and C^*-blocks

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The Harish-Chandra Schwartz algebra

Let G be a p-adic reductive group.

- If K is a compact open subgroup of G, we denote by $\mathcal{S}(G,K)$ be the space of rapidly decreasing K-biinvariant functions on G.
- The Harish-Chandra Schwartz algebra of G is defined to be

$$\mathcal{S}(G) := \bigcup_{K \text{ compact open subgroup of } G} \mathcal{S}(G,K).$$

- We have $\mathcal{S}(G,K) = e_K \mathcal{S}(G) e_K = \mathcal{S}(G)^{K\times K}$.
- $\mathcal{S}(G)$ consists of all uniformly locally constant and rapidly decreasing functions on G.
- Endowed with the inductive limit topology this is a complete locally convex topological algebra with separately continuous multiplication.
- $\mathcal{S}(G)$ is an idempotented algebra.
- The Hecke algebra $\mathcal{H}(G)$ is contained in $\mathcal{S}(G)$, and is dense in $\mathcal{S}(G)$.

Definition

A representation (π, V) of G is admissible if, for any compact open subgroup K of G, the subspace $V^K = e_K V$ of K-invariant vectors is finite dimensional.

Remark

- Every admissible representation in $\mathfrak{R}(G)$ which is "tempered" in the traditional sense, i.e., its matrix coefficients are tempered functions on G carries a unique $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure.
- Conversely, if an admissible representation (π, V) in $\mathfrak{R}(G)$ carries an $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure, then (π, V) is tempered.

Definition

The category $\mathfrak{R}^{\mathrm{t}}(G)$ of tempered representations of G is the category of nondegenerate left $\mathcal{S}(G)$-modules.

Definition

Let G be a p-adic group (or more generally alocally compact group), and let dg be a left Haar measure on G, and let $\mathcal{C}_c(G)$ denote be the convolution algebra of compactly supported continuous functions G with values in $\mathbb{C}$. The algebra $\mathcal{C}_c(G)$ acts as bounded operators on the Hilbert space $L^2(G)$: for $f \in C_c(G)$ and $\xi \in L^2(G)$, we define

$$\lambda(f)(\xi)(x) := \int_G f(g)\xi(g^{-1}x)dg.$$

The reduced C^*-algebra of G is the closure of $\lambda(\mathcal{C}_c(G))$ in the operator norm

$$C^*_{\mathrm{red}}(G) := \overline{\lambda(\mathcal{C}_c(G))}.$$

We have

$$\mathcal{H}(G) \subset \mathcal{S}(G) \subset C^*_{\mathrm{red}}(G)$$

and $\mathcal{S}(G)$ is dense in $C^*_{\mathrm{red}}(G)$.

Proposition

Let V be a simple $\mathcal{S}(G)$-module; we then have:

1. V is simple as an $\mathcal{H}(G)$-module;
2. every simple $\mathcal{S}$-module which is isomorphic to V as an $\mathcal{H}$-module is already isomorphic to V as an $\mathcal{S}(G)$-module.

Corollary

Simple $\mathcal{S}(G)$-modules coincide with irreducible tempered representations of G.

Notation

Let $\mathrm{Irr}^2(G)$ denote the set of equivalence classes of irreducible square integrable modulo the center representations (i.e., discrete series) of G. Every $\pi \in \mathrm{Irr}^2(G)$ is tempered.

Theorem [Harish-Chandra]

1. An irreducible representation π of G is tempered if and only if it occurs as an irreducible component of a parabolically induced representation $\mathrm{i}_{M,Q}^{G}(\delta)$, where Q is a parabolic subgroup of G with Levi factor M and $\delta \in \mathrm{Irr}^2(M)$.
2. The G-conjugacy class $(M,\delta)_G$ of (M,δ) is uniquely determined and is called the discrete support of π.

The Plancherel theorem

$\mathcal{S}(G)$ is the orthogonal direct sum of subspaces formed from series of representations induced unitarily from discrete series of the Levi factors of parabolic subgroups.
Moreover, if two such series of induced representations yield the same subspace of $\mathcal{S}(G)$, then the parabolic subgroups from which they are induced are associate, and the representations of the Levi factors are G-conjugate.

Remark

Harish-Chandra formulated a version of the Plancherel theorem which is similar to that for real groups, although he published only a sketch of the proof. The full proof was written by Waldspurger.

Notation

- $\mathfrak{X}_{\mathrm{unr}}(M)$ the group of the unitary unramified characters of M, with M a Levi subgroup of G
- $\Omega(G)$ set of G-conjugacy classes of pairs (L, σ) with $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$
- Σ_{c} supercuspidal support map
- $\Omega^2(G)$ set of G-conjugacy classes of pairs (M, δ) with $\delta \in \mathrm{Irr}^2(M)$
- Σ^2 discrete support map

Compatibility of the discrete and supercuspidal support maps

The following diagram commutes:

$$\begin{array}{ccc} & \mathrm{Irr}^{\mathrm{t}}(G) & \\ {\scriptstyle \Sigma^2} \swarrow & & \searrow {\scriptstyle \Sigma_{\mathrm{c}}} \\ \Omega^2(G) & \xrightarrow{\quad \mathfrak{z} \quad} & \Omega(G) \end{array}$$

where $\mathfrak{z}\colon (M,\delta)_G \mapsto$ supercuspidal support of δ.

Remark

For $G = \mathrm{GL}_n(F)$, the map $\mathfrak{z}$ is injective and the map Σ^2 is bijective (since unitary induction is irreducible for $\mathrm{GL}_n(F)$).

Definition/Notation

- Let $\delta \in \mathrm{Irr}^2(M)$, and let $\mathcal{O}^{\mathrm{t}} := \mathfrak{X}_{\mathrm{unr}}(M) \cdot \delta$. The G-conjugacy class $\mathfrak{d} := (M, \mathcal{O}^{\mathrm{t}})_G$ of $(M, \mathcal{O}^{\mathrm{t}})$ is called an inertial discrete pair in G. We write also $\mathfrak{d} := [M, \delta]_G$ for $\delta \in \mathcal{O}^{\mathrm{t}}$. We denote by $\mathfrak{B}^2(G)$ the set of inertial discrete pairs in G.
- Let Q be a parabolic subgroup of G with Levi factor M. We denote by $\mathrm{Irr}^{\mathrm{t}}_{\mathfrak{d}}(G)$ the set of irreducible representations of G that occur in one of the induced representations $\mathrm{i}_{M,Q}^G(\delta \otimes \chi)$, with $\chi \in \mathfrak{X}_{\mathrm{unr}}(M)$.

Theorem [Plymen]

We have a decomposition:

$$C^*_{\mathrm{r}}(G) = \bigoplus_{\mathfrak{d} \in \mathfrak{B}^2(G)} C^*_{\mathrm{r}}(G; \mathfrak{d}),$$

where $C^*_{\mathrm{r}}(G; \mathfrak{d})$ is a subalgebra of $C^*_{\mathrm{r}}(G)$ with spectrum $\mathrm{Irr}^{\mathrm{t}}_{\mathfrak{d}}(G)$.

The group $W_{\mathfrak{d}}$

Let $\mathfrak{d} = [M, \delta]_G \in \mathfrak{B}^2(G)$. We set

$$\mathrm{N}_G(\mathfrak{d}) := \{n \in \mathrm{N}_G(M) \,:\, {}^n\delta \simeq \delta \otimes \chi \ \text{ for some } \chi \in \mathfrak{X}_{\mathrm{unr}}(M)\}$$

and define the stabilizer of $\mathcal{O}^{\mathrm{t}}$ (so called the *inertial stabilizer* of δ) as:

$$W_{\mathfrak{d}} := \mathrm{N}_G(\mathfrak{d})/M.$$

The group W_δ

For $\delta \in \mathrm{Irr}^2(M)$, we denote by W_δ its stabilizer in $\mathrm{N}_G(M)/M$:

$$W_\delta := \{n \in \mathrm{N}_G(M) \mid \ {}^n\delta \simeq \delta \ \}/M,$$

where ${}^n\delta$: $m \mapsto \delta(n^{-1}mn)$ of M. We have

$$W_\delta = W_\delta^0 \rtimes R_\delta.$$

Remark

Obviously: $W_\delta \subset W_\mathfrak{d}$ for any $\delta \in \mathcal{O}^{\mathrm{t}}$.

Definition of W_δ^0 and R_δ

We have To each root α is attached a Levi subgroup $M_\alpha \supset M$. We have $\mu^{M_\alpha}: \mathcal{O}^{\mathrm{t}} \to \mathbb{R}^+$. The set

$$R^0 := \left\{\alpha \in R \,:\, \mu^{M_\alpha}(\delta) = 0\right\}$$

is itself a root system, and the group W_δ^0 is its Weyl group. We fix a positive system $R_+'^0$ in R^0, and set

$$R_\delta := \left\{w \in W_\delta \,:\, w(R_+^0) = R_+^0\right\}.$$

Definition

The fixed point δ is good if for every point $\delta' \in \mathcal{O}^{\mathrm{t}}$, the Knapp-Stein decompositions $W_{\delta'} = W^0_{\delta'} \rtimes R_{\delta'}$ and $W_\delta = W^0_\delta \rtimes R_\delta$ are compatible in the following sense:

1. we have $W^0_{\delta'} \subset W^0_\delta$,
2. and the R-group $R_{\delta'}$ is isomorphic with a subgroup of R_δ.

Question (open in general)

Does the action of $W_{\mathfrak{d}}$ on $\mathcal{O}^{\mathrm{t}}$ always admit a fixed point?

Theorem [Afgoustidis-A.]

Let G be a quasi-split symplectic, orthogonal or unitary group over a p-adic field F.
Then for every $\mathfrak{d} = (M, \mathcal{O}^{\mathrm{t}})_G \in \mathfrak{B}^2(G)$, the action of $W_{\mathfrak{d}}$ on $\mathcal{O}^{\mathrm{t}}$ has a fixed point.

Remark

However, the existence of a fixed point is not enough for our purpose, we will need a "good" fixed point.

Intertwining operators (works of Waldspurger, Langlands, Arthur)

For each $w \in W_{\mathfrak{d}}$ and every $\chi \in \mathfrak{X}_{\mathrm{unr}}(M)$, there is a certain intertwining operator $\mathcal{A}(w, \delta \otimes \chi)$ such that

$$\mathcal{A}(w_1 w_2, \delta \otimes \chi) = \eta_\delta(w_1, w_2)\, \mathcal{A}(w_1, \delta \otimes (w_2\chi))\, \mathcal{A}(w_2, \delta \otimes \chi),$$

for any w_1, w_2 in W_δ and $\chi \in \mathfrak{X}_{\mathrm{unr}}(M)$, where

$$\eta_\delta \colon W_\delta \times W_\delta \to \mathbb{C}^\times \text{ is a 2-cocycle.}$$

Proposition

For every $\delta' \in \mathcal{O}^{\mathrm{t}}$, the map $r \mapsto \mathcal{A}(r, \delta')$ defines a projective representation of $R_{\delta'}$. The multiplier of this projective representation is the restriction $\eta_{\delta'} \colon R_{\delta'} \times R_{\delta'} \to \mathbb{C}^\times$ of the cocycle η_δ.

Central extensions of R-groups [Arthur]

The theory of the R-group can be brought to full fruition once chosen a central extension

$$1 \to Z_\delta \to \widetilde{R}_\delta \to R_\delta \to 1$$

with the property that the 2-cocycle of $\widetilde{R}_\delta$ induced by η_δ is a coboundary. Such a central extension exists, and we can then choose a map $\xi_\delta \colon \widetilde{R}_\delta \to \mathbb{C}^\times$ that splits η_δ, in that:

$$\eta_\delta(w_1, w_2) = \frac{\xi_\delta(w_1 w_2)}{\xi_\delta(w_1)\xi_\delta(w_2)}, \quad \text{for any } w_1, w_2 \in \widetilde{R}_\delta.$$

Theorem [Arthur]

The irreducible components of $\mathrm{i}_{M,Q}^G(\delta)$ are in natural bijection with the set of irreducible representations of $\widetilde{R}_\delta$ with Z_δ-central character ζ_δ, where $\zeta_\delta \colon Z_\delta \to \mathbb{C}^\times$ is given by $\zeta_\delta(z) := \xi_\delta(z)^{-1}$ for $z \in Z_\delta$.

There is a central idempotent $\widetilde{p}$ acting on the group algebra of $\widetilde{R}_\delta$:

$$\widetilde{p} := |Z_\delta|^{-1} \sum_{z \in Z_\delta} \zeta_\delta(z)^{-1} z \in \mathbb{C}[Z_\delta].$$

Then we have

$$\widetilde{p}(\mathbb{C}[\widetilde{R}_\delta]) \simeq \mathbb{C}[R_\delta, \eta_\delta].$$

Theorem [Afgoustidis-A.]

Assume that δ is a good fixed-point for the action of $W_\mathfrak{d}$ on $\mathcal{O}^{\mathrm{t}}$. Then we have a strong Morita equivalence

$$C_r^*(G, \mathfrak{d}) \underset{\text{Morita}}{\sim} \widetilde{p}\left[\mathcal{C}(\mathcal{O}^{\mathrm{t}}/W_\delta^0) \rtimes \widetilde{R}_\delta\right].$$

How to interpret the result above:

- Strong Morita equivalence preserves spectra: if A and B are strongly Morita equivalent C^*-algebras, then their spectra are homeomorphic.
- We have obtained a strong Morita equivalence between a highly noncommutative C^*-block and (the image under $\tilde{p}$ of) an almost commutative C^*-algebra, namely the crossed product of a commutative C^*-algebra by a finite group. This allows one to infer the topology on the tempered dual, in a way which reflects reducibility of the induced representations.

Special case

If $\widetilde{R}_\delta = R_\delta$, then $\widetilde{p}$ is the identity, and we have

$$C_r^*(G, \mathfrak{d}) \underset{\text{Morita}}{\sim} \mathcal{C}(\mathcal{O}^{\mathrm{t}}/W_\delta^0) \rtimes R_\delta.$$