Strong Matrix Properties
and the
Inverse Eigenvalue Problem
Pre-meeting Materials

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1 Introduction

The following material provides some definitions, notation, and various background facts that we will use during the workshop. It may be helpful to review this information before the workshop begins. Section 2 covers matrices and patterns, Section 3 provides some basic information about matrix ranks and nullity, Section 4 covers matrices of graphs and digraphs, Section 5 reviews various facts about eigenvalues, Section 6 provides some results about nodal domains for trees, and finally, Section 7 gives a **general outline of topics to be covered in the workshop.**

2 Matrices and Patterns

An $m\times n$ matrix $A = [a_{ij}]$ is an array with entries in m rows and n columns, where a_{ij} is the entry in row i and column j. The *size of a matrix* is the number of rows and columns, expressed as $m\times n$. A *square matrix* is a matrix with $m = n$. The set of $m\times n$ matrices with entries from a field $\mathbb{F}$ is denoted by $\mathbb{F}^{m\times n}$. Matrices with entries in the real numbers $\mathbb{R}$ or in the complex numbers $\mathbb{C}$ are called *real matrices* or *complex matrices*, respectively.

size of a matrix
square matrix
$\mathbb{F}$
$\mathbb{F}^{m\times n}$
$\mathbb{R}$
$\mathbb{C}$
real matrices
complex matrices

A *vector* is a matrix with either one column (a column vector) or one row (a row vector). We indicate vectors in bold, as in $\mathbf{x}$, and all vectors are column vectors unless otherwise indicated. The zero vector $\mathbf{0}$ and the vector $\mathbf{1}$ with each entry equal to 1 will be taken to be of appropriate size when they appear, unless the size is specified via subscript as in $\mathbf{1}_m$.

vector
$\mathbf{0}$
$\mathbf{1}$

A *submatrix* B of a matrix A is a matrix obtained from A by deleting rows or columns of A. If R and C are subsets of row and column indices (possibly empty), then $A[R, C]$ is the submatrix that retains only rows R and columns C, while $A(R, C)$ deletes them. If A is a square matrix, then $A[S] = A[S, S]$ and $A(S) = A(S, S)$ are *principal submatrices*; in the case of a vector $\mathbf{v}$, the notation $\mathbf{v}[S]$ and $\mathbf{v}(S)$ have only one possible interpretation.

submatrix

principal submatrices

For a matrix $A = [a_{ij}] \in \mathbb{F}^{m\times n}$, its *transpose* is the matrix $A^\top$ given by $[a_{ji}] \in \mathbb{F}^{n\times m}$. Matrices A and B are *congruent matrices* if there exists an invertible matrix M such that $A = M^\top BM$. The matrix A is called a *symmetric matrix* if $A = A^\top$ and *skew-symmetric* if $A = -A^\top$. The set of all $n \times n$ real symmetric matrices is denoted $\mathrm{Sym}(n)$ and the $n \times n$ real skew-symmetric matrices by $\mathrm{Skw}(n)$. A complex matrix A is *Hermitian* if it equals its conjugate transpose (take the transpose of the matrix and conjugate of each element) A^*.

transpose

$A^\top$

congruent matrices

symmetric matrix

skew-symmetric

$\mathrm{Sym}(n)$

$\mathrm{Skw}(n)$

Hermitian

A^*

Exercise 2.1. Prove that a matrix congruent to a symmetric matrix is also symmetric.

Given a monic polynomial $p(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + x^n$, its (Frobenius) *companion matrix* is

companion matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix} \tag{1}$$

The companion matrix has $p(x)$ as its characteristic polynomial.

A *pattern* (or pattern matrix; or zero-nonzero pattern) is a matrix with entries from the set $\{0, *\}$, while a *sign pattern matrix* is a matrix with entries from the set $\{0, +, -\}$.

pattern

$*$

sign pattern matrix

$+$

$-$

The *sign-pattern of a real matrix* $A = [a_{ij}]$ is the sign-pattern matrix $\mathrm{Sgn}(A)$ of the same size with entries $\{0, +, -\}$ corresponding to whether a_{ij} is zero, positive, or negative, respectively. The pattern matrix $\mathrm{Pat}(A)$ of A is defined in a similar way. All real matrices with the same sign-pattern as A from the *qualitative class* of A, $\mathrm{Q}(A)$. The *zero-nonzero class*, $\mathrm{Z}(A)$, is defined similarly when the underlying field is understood.

sign-pattern of a real matrix

$\mathrm{Sgn}(A)$

$\mathrm{Pat}(A)$

qualitative class

$\mathrm{Q}(A)$

zero-nonzero class

$\mathrm{Z}(A)$

Exercise 2.2. Find a 3×3 real symmetric matrix $\boldsymbol{A}$ and a matrix $\boldsymbol{B}$ that belongs to $\mathrm{Z}(\boldsymbol{A})$ but not to $\mathrm{Q}(\boldsymbol{A})$.

The following are non-symmetric 3×3 examples, one a sign-pattern and one a zero-nonzero pattern:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & - \\ + & 0 & - \\ 0 & + & - \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & * & * \end{bmatrix}. \tag{2}$$

We let $\boldsymbol{Q}(\mathcal{A})$, represent the set of matrices with sign (or zero-nonzero) pattern $\mathcal{A}$. We say a sign or zero-nonzero pattern $\mathcal{A}$ allows a property $\boldsymbol{P}$ if there is a matrix in $\mathrm{Q}(\mathcal{A})$ that has property $\boldsymbol{P}$. Since the example patterns in line (2) have the structure of a companion matrix, the pattern $\mathcal{A}$ only allows matrices having a characteristic polynomial with all positive coefficients, and $\mathcal{B}$ only allows matrices having a characteristic polynomial with all nonzero coefficients.

A pattern or matrix $\boldsymbol{A} = [a_{ij}]$ is *upper triangular* if $a_{ij} = 0$ for all entries with $i > j$ and *upper Hessenberg* if $a_{ij} = 0$ for all entries with $i > j + 1$; *lower triangular* and *lower Hessenberg* patterns and matrices are defined in a similar way. A *triangular matrix or pattern* is either upper or lower triangular, and a *Hessenberg matrix or pattern* is either upper or lower Hessenberg. A pattern or matrix is *nowhere zero* provided each of its entries is nonzero.

upper triangular
upper Hessenberg
lower triangular
lower Hessenberg
triangular matrix or pattern
Hessenberg matrix or pattern
nowhere zero

The *diagonal of a matrix* $\boldsymbol{A} = [a_{ij}]$ consists of the entries with $i = j$. $\boldsymbol{A}$ is a *diagonal matrix* if $a_{ij} = 0$ when $i \neq j$. The construction $\operatorname{diag}(d_1, \ldots, d_n)$ represents a diagonal matrix $\boldsymbol{D} = [d_{ij}]$ with $d_{ii} = d_i$ for each i (and all other entries zero). A *Z-matrix* is a real square matrix each of whose off-diagonal entries is non-positive.

diagonal of a matrix
diagonal matrix
$\operatorname{diag}(d_1, \ldots, d_n)$
Z-matrix

The *direct sum* $\boldsymbol{A} \oplus \boldsymbol{B}$ of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the *block diagonal* matrix

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

direct sum
$\boldsymbol{A} \oplus \boldsymbol{B}$
block diagonal

The $n \times n$ *identity matrix*, I_n or just I if n is clear, is $\operatorname{diag}(1, \ldots, 1)$. A square matrix $\boldsymbol{A}$ is *invertible* if there exists a (unique) square matrix $\boldsymbol{A}^{-1}$ with $\boldsymbol{A}\boldsymbol{A}^{-1} = \mathrm{I}$. Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar* if there exists an invertible matrix $\boldsymbol{M}$ with $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{M}$.

identity matrix
I_n
invertible
$\boldsymbol{A}^{-1}$
similar

A *permutation matrix* is an $n \times n$ matrix with exactly one 1 in each row

permutation matrix

and column and 0 entries otherwise.

3 Rank and Nullity

The *nullspace* $\mathrm{Nul}(\mathbf{A})$ of a matrix A is the subspace of vectors $\mathbf{v}$, called *null vectors*, such that $A\mathbf{v} = \mathbf{0}$. The dimension of the nullspace is the *nullity*, $\mathrm{nul}(A)$. The *rank* of A, $\mathrm{rank}(A)$, is the dimension of the subspace spanned by the columns of A (the *column space*) $\mathrm{CS}(A)$, which is also equal to the dimension of the rowspace.

nullspace $\mathrm{Nul}(\mathbf{A})$ null vectors nullity $\mathrm{nul}(A)$ rank $\mathrm{rank}(A)$ column space

Theorem 3.1 (Rank-Nullity Theorem)**.**
For an $m \times n$ matrix A, $\mathrm{rank}(A) + \mathrm{nul}(A) = n$.

If $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is an $n\times n$ matrix, and B is an $m\times m$ invertible submatrix of A, then the *Schur complement* of A with respect to B is defined as $A/B = E - DB^{-1}C$. It is known that $\mathrm{rank}(A) = m + \mathrm{rank}(A/B)$.

Schur complement

Two $n \times n$ matrices A and B are *similar matrices* if there exists an invertible matrix P such that $A = PBP^{-1}$. The matrix A is *diagonalizable* if it is similar to a diagonal matrix.

similar matrices diagonalizable

Theorem 3.2 (Rank-One Decomposition)**.** Suppose that A is an $n \times n$ diagonalizable matrix with rank r. Then there exist r rank-one matrices $A_1, \ldots, A_r$ such that $A = A_1 + \cdots + A_r$. Each $A_i = \lambda_i \mathbf{x}_i \mathbf{y}_i^\top$ where λ_i is a non-zero eigenvalue of A_i and $\mathbf{x}_i$ and $\mathbf{y}_i$ are right and left eigenvectors, respectively, and $\mathbf{y}_i^\top \mathbf{x}_i = \delta_{ij}$.

4 Graphs, Digraphs and Related Matrices

A *graph* $G = (V, E)$ consists of a set of vertices $V = V(G)$ and a set of edges $E = E(G)$, where each edge is an unordered pair of vertices. We often assume the vertices are $\{1, 2, \ldots, n\}$ and write, for example, $1 - 2$ for an edge; we say vertices 1 and 2 are the *endpoints* of this edge and the edge is *incident* to vertices 1 and 2. The *degree* of a vertex is the number of incident edges.

graph $1-2$ endpoints incident degree

When $u - v$ is an edge of a graph G, we say that the vertices u and v are *adjacent* in G. The *neighborhood* of a vertex v in a graph G, denoted $\mathrm{N}_G(v)$, is the set of vertices adjacent to v in G. Two vertices u and v are called *twin vertices* provided they have the same neighbors except possibly

adjacent neighborhood $\mathrm{N}_G(v)$ twin vertices one another.

A set of vertices is an *independent set* if those vertices are mutually non-adjacent. A graph G is a *bipartite graph* if its vertex set can be partitioned into two independent sets, called the *partite sets* of G.

independent set
bipartite graph
partite sets

A *simple graph* has no *loops* (an edge from a vertex to itself) or multiple edges (between the same two vertices). Unless otherwise stated, all graphs are assumed to be simple graphs.

simple graph
loops

A graph H is a *subgraph* of a graph G if the vertices and edges of H are subsets of those of G, in which case we write $H \subseteq G$. Special subgraphs of G can be obtained by *edge removal* of an edge e from $E(G)$, denoted $G \setminus e$; or *vertex removal* of a vertex v, denoted $G \setminus v$, in which case all edges incident to v are also removed; or *edge contraction* of an edge e, denoted G/e, that merges the two endpoints of e into a single vertex, keeping all other edges. An *induced subgraph* is a subgraph obtained by removing vertices; the subgraph induced by vertices $w_1, \ldots, w_m$, denoted $G[w_1, \ldots, w_m]$, is the induced subgraph obtained by removing all other vertices of G. A *minor* of a graph G is a subgraph that can be obtained through use of a sequence of the three operations of edge removal, vertex removal and edge contraction. If H is a minor of G, we write $H \leq G$.

subgraph
$H \subseteq G$
edge removal
$G \setminus e$
vertex removal
$G \setminus v$
edge contraction
G/e
induced subgraph
$G[w_1, \ldots, w_m]$
minor
$H \leq G$

Starting with graphs $G_1 = (V_1, E_2)$ and $G_2 = (V_2, E_2)$, their *join* is $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{(a, b) : a \in V_1, b \in V_2\})$, their *disjoint union* is $G_1 \sqcup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and their *graph direct sum* is $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \vartriangle E_2)$, where $E_1 \vartriangle E_2$ is the symmetric difference of the edge sets.

join
disjoint union
graph direct sum

The *complement of a graph* G, G^{c}, is the graph with the same vertex set as G and edge $i{-}j$ if and only if $i \neq j$ and $i{-}j$ is not an edge of G.

complement of a graph
G^{c}

Two graphs are *isomorphic* if there exists a bijection between their sets of vertices that respects edges. A *drawing of a graph* on a surface is a representation of the graph using points on the surface as vertices and simple arcs[1] as edges. An *embedding of a graph* is a drawing for which arcs include only vertices that are their endpoints and do not intersect other arcs except at vertices.

isomorphic
drawing of a graph
embedding of a graph

The *graph of a symmetric matrix* $A = [a_{ij}]$ that is $n \times n$, denoted by $\mathrm{G}(A)$, has vertex set $\{1, \ldots, n\}$ and edge set $\{i{-}j : a_{ij} \neq 0 \, (i \neq j)\}$. Note that the diagonal entries of A do not play a role in determining $\mathrm{G}(A)$. For

graph of a symmetric matrix
$\mathrm{G}(A)$

[1] that is, non-self-intersecting continuous curves ($[0, 1] \to \mathbb{R}^2$) connecting two distinct points in a 2D plane

a given graph G, $\mathcal{S}(G)$ is the set of real symmetric matrices whose graph is G. $\mathcal{S}(G)$ is always nonempty and a distinguished member is the *adjacency matrix* of G, $\mathrm{A}(G)$, whose ij entry is 1 if i—j is an edge of G and 0 otherwise. Note that, for a simple graph, the diagonal entries of $\mathrm{A}(G)$ are all zeros. $\mathcal{S}(G)$ adjacency matrix $\mathrm{A}(G)$

The *minimum rank* of a graph G, $\mathrm{mr}(G)$, is the smallest rank among matrices in $\mathcal{S}(G)$. By Theorem 3.1, determining the minimum rank of a graph G is equivalent to finding the *maximum nullity* of the graph, $\mathrm{M}(G)$. Note that $M(G)$ is also the maximum multiplicity of a matrix in $\mathcal{S}(G)$. minimum rank $\mathrm{mr}(G)$ maximum nullity $\mathrm{M}(G)$

An often used observation is that if A is an $n \times n$ matrix of nullity $k > 0$ and $R \subseteq \{1, \ldots, n\}$, then there is a nonzero vector $\mathbf{x}$ in the nullspace of A with $\mathbf{x}[R] = O$.

Exercise 4.1. Let

$$A = \begin{bmatrix} a & f & 0 & 0 & 0 \\ f & b & g & 0 & 0 \\ 0 & g & c & h & 0 \\ 0 & 0 & h & d & i \\ 0 & 0 & 0 & i & e \end{bmatrix},$$

where each of f, g, h, i is nonzero.

(a) Show that if $\mathbf{x}$ is a nullvector whose first coordinate is zero, then $\mathbf{x} = \mathbf{0}$.

(b) Explain why each eigenvalue of A is simple (that is, its multiplicity is 1).

(c) What does this exercise say about the maximum multiplicity of an eigenvalue of a matrix whose graph is a path?

The *Laplacian matrix* of a graph G, $\mathrm{L}(G)$, is $\mathrm{L}(G) = D - \mathrm{A}(G)$, constructed from the adjacency matrix and a diagonal degree matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, where each d_i is the degree of vertex i. Laplacian matrix $\mathrm{L}(G)$

Exercise 4.2. Construct the Laplacian matrix for several examples of graphs defined above.

Exercise 4.3. Prove that $\mathrm{mr}(G) \le n - 1$ for every graph G. Hint: show that $\mathrm{L}(G)$ has a null vector consisting of all ones and so has rank at most $n - 1$.

Common Families of Graphs

The path P_n on n vertices is the graph with edges i—$(i+1)$ for $i = 1, \ldots, n-$ P_n

1. The cycle C_n on $n \geq 3$ vertices has edges $1{-}2, \ldots, (n-1){-}n$, and $n{-}1$. The complete graph K_n on $n \geq 2$ vertices has each pair of vertices joined by an edge; K_1 is a single vertex with no edges. The complete bipartite graph $\mathrm{K}_{a,b}$ has partite sets of size a and b and all possible edges between them.

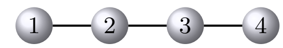

Figure 1: A drawing of P_4.

We say that a graph G *contains* a graph H if H is isomorphic to a subgraph of G. In this case, we refer to that subgraph as an H-subgraph or *copy* of H in G.

contains
copy

A *path* in a graph G is a copy of a P_n, a *cycle* is a copy of a C_n, and a *clique* is a copy of a K_n for some $n \geq 2$. A copy of K_1 is an *isolated vertex*.

path
cycle
clique
isolated vertex

Exercise 4.4. Compute the adjacency matrices for several examples of graphs defined above.

A *pendant vertex* is a vertex of degree one. An edge incident to a pendant vertex is a pendant edge. The two pendant vertices of a path are called its endpoints and we often say the path goes from one endpoint to the other or goes between the two vertices.

pendant vertex

A graph is *connected* if there is a path between any two vertices. Maximal connected subgraphs of a graph are its *connected components*. The *vertex connectivity*, $\kappa(G)$ of a graph is the minimum number of vertices whose removal results in a disconnected graph.

connected
connected components
vertex connectivity
$\kappa(G)$

A *tree* is a graph that contains no cycles. A *forest* is a graph in which every connected component is a tree. A symmetric matrix whose graph is a tree is called an *acyclic matrix*.

tree
forest
acyclic matrix

Theorem 4.5 (Graph Minor Theorem)**.** Every family of graphs that is closed under taking minors (*minor-closed*) can be characterized by a finite set of forbidden minors.

minor-closed

A graph is *planar* if it can be drawn in the plane without any crossing edges. Planar graphs are minor closed, and characterized by K_5 and $\mathrm{K}_{3,3}$ as forbidden minors.

planar

König-Egervary Theorem

A *matching* of a graph is a collection of vertex-disjoint edges. A *vertex cover* of a graph is a collection of vertices such that every edge of the graph is incident to at least one of the vertices in the collection.

matching
vertex cover

Theorem 4.6 (König–Egervary Theorem)**.** For bipartite graphs G, the maximum number of edges in a matching of G equals the minimum number of vertices in a vertex cover of G.

The König–Egervary theorem has implications for $m \times n$ matrices. Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *bigraph* of A, $\mathcal{B}_A$, has vertices $r_1, \ldots r_m; c_1, \ldots, c_n$ and an edge joining r_i and c_j if and only if $a_{ij} \neq 0$. Thus, $\mathcal{B}_A$ is a bipartite graph. Note that a matching of $\mathcal{B}_A$ corresponds to a collection of nonzero entries of A with no two in the same row or column. The *term-rank* of a matrix $A = [a_{ij}]$ is the maximum cardinality of such a collection; hence, the term-rank A equals the maximum size of a matching in the bigraph of A. A *line-cover* of A is a collection of rows and columns of A such that each nonzero entry of A is contained in at least one of the rows or columns in the collection. Thus, a line cover of A corresponds to a vertex cover of $\mathcal{B}_A$, and the minimum size of a line cover of A equals the minimum size of a vertex cover of $\mathcal{B}_A$. Thus we have:

bigraph
$\mathcal{B}_A$
term-rank
line-cover

Theorem 4.7 (Matrix form of König-Egervary)**.** Let A be an $m \times n$ matrix. Then the term-rank of A equals the minimum size of a line cover of A.

Two fun exercises related to term-rank are the following.

Exercise 4.8. Let P be an $m \times n$ zero-nonzero pattern. Show that the term-rank of P is the largest rank of a real matrix with zero-nonzero pattern P. (Where is the assumption that the matrix is real used?)

Exercise 4.9. Let A be an $m \times n$ matrix. Show that the term rank of A is at most k if and only if A has a zero submatrix O whose dimensions sum to at least $m + n - k$.

Digraphs

A *directed graph* or *digraph* is a graph where the edges (or *arc*) are ordered pairs of vertices, written $1 \longrightarrow 2$ or simply as $(1, 2)$. A digraph is *strongly connnected* if there exists a directed path between any two vertices. A square matrix M defines a digraph $\mathrm{D}(M)$ as in the construction of a graph for symmetric matrices, and a directed graph D has a (not-necessarily symmetric) adjacency matrix $\mathrm{A}(D)$.

directed graph
digraph
arc
$1 \longrightarrow 2$
strongly connnected
$\mathrm{D}(M)$

A *walk* in a digraph is a path that respects the directions of the edges:

walk

$$v_1, v_1 \longrightarrow v_2, v_2, \ldots, v_{k-1} \longrightarrow v_k, v_k.$$

The length of a walk is the number of arcs in the walk. A *path* in a digraph is a walk with distinct vertices. A *cycle* in a digraph is a walk whose vertices are distinct, except for its first and last.

path
cycle

5 Eigenvalues

An *eigenvalue* of a complex matrix $\boldsymbol{A}$ is a complex number λ such that there exists a nonzero vector $\mathbf{v}$ for which $\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$. The dimension of the subspace of vectors for which this equation holds true is the *geometric multiplicty* of λ as an eigenvalue, written $\mathrm{gmult}_{\boldsymbol{A}}(\lambda)$. The eigenvalues of $\boldsymbol{A}$ are exactly the roots of the *characteristic polynomial* $\mathrm{char}_{\boldsymbol{A}}(x)$ of $\boldsymbol{A}$; the multiplicity of λ as a root of $\mathrm{char}_{\boldsymbol{A}}(x)$ is the *algebraic multiplicity* $\mathrm{amult}_{\boldsymbol{A}}(\lambda)$, which is always at least the geometric multiplicity. An eigenvalue of algebraic multiplicity one (and therefore geometric multiplicity one) is a *simple eigenvalue*. $\Lambda(\boldsymbol{A})$ is the multiset of eigenvalues of the square matrix $\boldsymbol{A}$ listed with their algebraic multiplicities.

eigenvalue
geometric multiplicty
$\mathrm{gmult}_{A}(\lambda)$
characteristic polynomial
$\mathrm{char}_{A}(x)$
algebraic multiplicity
$\mathrm{amult}_{A}(\lambda)$
simple eigenvalue
$\Lambda(A)$

Exercise 5.1. Show that the characteristic polynomial of the companion matrix of $p(x)$ is simply $p(x)$.

If $\boldsymbol{A}$ is complex Hermitian or real symmetric, then all eigenvalues are real and the two multiplicities are equal for each eigenvalue. Such a matrix is *positive definite* if all eigenvalues are positive and *positive semidefinite* if all eigenvalues are nonnegative.

positive definite
positive semidefinite

Theorem 5.2 (Cauchy's Interlacing)**.** If $\boldsymbol{A}$ is a real $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ and $\boldsymbol{B}$ is an $m \times m$ principal submatrix with eigenvalues $\beta_1 \le \cdots \le \beta_m$, then $\lambda_k \le \beta_k \le \lambda_{k+n-m}$ for $k \in \{1, \ldots, m\}$. If $m = n - 1$, then

$$\lambda_1 \le \beta_1 \le \lambda_2 \le \beta_2 \le \cdots \le \beta_{n-1} \le \lambda_n.$$

The *inertia* $\mathrm{i}(\boldsymbol{A})$ of an $n \times n$ complex matrix $\boldsymbol{A}$ is $(\mathrm{i}_+(\boldsymbol{A}), \mathrm{i}_-(\boldsymbol{A}), \mathrm{i}_0(\boldsymbol{A}))$, where $\mathrm{i}_+(\boldsymbol{A})$, $\mathrm{i}_-(\boldsymbol{A})$, $\mathrm{i}_0(\boldsymbol{A})$ are the number of eigenvalues of $\boldsymbol{A}$, according to algebraic multiplicity, whose real part is positive, negative, or zero, respectively.

inertia
$\mathrm{i}(A)$
$\mathrm{i}_+(A)$
$\mathrm{i}_-(A)$
$\mathrm{i}_0(A)$

Theorem 5.3 (Sylvester's Law of Inertia)**.** If $\boldsymbol{A}$ is a real symmetric matrix and $\boldsymbol{B}$ is congruent to $\boldsymbol{A}$ then $\mathrm{i}(\boldsymbol{B}) = \mathrm{i}(\boldsymbol{A})$.

The $m \times n$ matrix $\boldsymbol{X}$ *intertwines* the $m \times m$ matrix $\boldsymbol{A}$ and the $n \times n$ matrix $\boldsymbol{B}$ provided $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{B}$.

intertwines

Theorem 5.4 (Sylvester)**.** Let $\boldsymbol{A}$ be an $m \times m$ real matrix, and $\boldsymbol{B}$ be an $n \times n$ real matrix. Then the only matrix that intertwines $\boldsymbol{A}$ and $\boldsymbol{B}$ is the zero matrix if and only if A and B have no common eigenvalue. Moreover, if both A and B are symmetric, then X intertwines A and B if and only if X is in the span of the matrices of the form $\mathbf{u}\mathbf{v}^\top$ where $\mathbf{u}$ (respectively $\mathbf{v}$) is an eigenvector of A (respectively B)) corresponding to some common eigenvalue of A and B.

Theorem 5.5 (Parter-Wiener)**.** Suppose T is a tree, $A \in \mathcal{S}(T)$, and λ is an eigenvalue of A with $\operatorname{gmult}_A(\lambda) \geq 2$. Then there exists a vertex v of T such that $\operatorname{gmult}_{A(v)}(\lambda) = \operatorname{gmult}_A(\lambda) + 1$, and λ is an eigenvalue of at least three diagonal blocks of $A(v)$.

A matrix A is a *positive matrix* if all of its entries are positive real numbers; a *non-negative matrix* is defined similarly. The largest absolute value of an eigenvalue of A is called the *spectral radius*, $\rho(A)$ of A.

positive matrix
non-negative matrix
spectral radius
$\rho(A)$
irreducible

A matrix A is *irreducible* if it is not permutation conjugate to a block upper triangular matrix with at least two diagonal blocks. Equivalently, A is irreducible if $\mathrm{D}(A)$ is strongly connected.

Theorem 5.6 (Perron-Frobenius Theorem)**.** Let A be an $n \times n$ matrix. If A is either positive or irreducible non-negative, then $\rho(A)$ is a simple eigenvalue of A and has a corresponding positive eigenvector.

minimal polynomial
$\operatorname{minp}_A(x)$

The *minimal polynomial* of a matrix A, $\operatorname{minp}_A(x)$, is the monic polynomial of least degree for which $\operatorname{minp}_A(A)$ equals the zero matrix. The minimal polynomial always divides the characteristic polynomial, and the degree of minp_A is the dimension of the subspace consisting of the polynomials in A. The number of distinct eigenvalues of A is denoted by $q(A)$. It is known that $q(A) \leq \deg \operatorname{minp}_A(x)$, and equality holds if A is a Hermitian matrix.

Weighted Digraphs

edge weights
weighted adjacency matrix

If the digraph has *edge weights* assigned to each edge, the *weighted adjacency matrix* A can be constructed using those weights as the non-zero entries. The entries of powers $A^k = [a_{ij}^{(k)}]$ of the weighted adjacency matrix can be interpreted as the sum of the weighted directed walks from vertex i to vertex j of length k, with $a_{ij}^{(k)} = 0$ if there is no walk of length k from vertex i to j.

Let B be an $n \times n$ matrix. Then B determines a weighted digraph D (with vertices, $1, \ldots, n$; arcs $i \longrightarrow j$ if and only if $b_{ij} \neq 0$, and the weight of the arc $i \longrightarrow j$ is b_{ij}). A *composite cycle* of D is a collection of vertex disjoint cycles. The *weight of composite cycle* γ of D is the product of the weights the arcs on its cycles and is denoted by $\mathrm{wt}(\gamma)$. If a composite cycle γ consists of γ_v vertices and γ_t cycles, then the *sign* of γ, $\mathrm{sgn}(\gamma)$, is $(-1)^{\gamma_v - \gamma_t}$. The set of all composite cycles with v vertices is denoted by $\mathcal{C}_v(D)$.

composite cycle
weight of composite cycle
$\mathrm{wt}(\gamma)$
sign
$\mathrm{sgn}(\gamma)$
$\mathcal{C}_v(D)$

Theorem 5.7. Let $B = [b_{ij}]$ be an $n \times n$ matrix with weighted digraph D. Then

(a) $\det B = \sum_{\gamma \in \mathcal{C}_n(D)} \mathrm{sgn}(\gamma)\,\mathrm{wt}(\gamma)$.

(b) The coefficient of x^{n-v} in $\mathrm{char}_B(x)$ is

$$\sum_{\gamma \in \mathcal{C}_v(D)} (-1)^{\gamma_t}\,\mathrm{wt}(\gamma).$$

As an example, consider the matrix $A = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Its weighted digraph is given in Figure 2. The composite cycles on four vertices of these weighed digraph are given in Figures 3-5.

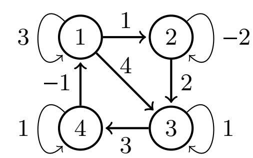

Figure 2: Digraph of A

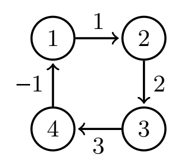

Figure 3: A composite cycle

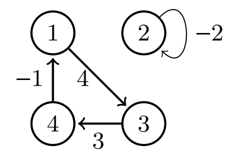

Figure 4: A composite cycle

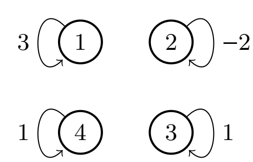

Figure 5: Another composite cycle

These respectively contribute

$$\begin{aligned} (-1)^{4-1}(1)(2)(3)(-1) &= 6, \\ (-1)^{4-2}(-1)(4)(3)(-2) &= 24, \\ (-1)^{4-4}(3)(-2)(1)(1) &= -6 \end{aligned}$$

Thus, $\det A = 6 + 24 - 6 = 24$.

6 Nodal Domain Theorem for Trees

We begin by noting that each combinatorially symmetric matrix whose graph is a tree is diagonally similar to a symmetric matrix with the same graph.

Lemma 6.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij}a_{ji} > 0$ whenever at least one of a_{ij} and a_{ji} is nonzero such that the graph with edge $i - j$ if and only if $a_{ij} \neq 0$ and $i \neq j$ is a tree T. Then there exists $D = \mathrm{diag}(d_1, \ldots, d_n)$ with each $d_i > 0$ such that DAD^{-1} is symmetric.

Proof. The proof is by induction on n. If $n = 1$ there is nothing to show. If $n = 2$, then $D = \operatorname{diag}(\sqrt{|a_{21}|}, \sqrt{|a_{12}|})$ works. Assume $n \geq 3$ and proceed by induction. Without of generality, assume $(n-1)$—n is a pendant edge in $\mathrm{G}(A)$. By the inductive hypothesis there exists $E = \operatorname{diag}(e_1, \ldots, e_{n-1})$ such that $E(A(1))E^{-1}$ is symmetric. Let a and b be the $(n-1, n)$ and $(n, n-1)$ entries of $E(A(1))E^{-1}$ respectively, and $F = \operatorname{diag}(1, \ldots, 1, \sqrt{|b|}, \sqrt{|a|})$. Then $D = FE$ is the desired diagonal matrix for A. □

An argument like that for Theorem 6.1 can be used to show the following.

Lemma 6.2. Let $A \in \mathcal{S}(T)$ where T is a tree. Then there is a diagonal matrix D, each of whose diagonal entries lie in $\{\pm 1\}$, such that DAD is a Z-matrix.

We wish to describe the relationship between eigenvectors of A corresponding to an eigenvalue λ and the location of λ_k among the eigenvalues of A (i.e., how many eigenvalues of A are less than λ_k?). We restrict to the case that A is a Z-matrix, as the general case can be readily interpreted from this. Additionally, without loss of generality we assume $\lambda = 0$.

Let $\mathbf{v}$ be a null vector of A having the largest number of nonzeros. Then $\mathbf{v}$ partitions the vertices of A into 3 sets:

$$
\begin{aligned}
\mathcal{I} &= \{i : \ v_i = 0 \text{ and } v_j = 0 \text{ for all } j \text{ adjacent to } i\},\\
\mathcal{P} &= \{i : \ v_i = 0 \text{ and } v_j \neq 0 \text{ for some } j \text{ adjacent to } i\},\\
\mathcal{S} &= \{i : v_i \neq 0\}.
\end{aligned} \tag{3}
$$

The vertices in $\mathcal{I}$ are called isolated vertices.

Theorem 6.3. Let T be a tree on n vertices, $A \in \mathcal{S}(T)$ be a singular Z-matrix, and $\mathbf{v}$ be a null vector of A with the largest number of nonzero entries. Let $\mathcal{I}$, $\mathcal{P}$ and $\mathcal{S}$ be as defined in (3), and a_+ and a_- be the number of edges i—j in T with $v_i v_j > 0$ or $v_i v_j < 0$, respectively. If $\mathcal{I} = \emptyset$, then

(a) $n_+(\lambda) = |\mathcal{P}| + a_+$.

(b) $n_-(\lambda) = |\mathcal{P}| + a_-$.

(c) $n_0(\lambda) = (\text{number of connected components of } T[S]) - |\mathcal{P}|$.

(d) The number of connected components of $T[S]$ is at most $n_-(A)$.

Proof. By permutation similarity, we may assume that

$$\mathbf{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w} \end{bmatrix},$$

where $\mathbf{w}$ is a nowhere-zero $m \times 1$ vector. Set $\widehat{A} = EAE$ and $\hat{w} = \mathbf{1}$. where $E = \operatorname{diag}(1, \dots, 1, 1/w_1, \dots, 1/w_m)$, Note that $\widehat{A}$ and A are congruent, and thus have the same inertia. With this replacement, $\widehat{A}$ has the form

$$\begin{bmatrix} B & D \\ D^\top & C_1 \oplus \cdots \oplus C_t \end{bmatrix},$$

where B is $(n-m) \times (n-m)$ and $C_1, \dots, C_t$ correspond to the connected components of $T[\{n-m+1, \dots, n\}]$, and each row sum of D and of C is 0.

Let $p = n - m$ be the order of B and $C = C_1 \oplus \cdots \oplus C_t$. The nullspace of C is the direct sum of the nullspace of the C_i $(i = 1, \dots, t)$. Thus, C_i has a nowhere zero null vector, and by Parter's theorem, C_i must have nullity 1. Hence $\operatorname{nul}(C) = t$.

Furthermore, for $i \in \{p, p-1, \dots, 1\}$, the matrix $A[\{i, \dots, n\}]$ has a graph that is a forest, and a nullvector $\mathbf{v}[\{i, \dots, n\}]$ that is zero on i and nonzero on a neighbor of i. Hence by Parter's theorem,

$$\operatorname{nul}(A[\{i, \dots, n\}]) \geq \operatorname{nul}(A[\{i+1, \dots, n\}]) + 1.$$

It follows that $\operatorname{nul}(A) = \operatorname{nul}(C) - p = t - p$, and $\operatorname{CS}(D) \cap \operatorname{CS}(C) = \{\mathbf{0}\}$.

As C has row sums 0, C is the sum of the matrices $v_i v_j (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top$ where i and j are adjacent vertices in $\{p+1, \dots, n\}$. Such a matrix has inertia $(1, 0, 1)$ if $v_i v_j$ is positive, and $(0, 1, 1)$ if $v_i v_j < 0$. Also

$$\begin{bmatrix} B & D \\ D^\top & O \end{bmatrix}$$

the sum of the matrices M_i $(i = 1, \dots, p)$ obtained by replacing each entry neither in row i nor column i by 0. As each row of D is nonzero, M_+ has one positive eigenvalue and one negative eigenvalue, and all other eigenvalues are 0. It follows that A is a sum of a_+ matrices of inertia $(0, 1, 1)$ and a_- matrices of inertia $(0, 1, 1)$ and p matrices of inertia $(1, 1, n-2)$. By the sub-additivity of inertia, $(\mathrm{i}_+(A), \mathrm{i}_-(A)) \leq (p + a_+, p + a_-)$. Note that

$$\begin{aligned} p + a_+ + p + a- &= 2p + \#\text{ of edges of } T[\{p+1, \ldots, n\}] \\ &= 2p + m - t = p + m - (t - p) \\ &= p + m - \mathrm{nul}(A). \end{aligned}$$

Thus $\mathrm{i}(A)$ is as claimed. □

7 Topics to be covered

1. **Overview of inverse eigenvalue problems.**

   This lecture hopes to discuss questions like: What's an inverse problem? How do inverse problems give rise to matrix theoretic problems? How are the analytic, geometric and algebraic properties of a matrix impacted by the combinatorial arrangement of its zero and nonzero entries? What does the topic this conference have to do with skyscrapers, archery, power lines, and F1-racecars?

2. **Foundations of the Inverse Eigenvalue Problem of a Graph.**

   This lecture will describe the early work that led to the Inverse Eigenvalue Problem for a Graph, the historical roadblocks encountered that frustrated many a mathematician, and the breakthrough idea of using the Inverse Function Theorem to implicitly solve many inverse eigenvalue problems.

3-4. **Theoretical Underpinnings of Strong Properties**

   Historically, the study of inverse eigenvalue problems focused on explicit methods. A major turning point was the introduction of implicit methods for inferring the solvability of families of inverse problems from an "especially nice" solution to a specific inverse problem. Here, meaning of "especially nice" the solution is generic in a sense that can expressed as a certain linear algebraic condition. This genericity condition leads to the notion of strong properties.

   In this lecture we motivate and establish the theoretical underpinnings of strong properties. This underpinnings and their consequence will be a common thread in the remaining lectures.

   The key theorems are:

- The Supergraph Theorem:
  If $A\mathcal{S}(G)$ has the SAP and H is a supergraph of G, there there is matrix in $\mathcal{S}(G)$ with the same nullity as A.
- The Proximity Theorem: if A has the SAP and graph G, and B is sufficiently close to A, there there is a matrix with graph G and the same nullity as B.
- Minor Monotonicity Theorem: If G is a graph minor of H, and $\mathcal{S}(G)$ contains a matrix with the SAP and nullity k, then so does $\mathcal{S}(H)$.

5. **SAP, ξ and graph minors.**
   $M(G)$ is an ill-behaved graph parameter. A nicer parameter, $\xi(G)$ is the largest nullity of a matrix in $\mathcal{S}(G)$ that has the SAP.

   We'll show that $\xi(G)$ is minor monotone; that is, if G is a graph minor of H, then $\xi(G) \leq \xi(H)$.

   Then we survey results about graphs with extreme values of $\xi(G)$.

6. **Jacobian Methods**

   We explore a number of different Jacobian methods for providing information about eigenvalues allowed by a matrix sign pattern or zero-nonzero pattern. The methods are applied to a number of inverse eigenvalue problems, such as determining if a sign pattern or zero-nonzero matrix pattern is inertially or spectrally arbitrary.

7. **The Strong Spectral Property.**
   The Strong Arnol'd Property allows one to study the maximum multiplicity of one eigenvalue at a time. Yet, inverse eigenvalue problems often concern the entire spectrum of a matrix.

   In this talk, we introduce and develop a property, known as the Strong Spectral Property (SSP) that provides a tool for implicitly inferring the existence to a family of inverse eigenvalue problems from an "especially nice" solution to one inverse problem.

8. **Zero Forcing and Invariants**
   This talk will discuss various combinatorial and geometric concepts and invariants related to maximum nullity, including orthogonal representations of graphs, and zero forcing (which can be used to provide an upper bound on the maximum multiplicity of a graph).

9. **The ν and μ parameters related to the Strong Arnol'd Property**

   $\xi(G)$ is related to <u>all</u> symmetric matrices graph G.

   For many applications, the set matrices of interest is more constrained. $\nu(G)$ is the maximum nullity among the positive semidefinite matrices with graph G. $\mu(G)$ is the maximum nullity among the so-called Colin de Verdiere (CDV)-matrices; that is, the matrices have the SAP, graph G, all off-diagonal entries non-positive, and exactly one negative eigenvalue.

   We survey results about $\mu(G)$ and $\nu(G)$; many of these are related to planarity or embedability properties of G.

10. **Eigenvectors and Strong Properties.**

    In the study of inverse eigenvalue problems, constructions of matrices with prescribed spectra often depend on methods that build larger matrices from smaller components. These constructions frequently depend on the specific eigenvectors of the matrices that are used as building blocks. While eigenvectors are even harder to study than eigenvalues, we show how the SSP allows us to approach them, at least in the case of distinct eigenvalues. As an application of these results, we demonstrate a method for constructing symmetric orthogonal matrices with prescribed patterns.

11. **Inverse problems for not necessarily symmetric matrices.**

    We describe a strong spectral property and a strong multiplicity property for not necessarily symmetric matrices. Considering matrix sign patterns or zero-nonzero patterns, we describe some classes of patterns that allow or require these strong properties. We also explore how the methods provide information about the number of distinct eigenvalues allowed by a pattern.

12. **Nonnegative Inverse Eigenvalue Problem.**

    The Nonnegative Inverse Eigenvalue Problem (NIEP), seeks to characterize all the multi-sets of eigenvalues that can be realized as the spectrum of an entry-wise nonnegative matrix. After reviewing selected highlights from the current state-of-the-art, we recast the problem to characterize the characteristic polynomials of nonnegative matrices. We retrace the steps taken in the previous lectures, and investigate how strong properties can help understand characteristic polynomials of nonnegative Hessenberg matrices.

13. **Inverse problems nullity pairs.**

    Given an $n \times n$ symmetric matrix A and an index r, the *nullity pair* of A with respect to r is $(\mathrm{nul}(A), \mathrm{nul}(A(r)))$.

    In this lecture we introduce and discuss a strong property for nullity pairs, and then use it to give characterizations of which graphs allow certain nullity pairs.

14. **Minimum number of distinct eigenvalues.**

    We explore implications of strong properties (and Jacobian methods) for the minimum number of distinct eigenvalues among matrices with a given graph or pattern. We also describe other techniques for providing information about the eigenvalues allowed by a graph or pattern.

15. **The Strong Inner Product Property (SIPP)**

    One of the oldest (still open) problems in combinatorial matrix theory is to characterize the zero-nonzero patterns of orthogonal matrices.

    This talk introduces a strong property that allows one to infer the existence of orthogonal matrices with given patterns from a single orthogonal matrix with the strong property,

16. **Patterns of orthogonal matrices**

    We highlight some results on graphs G with $q(G) = 2$.

17-18.5 **Inverse Singular Value Problems.**

In these 1.5 lectures we discuss the inverse signular value problem for patterns:

Given an $m \times n$ pattern matrix P with $m \le n$, which multi-sets of m nonnegative real numbers are the multiset of singular values of some matrix with pattern P?

18.5-20 **Inverse Symplectic Eigenvalue Problems.**

The appropriate setting for Newtonian physics is Euclidean space (that is, $\mathbb{R}^n$ endowed with the standard inner product). Mathematically, this gives rise to symmetric matrices, their eigenvalues, and orthogonal similarities that preserve the above.

Hamiltonian mechanices is set in symplectic space (that is, $\mathbb{R}^{2p}$ endowed with a nondegenerate alternating bilinear form). Mathematically, this gives rise to positive definite matrices, their so-called symplectic eigenvalues, and symplectic similarity which preserves the above.

In this lecture, we will introduce the needed background, and then study the inverse symplectic eigenvalue problem.

8 Resources

You may find the following additional resources helpful for the foundational notions and results that will be used during the conference.

Graph theory

R. Diestel, *Graph Theory*

- Chapter 1: The Basics
- Section 1.1: Graphs
- Section 1.5: Trees
- Section 1.6: Bipartite Graphs
- Section 1.7: Contraction and minors

- Sections 1-4.

Basics about functions of several real variables.

T. Garrity, *All the Mathematics You Missed [But needed to Know for Graduate School]*, Cambridge University Press, 2002.

- Section 3.3: Differentiation and Jacobians
- Section 3.4 The Inverse Function Theorem
- Section 3.5 The Implicit Function theorem

Matrix Theory R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 3rd Edition, 2013.

- Section 0.2: Matrices
- Chapter 1: Eigenvalues and Eigenvectors
- Section 2.6: The singular value decomposition
- Section 7.1-2 Positive definite and Semidefinite Matrices
- Section 8.2-8.3 Postive and nonnegative matrices.

Index

List of Symbols