

Examining Students' Understanding of Short-Term Exponential Behavior

Task

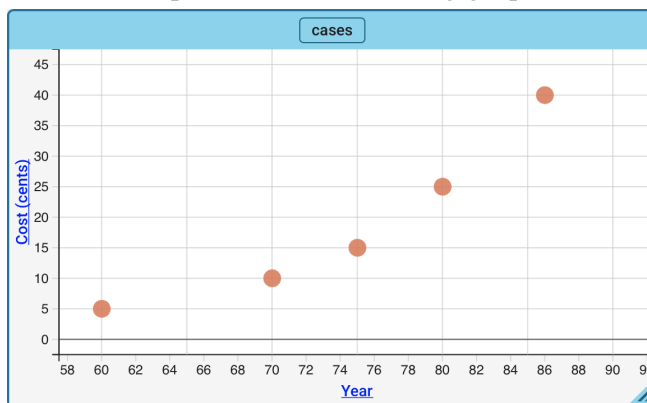
1. Read the scenario below and then discuss with a neighbor:
 - a. How would you describe Student A's approach and Student B's approach?
 - b. How would you explain to a student the difference between rates of change and ratios of change, and what each one means for modeling short-term behavior?
2. Using a graphing tool, plot some exponential curves. Then change the view of the graph so that the x -axis is linear and the y -axis is logarithmic.
 - a. What do you notice? How can what you see be explained in terms of ratios of change?
 - b. How would you explain log scales (e.g., decibels, Richter scale) to a student?

Scenario

A high school mathematics class is exploring how prices increase as an example of short-term exponential behavior, considering the cost of a Hershey Bar over the years as shown in the table¹:

Year	Cost
1960	5
1970	10
1975	15
1980	25
1986	40

As they examine data, the students produce the following graph.



The teacher then asks them to work on the following task.

¹ Based on data from Olver (2024) <https://www.foodtimeline.org/foodfaq5.html#candybar>.

Chocolate Index. In 1960, the cost of a Hershey Bar was 5 cents. 26 years later its cost has increased to 40 cents. Create a model that will predict the cost of a Hershey Bar in 52 more years in 2038.

The teacher overhears their students talking:

Student A: I computed $40 + 2 \times (40 - 5)$ to get 110 cents or \$1.10.

Student B: I disagree. It is $40 \times (40 \div 5)^2$ which is 2560 cents or \$25.60.

Student C: We just talked about log scales last week. Does that have anything to do with this?

Solution

In the scenario, Student A assumes a constant *rate of change* to predict the cost of a candy bar in 2038, whereas Student B assumes a constant *ratio of change* to predict this cost. When a function has constant *rate of change*, it is linear; when a function has constant *ratio of change*, it is exponential. Hence assuming a constant *rate of change* is more appropriate when there is reason to believe in linear short term behavior, and assuming a constant *ratio of change* is more appropriate when there is reason to believe in exponential short term behavior.

The selection of data suggests that the function of candy bar price over time is not linear, and that it could be exponential. (This can be a topic of further exploration with the class using more data from the same source, which we provide below.)

When graphing exponential curves on a view where the output axis is logarithmic and the input axis is linear, the curves appear as lines. The apparent “rate of change” is not actually a rate of change, but a ratio of change. Log scales reference an exponent to give a sense of the “scale” of what is being measured.

Commentary

There are two layers to this simulation of practice. First, it shows a common occurrence in high school teaching, when students turn to rates of change instead of ratios of change when first learning about exponential behavior. The discussion questions for the teachers are intended to sharpen their ability to explain these ideas in a student-friendly context.

Second, it opens the idea of different ways to represent data, including log scale graphs. When looking at exponential behavior with a log scale graph, the data points will approximate a line, and the “slope” can be interpreted as the ratio of change.

Overall, the big questions driving this simulation of practice are: What mathematical conceptions may underlie different approaches that students may use? How do we set up, represent, examine, and interpret mathematical quantities to represent contexts both within and outside of mathematics?

Potential extensions to this task include asking teachers to analyze the historical data on candy bars from Olver (2024) and using this to improve a prediction of future candy bar price, or comparing the appearance of linear and exponential data on a standard cartesian graph and on a logarithmic scale, and using the definitions of constant rate of change and constant ratio of change to explain the appearances.

One plot of the data from Olver (2024) is available at <https://www.desmos.com/calculator/szfbdlxq3n>.

Candy data

Source: Olver (2024) <https://www.foodtimeline.org/foodfaq5.html#candybar>

year	weight	cost
1908	9/16 oz	2 cents
1918	16/16 oz	3 cents
1920	9/16 oz	3 cents
1921	1 oz	5 cents
1924	1 3/8 oz	5 cents
1930	2 oz	5 cents
1933	1 7/8 oz	5 cents
1936	1 1/2 oz	5 cents
1937	1 5/8 oz	5 cents
1938	1 3/8 oz	5 cents
1939	1 5/8 oz	5 cents
1941	1 1/4 oz	5 cents
1944	1 5/8 oz	5 cents
1946	1 1/2 oz	5 cents
1947	1 oz	5 cents
1954	7/8 oz	5 cents
1955	1 oz	5 cents
1958	7/8 oz	5 cents
1960	1 oz	5 cents
1963	7/8 oz	.5 cents
1965	1 oz	5 cents
1966	7/8 oz	5 cents
1968	3/4 oz	5 cents

1969	1 1/2 oz	10 cents
1970	1 3/8 oz	10 cents
1973	1.26 oz	.10 cents
1974	1.4 oz	15 cents
1975	1.05 oz	15 cents
1976	1.2 oz	15 cents
1977	1.2 oz	.20 cents
1978	1.2 oz	25 cents
1980	1.05 oz	25 cents
1982	1.45 oz	30 cents
1983	1.45 oz	35 cents
1986	1.45 oz	40 cents
1986	1.65 oz	40 cents