Lecture 9: The ABPS conjecture at the level of K-theory

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Introduction

Let G be a p-adic group. It follows from the work of Harish–Chandra that the irreducible representations of the reduced C^*-algebra $C^*_{\rm red}(G)$ can be identified with those of the Harish-Chandra-Schwartz algebra $\mathcal{S}(G)$. Thus we get

$$\mathrm{Irr}(C^*_{\rm red}(G)) = \mathrm{Irr}^{\rm t}(G)$$

which means that $C^*_{\rm red}(G)$ is the correct C^*-algebra to study the noncommutative geometry of the tempered dual of G.

More properties of the Bruhat-Tits building of G

The Bruhat-Tits building of G, denoted by $\beta(G)$, is a proper G-space, satisfying the following properties

- $\beta(G)$ satisfies the negative curvature inequality [**?**, 2.3] and hence is contractible and has unique geodesics
- every compact subgroup of G fixes a point of $\beta(G)$

These properties make $\beta(G)$ into a universal space for proper G-actions.

The Baum–Connes conjecture

Let $K_*^G(\beta(G))$ denote the G-equivariant K-homology of $\beta(G)$ as defined by Kasparov. The Baum–Connes conjecture asserts that the canonical assembly map

$$K_*^G(\beta(G)) \to K_*(C^*_{\mathrm{red}}(G))$$

is an isomorphism. This was proven by V. Lafforgue for a large class of groups which includes all p-adic reductive groups.

The Bernstein decomposition of the reduced C^*-algebra of G

We have

$$C_{\mathrm{r}}^*(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} C_{\mathrm{r}}^*(G)^{\mathfrak{s}}$$

with

$$\mathrm{Irr}(C_{\mathrm{r}}^*(G)^{\mathfrak{s}}) \simeq \mathrm{Irr}^{\mathrm{t}}(G) \cap \mathrm{Irr}^{\mathfrak{s}}(G).$$

Notation

- Let L be a Levi subgroup of G and σ a supercuspidal irreducible representation of L.
- Recall that $\mathfrak{X}_{\mathrm{nr}}(L)$ denotes the group of unramified characters of L and $\mathfrak{X}_{\mathrm{unr}}(L) \subset \mathfrak{X}_{\mathrm{nr}}(L)$ the subgroup of unitary unramified characters
- Recall $\mathfrak{s} = (L, \mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma)_G \in \mathfrak{B}(G)$ and the stabilizer of $\mathfrak{s}$ is

$$W^{\mathfrak{s}} := \{n \in \mathrm{N}_G(L) \,:\, {}^n\sigma \simeq \sigma \otimes \chi \text{ for some } \chi \in \mathfrak{X}_{\mathrm{nr}}(L)\} / L.$$

- $\mathcal{O}^{\mathrm{t}} := \mathfrak{X}_{\mathrm{unr}}(L) \cdot \sigma$, where σ is chosen to be unitary.
- The orbit $\mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma$ has the structure of a complex torus $T^{\mathfrak{s}}$.
- $\mathcal{O}^{\mathrm{t}}$ has the structure of a compact torus: it is the maximal compact subgroup, denoted by $T^{\mathfrak{s}}_{\mathrm{u}}$, of $T^{\mathfrak{s}}$.

Remark

If σ is unitary, it is a discrete series representation of L. Hence, we are considering the case of $\mathfrak{d} = (\mathcal{O}^{\mathrm{t}}, \sigma)_G = [L, \sigma]_G$.

Conjecture [A-Baum-Plymen (2011)]

If G is quasi-split, then we have

$$K_j(C_r^*(G)^{\mathfrak{s}}) \simeq K^j_{W^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \quad \text{for } j = 0, 1 \qquad (1)$$

where $K^j_{W_{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}})$ is the classical topological equivariant K-theory for the group $W^{\mathfrak{s}}$ acting on the compact torus $T^{\mathfrak{s}}_{\mathrm{u}}$.

The case of an arbitray p-adic group G:

Notation

- Let $C_0(T^{\mathfrak{s}}_{\mathrm{u}}) := \{f \in C(T^{\mathfrak{s}}_{\mathrm{u}} \sqcup \{\infty\}) : f(\infty) = 0\}$.
- For $\natural^{\mathfrak{s}} \colon W^{\mathfrak{s}} \times W^{\mathfrak{s}} \to \mathbb{C}^\times$ a 2-cocycle, there is $p_{\natural^{\mathfrak{s}}} \in \mathbb{C}[W^{\mathfrak{s}}]$ be a idempotent such that

$$p_{\natural^{\mathfrak{s}}}\mathbb{C}[\widetilde{W^{\mathfrak{s}}}] \simeq \mathbb{C}[W^{\mathfrak{s}}, \natural^{\mathfrak{s}}]$$

for a central extension $\widetilde{W^{\mathfrak{s}}}$ of $W^{\mathfrak{s}}$.

Conjecture [A-Baum-Plymen-Solleveld (2017)]

Let G be an arbitrary p-adic group. For each $\mathfrak{s} \in \mathfrak{B}(G)$, there exists a 2-cocycle $\natural^{\mathfrak{s}} \colon W^{\mathfrak{s}} \times W^{\mathfrak{s}} \to \mathbb{C}^{\times}$ such that

$$K^{j}_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \simeq K_j(C^*_{\mathrm{r}}(G)^{\mathfrak{s}}) \quad \text{for } j = 0, 1$$

where

$$K^{j}_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) := p_{\natural^{\mathfrak{s}}} K^{j}_{\widetilde{W^{\mathfrak{s}}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \simeq K_j(C_0(T^{\mathfrak{s}}_{\mathrm{u}}) \rtimes \mathbb{C}[W^{\mathfrak{s}}, \natural^{\mathfrak{s}}]).$$

Consequence

The ABPS conjecture at the level of K-theory asserts the existence of a bijection

$$K^{G}_{*}(\beta(G)) \to \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K^{j}_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}).$$

The local Langlands correspondence (enhanced version)

Notation

Groups attached to the L-parameter φ:

- $\mathrm{Z}_{G^\vee}(\varphi) := \{g \in G^\vee \,:\, g\varphi(w')g^{-1} = \varphi(w') \text{ for all } w' \in W'_F\}$
- $\mathcal{S}_\varphi := \mathrm{Z}_{G^\vee}(\varphi)/\mathrm{Z}_{G^\vee}(\varphi)^\circ$ component group of $\mathrm{Z}_{G^\vee}(\varphi)$

Definition

For the simplicity of exposition, we suppose in the lecture that G is quasi-split (an analogous but more technical definition works for an arbitrary G).

An enhanced L-parameter is a pair (φ, ρ), where φ is an L-parameter and ρ an irreducible representation of $\mathcal{S}_\varphi$.

The representation ρ is called an enhancement of φ.

Action of $G^\vee$ on the set of enhanced L-parameters:

For $g \in G^\vee$, we define

$$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho)$$

where ${}^g\rho$: $h \mapsto \rho(g^{-1}hg)$ for $h \in \mathrm{Z}_{G^\vee}(\varphi)$.

We set

$$\Phi_{\mathrm{e}}(G) := \{\text{enhanced } L\text{-parameters}\}/G^\vee.$$

A bijective LLC

has been constructed in the following cases (in addition to the $\mathrm{GL}_n(\mathbb{F})$-case):

- $\mathbb{F}$ archimedean: for any G (Langlands)
- $\mathbb{F}$ nonarchimedean:
  - $G = \mathrm{SL}_n(\mathbb{F})$ (and its inner twists): Hiraga-Saito (2012) $\mathrm{char}(\mathbb{F}) = 0$; A.-Baum-Plymen-Solleveld (2016) $\mathrm{char}(\mathbb{F}) > 0$
  - $G = \mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_{2n+1}(\mathbb{F})$ ($\mathrm{char}(\mathbb{F}) = 0$): Arthur (2013)
  - $G = \mathrm{G}_2(\mathbb{F})$: A.-Xu (2022), with $p \neq 2, 3$, Gan-Savin (2022).

The generalized Springer correspondence

Notation

Let $\mathcal{G}$ be a connected reductive group over $\mathbb{C}$ and let $\mathrm{Unip}_{\mathcal{G}}$ denote the unipotent variety of $\mathcal{G}$.
The category of $\mathcal{G}$-equivariant sheaves on $\mathrm{Unip}_{\mathcal{G}}$ has been the subject of many studies in geometric representation theory:

- Springer (1969, 1973) constructed the "Springer resolution" of $\mathrm{Unip}_{\mathcal{G}}$, and used it to give a geometric construction of the Weyl group representations from their actions on the cohomology of Springer fibers.
- Lusztig (1984) generalized the method and gave a description of all perverse sheaves on $\mathrm{Unip}_{\mathcal{G}}$ in terms representations of various relative Weyl groups.

Generalized Springer variety [Lusztig, Invent. math. 1984]

- $\mathcal{P} = \mathcal{L}\mathcal{U}$ parabolic subgroup of $\mathcal{G}$
- $u \in \mathcal{G}$ and $v \in \mathcal{L}$ unipotent elements.

The group $\mathrm{Z}_{\mathcal{G}}(u) \times \mathrm{Z}_{\mathcal{L}}(v)\mathcal{U}$ acts on the variety

$$Y_{u,v} := \left\{ y \in \mathcal{G} \,:\, y^{-1}uy \in v\mathcal{U} \right\}$$

by $(g,p) \cdot y := gyp^{-1}$, with $g \in \mathrm{Z}_{\mathcal{G}}(u)$, $p \in \mathrm{Z}_{\mathcal{L}}(v)\mathcal{U}$ and $y \in Y_{u,v}$.

- Let $A_{\mathcal{G}}(u) := \pi_0(\mathrm{Z}_{\mathcal{G}}(u))$ and $A_{\mathcal{L}}(v) := \pi_0(\mathrm{Z}_{\mathcal{L}}(v))$
- The group $A_{\mathcal{G}}(u) \times A_{\mathcal{L}}(v)$ acts on the set of irreducible components of $Y_{u,v}$ of maximal dimension, i.e. of dimension

$$\dim \mathcal{U} + \frac{1}{2}(\dim \mathrm{Z}_{\mathcal{G}}(u) + \dim \mathrm{Z}_{\mathcal{L}}(v)).$$

- Let $\sigma_{u,v}$ denote the corresponding permutation representation.

Definition [Lusztig, Invent. math. 1984]

Let $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$. Then ρ is called cuspidal if, for every unipotent element $v \in \mathcal{L}$

$$\langle \rho, \sigma_{u,v} \rangle_{A_{\mathcal{G}}(u)} \neq 0 \quad \Rightarrow \quad \mathcal{P} = \mathcal{G}$$

where $\langle\ ,\ \rangle_{A_{\mathcal{G}}(u)}$ is the usual scalar product on the space of class functions on $A_{\mathcal{G}}(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

Disconnected complex reductive groups [A.-Moussaoui-Solleveld, 2018]

Let $\mathcal{G}$ be a possibly disconnected reductive group over $\mathbb{C}$, with identity component $\mathcal{G}^\circ$. Let $u \in \mathrm{U}(\mathcal{G})$ and $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$.

- We have $A_{\mathcal{G}^\circ}(u) \subset A_{\mathcal{G}}(u)$.
- The pair (u, ρ) is called cuspidal if the restriction of ρ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ρ° such that one (or equivalently any) of the pairs (u, ρ°) is cuspidal.

Remark

Let $\mathcal{C}$ denote the $\mathcal{G}$-conjugacy class of u. If (u, ρ) is cuspidal, then $\mathcal{C}$ is a distinguished (i.e. $\mathcal{C}$ does not meet the unipotent variety of $\mathcal{L}$ for any $\mathcal{L} \neq \mathcal{G}$). However, in general not every distinguished unipotent class supports a cuspidal representation.

Notation/Remark

Let $\mathcal{C}$ denote the unipotent $\mathcal{G}$-conjugacy class of u. If $\mathcal{G}$ is connected, since its action on $\mathcal{C}$ is transitive, the (irreducible) $\mathcal{G}$-equivariant local systems on $\mathcal{C}$ are in $1-1$ correspondence with the (irreducible) representations of $A_{\mathcal{G}}(u)$:

$$\mathcal{F}_\rho \longleftrightarrow \rho.$$

We will say that $\mathcal{F}_\rho$ is cuspidal and $(\mathcal{C}, \mathcal{F}_\rho)$ is a cuspidal unipotent pair whenever ρ is cuspidal.

Remark

Let $\mathcal{C}$ be unipotent $\mathcal{G}$-conjugacy class such that $u \in \mathcal{C}$. If $\mathcal{G}$ is connected, since its action on $\mathcal{C}$ is transitive, the (irreducible) $\mathcal{G}$-equivariant local systems on $\mathcal{C}$ are in $1-1$ correspondence with the (irreducible) representations of $A_{\mathcal{G}}(u)$:

$$\mathcal{F}_\rho \longleftrightarrow \rho.$$

We will say that $\mathcal{F}_\rho$ is cuspidal and $(\mathcal{C}, \mathcal{F}_\rho)$ is a cuspidal unipotent pair whenever ρ is cuspidal.

Notation

- $\mathcal{P}$ parabolic subgroup of $\mathcal{G}$ (i.e., subgroup of $\mathcal{G}$ s.t. $\mathcal{P}^\circ$ is a parabolic subgroup of $\mathcal{G}^\circ$)
- $\mathcal{L}$ complement in $\mathcal{P}$ of its unipotent radical $\mathcal{U}$
- $\mathrm{Unip}_{\mathcal{G}}$ unipotent variety of $\mathcal{G}$, similarly, $\mathrm{Unip}_{\mathcal{P}}$, $\mathrm{Unip}_{\mathcal{L}}$
- $D_c^b(X)$ category of bounded constructible ℓ-adic sheaves on the algebraic stack X
- $\mathrm{Perv}_{\mathcal{G}}(\mathrm{Unip}_{\mathcal{G}})$ category of $\mathcal{G}$-equivariant perverse sheaves on $\mathrm{Unip}_{\mathcal{G}}$.

We consider the correspondence of algebraic stacks

$$\mathrm{Unip}_{\mathcal{L}}/\mathcal{L} \xleftarrow{\pi} \mathrm{Unip}_{\mathcal{P}}/\mathcal{P} \xrightarrow{\iota} \mathrm{Unip}_{\mathcal{G}}/\mathcal{G}$$

induced by the natural maps $\pi\colon \mathcal{P} \twoheadrightarrow \mathcal{L}$ and $\mathcal{P} \hookrightarrow \mathcal{G}$.

Geometric parabolic induction

The functor $\mathrm{i}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\colon D_c^b(\mathrm{Unip}_{\mathcal{L}}/\mathcal{L}) \to D_c^b(\mathrm{Unip}_{\mathcal{G}}/\mathcal{G})$ is defined by

$$\mathrm{i}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}} := \iota_! \circ \pi^*.$$

Interpretation of cuspidality

Let $\mathcal{F}$ be an irreducible $\mathcal{G}$-equivariant local system on a unipotent class $\mathcal{C}$ in $\mathcal{G}$.
The pair $(\mathcal{C},\mathcal{F})$ is cuspidal if and only if the Deligne-Goresky-MacPherson intersection cohomology complex $\mathrm{IC}(\mathcal{C},\mathcal{F})$ does not occur in $\mathrm{i}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}(D_c^b(\mathrm{Unip}_{\mathcal{L}}/\mathcal{L}))$ for any proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}$.

Theorem [Lusztig, 1984]

Let $\mathcal{C}$ be a unipotent class in $\mathcal{G} = \mathcal{G}^\circ$ and $\mathcal{F}$ an irreducible $\mathcal{G}$-equivariant local system on $\mathcal{C}$. Then $\mathrm{IC}(\mathcal{C}, \mathcal{F})$ occurs as a summand of $\mathrm{i}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\mathrm{IC}(\mathcal{C}_{\mathrm{cusp}}, \mathcal{F}_{\mathrm{cusp}}))$, for some triple $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{F}_{\mathrm{cusp}}))$, where $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$ with Levi subgroup $\mathcal{L}$ and $(\mathcal{C}_{\mathrm{cusp}}, \mathcal{F}_{\mathrm{cusp}})$ is a cuspidal unipotent pair in $\mathcal{L}$. Moreover, $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{F}_{\mathrm{cusp}}))$ is unique up to $\mathcal{G}$-conjugation.

Exists also for disconnected groups [A.-Moussaoui-Solleveld].

Definition

When $\mathcal{G} = \mathcal{G}^\circ$:

- The $\mathcal{G}$-conjugacy class of $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{F}_{\mathrm{cusp}}))$ is called the cuspidal support of $(\mathcal{C}, \mathcal{F})$.
- Let $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$. The cuspidal support of (u, ρ), denoted by $\mathrm{Sc}^{\mathcal{G}}(u, \rho)$, is defined to be

$$(\mathcal{L}, (v, \rho_{\mathrm{cusp}}))_{\mathcal{G}}, \quad \text{where } v \in \mathcal{C}_{\mathrm{cusp}} \text{ and } \mathcal{F}_{\mathrm{cusp}} = \mathcal{F}_{\rho_{\mathrm{cusp}}}. \qquad (2)$$

Definitions

Let φ be an L-parameter for G. For simplicity, we suppose that G is quasi-split (an analogous but more technical definition works for an arbitrary G). We set

- $\mathcal{G}_\varphi := \mathrm{Z}_{G^\vee}(\varphi(W_F))$: a (possibly disconnected) complex reductive group
- $u = u_\varphi := \varphi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$: unipotent element of $\mathcal{G}_\varphi$
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi))$.

We have

$$\mathcal{S}_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi). \tag{3}$$

Main idea:

(3) will allow us to use the generalized Springer correspondence for the complex group $\mathcal{G}_\varphi$ in order to understand the structure of the L-packets for the p-adic group G.

Definition [A.-Moussaoui-Solleveld, 2018]

An enhanced L-parameter $(\varphi, \rho) \in \Phi_{\mathrm{e}}$ is called cuspidal if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$, where $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$),
- (u_φ, ρ) is a *cuspidal pair* in $\mathcal{G}_\phi$.

We denote by $\Phi_{\mathrm{e,cusp}}(G)$ the set of $G^\vee$-conjugacy of cuspidal enhanced L-parameters for G.

The role of the generalized Springer correspondence

The generalized Springer correspondence allows us to define a cuspidal support map

$$\mathrm{Sc}\colon \Phi_{\mathrm{e}}(G) \to \bigcup_{L \text{ Levi de } G} \Phi_{\mathrm{e,cusp}}(L). \tag{4}$$

Remark

By the Jacobson–Morozov theorem, any unipotent element v of $\mathcal{L}$ can be extended (in a unique way up to $\mathrm{Z}_{\mathcal{L}}(v)^\circ$-conjugation) to a homomorphism of algebraic groups

$$j_v \colon \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L} \text{ satisfying } j_v \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = v.$$

Definition of the map Sc

Let (φ, ρ) be an enhanced L-parameter for G. Its cuspidal support is defined to be

$$\mathrm{Sc}(\varphi, \rho) := (\mathrm{Z}_{G^\vee}(\mathcal{T}), (\varphi_v, \rho_{\mathrm{cusp}}))$$

where $(\mathcal{L}, v, \rho_{\mathrm{cusp}})_{\mathcal{G}_\varphi}$ is the cuspidal support of (u_φ, ρ), $\mathcal{T} := \mathrm{Z}^\circ_{\mathcal{L}}$, and φ_v is defined by

$$\varphi_v(w, x) := \varphi(w, 1) \cdot \chi_{\varphi,v}(\|w\|^{1/2}) \cdot j_v(x) \quad \text{for all } w \in W_F,\ x \in \mathrm{SL}_2(\mathbb{C})$$

with

$$\chi_{\varphi,v} \colon z \mapsto \varphi\left(1, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) \cdot j_v \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \qquad \text{for } z \in \mathbb{C}^\times.$$