

Connecting Error Bounds and Limit Theorems to the Concept of Square Root¹

Task

Read the scenario below. Then work on the following questions.

1. Describe a way to construct many different right triangles whose hypotenuse has length $\sqrt{2}$ units.
2. Do you agree or disagree with Student B that rounding early in this problem explains the difference in leg lengths?
3. Apply lessons learned about potential error accumulation from the Algebraic Limit Theorems to answer:
 - a. How much potential error was introduced in Student A's steps to find y ?
 - b. Why would the lessons learned about potential error accumulation from the Algebraic Limit Theorems not apply to Student A's steps to solve for x ?
4. Record a video of yourself providing a response to both Student A and Student B where you include a summary of what each student might be thinking and what is worthwhile or reasonable about that student's thinking, and a response to each student that does one or more of the following: helps the student finish their thought, prompts the student to investigate an error, or helps the student move forward in their thinking.

Scenario

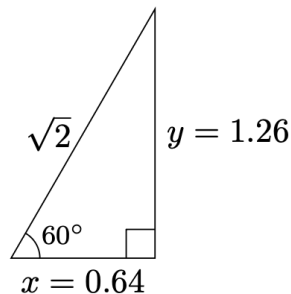
In a high school algebra class, students were asked to work on the following problem:
What are three different right triangles whose hypotenuse has length $\sqrt{2}$ units? Draw the triangles and label the legs with lengths.

After agreeing on the example of a right triangle with leg lengths of 1 and 1 units, two students work on finding lengths of another triangle with hypotenuse $\sqrt{2}$ units. They share their work with each other; their drawings and calculations follow:

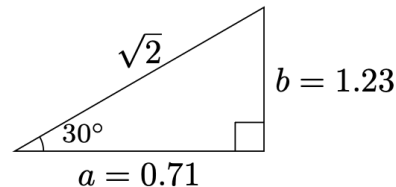
¹ This activity builds on a prompt in Wasserman et al. (2022) using ideas from MODULES² (n.d.)

Student A's calculations

$$\begin{aligned}\sin 60^\circ &= \frac{y}{\sqrt{2}} \\ 0.9 &= \frac{y}{1.4} \\ y &= 1.26 \\ (1.26)^2 + x^2 &= 2 \\ x^2 &= 2 - (1.26)^2 \\ x &= 0.64\end{aligned}$$

Student A's drawing**Student B's calculations**

$$\begin{aligned}\frac{b}{\sqrt{2}} &= \sin 30^\circ \\ b &= 0.5 \times \sqrt{2} \\ a^2 + b^2 &= 2 \\ a^2 &= 2 - \left(\frac{\sqrt{2}}{2}\right)^2 \\ a &= \sqrt{2 - \frac{1}{2}} \\ &= 1.23 \\ b &= 0.5\sqrt{2} = 0.71\end{aligned}$$

Student B's drawing

Student B: We found different triangles! We're done!

Student A: Wait. Our triangles should be the same. See, you have 30 degrees, so the other angle should be 60 degrees.

Student B: But our leg lengths are totally different!

Student A: But our work is the same.

Student B: You rounded early. Remember our teacher last year said don't round until the end.

Student A: I used a calculator. The numbers are right.

Solution and Commentary

One way to construct a right triangle with a particular hypotenuse is to construct a circle with the hypotenuse as its diameter and then taking any point on the circle as the third vertex. Students might also draw a side that creates an angle of less than 90 degrees from one end of the hypotenuse and construct the third side using a complementary angle, extending or truncating sides as needed.

As discussed in more detail in Wasserman et al. (2022), a consequence of the proof of the Algebraic Limit Theorem (that given a constant real c and two real convergent sequences a_n and b_n with respective limits a and b , we have $\lim (ca_n) = ca$, $\lim (a_n + b_n) = a + b$, $\lim (a_nb_n) = ab$, and $\lim (a_n/b_n) = a/b$ assuming b is nonzero) has to do with error accumulation. Specifically, if a' is an approximation for a , and b' is an approximation of b , then the error accumulation of ca' is bounded above by $|c(a-a')|$, the error accumulation of $a'b'$ is bounded above by $|b'(a-a')| + |a(b-b')|$, and the error accumulation of $1/b$ is bounded above by $2|b-b'|/b^2$. These ideas can explain why Student A's errors compound to result in values very different from Student B's, even though they in principle had found the same triangle.

There are many ways that prospective teachers might respond to Student A and Student B. Overall, when providing feedback on these videos, qualities to consider include:

- *Does the response respectfully attend to student thinking?* Ideally, prospective teachers will not only discuss specifics of each student's thinking and represented work, but also describe students' ideas with care and respect. This means pointing out both promising aspects of the student work while being compassionately honest about any errors or misconceptions.
- *Does the response discuss the logic underlying relationships, methods, or procedures used?* A good response will describe key mathematical concepts and how they show up in this specific problem.
- *Does the response position students to understand the logic underlying relationships, methods, or procedures used?* Responses that are helpful not only describe the key mathematical concepts but give the students something specific to think about in relation to these ideas and the particular work they showed.
- *Is the mathematical language used appropriate?* Ideas are described with accurate mathematical language.

References

Mathematics Of Doing, Understanding, Learning and Educating for Secondary Schools. (n.d.). <https://modules2.com>

Wasserman, N. H., Fukawa-Connelly, T., Weber, K., Pablo Mejia-Ramos, J., & Abbott, S. (2021). Algebraic Limit Theorems and Error Accumulation. In *Understanding Analysis and its Connections to Secondary Mathematics Teaching* (pp. 41-54). Springer.