Strong Properties of Matrices and Inverse Eigenvalue Problems

Bryan Shader
University of Wyoming

Helena Šmigoc
University College Dublin

Kevin N. Vander Meulen

Redeemer University Lon Mitchell
Eastern Michigan University

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Goals for the conference

Provide an on-ramp to a developing, rich area of mathematics.

Stimulate and nurture collaborations.

Influence your thinking and perspectives.

Become a community focused on attaining a deep understanding of "Strong Properties of Matrices & Inverse Eigenvalue Problems."

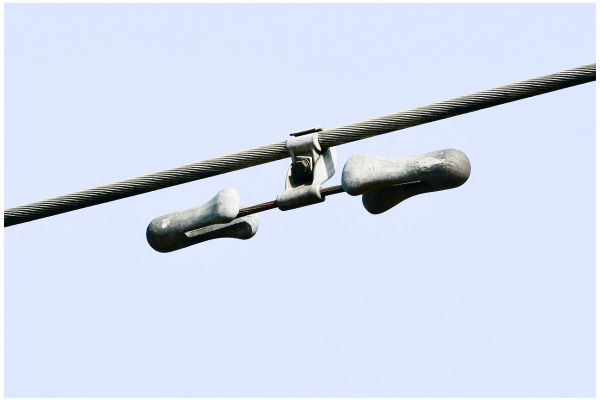
As a community, we will support each other to succeed this week, and beyond.

Provide underlying motivation for the subject.

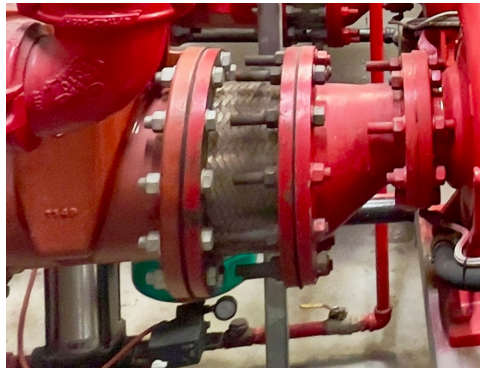
> *"In teaching, you do not want to cover things, you want to uncover them. The best way to get a good idea is to get a lot of them."* —*Linus Pauling*

Goals for this lecture

Tie the mathematics to the world around us.



Stockbridge damper



Pipe vibration mitigator



Archery stablilizer

Images from [S], [P], [B]

Top Ten

What items should be on the

"Top 10 Mathematical Inventions of All Time" list?

Top Ten

What items should be on the

"Top 10 Mathematical Inventions of All Time" list?

Find x.

Brilliant and gutsy idea–mathematizing

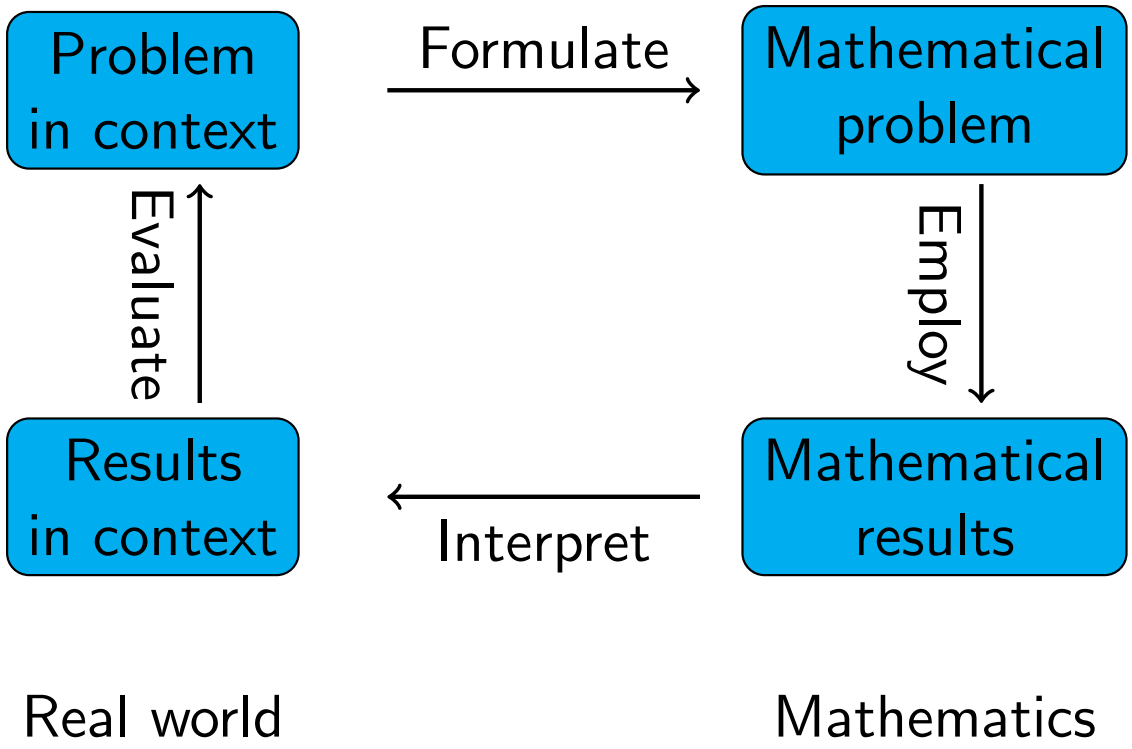


Image from [SFL]

Mathematizing leads to two types of problems.

Forward problems
In simplest terms, a forward problem seeks to find and analyze the output of a mathematical model given the governing equations and the inputs.

Solutions to forward problems help in predicting and establishing fundamental understandings.

Inverse problems
An inverse problem seeks to find the inputs or adjust the parameters of a model to achieve desired outputs. Inverse problems are inherently engineering or design problems.

Example: Population Model

Red maple trees fall into three age classes:
Seedlings (S), Juveniles (J), and Adults (A).

The dynamics of a population of red maple trees can be modeled by the matrix equation:

$$\mathbf{p}(t+1) = \begin{bmatrix} S(t+1) \\ J(t+1) \\ A(t+1) \end{bmatrix} = \begin{bmatrix} 0 & .9 & 1.4 \\ .9 & 0 & 0 \\ 0 & .6 & .8 \end{bmatrix} \begin{bmatrix} S(t) \\ J(t) \\ A(t) \end{bmatrix}$$
$$= L\mathbf{p}(t).$$

Here

$\mathbf{p}(t)$ is the population vector at time t.
L is the Leslie (or transition) matrix.
$\mathbf{p}(0)$ is the initial population.

Model equations

$$\mathbf{p}(t+1) = \begin{bmatrix} 0 & .9 & 1.4 \\ .9 & 0 & 0 \\ 0 & .6 & .8 \end{bmatrix} \mathbf{p}(t).$$

First row represents the fecundities (new trees). E.g., on average a juvenile creates .9 seedlings per time step; and adult 1.4 seedlings per time step.

Second row indicates that 90% seedlings survive to become juveniles; so 10% die-off.

The third row indicates that 60% of juveniles survive to adulthood, and 80% of adults survive the time step.

Forward problems

(a) Given an initial population $\mathbf{p}(0)$, what is the population at time $t = 10$?

Answer: $\mathbf{p}(10) = L\mathbf{p}(9) = L(L\mathbf{p}(8)) = \cdots = L \cdots L(\mathbf{p}(0)) = L^{10}\mathbf{p}(0)$.

(b) What's the long-term behavior of the population?

L has eigenpairs $(\lambda, \mathbf{u})$, $(\mu, \mathbf{v})$, $(\tau, \mathbf{w})$ with $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ linearly independent. So

$$\mathbf{p}(t) = L^t\mathbf{p}(0) = c\lambda^t\mathbf{u} + d\mu^t\mathbf{v} + e\tau^t\mathbf{w},$$

where $L(0) = c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$.

$$\mathbf{p}(t) = L^t\mathbf{p}(0) = c\lambda^t\mathbf{u} + d\mu^t\mathbf{v} + e\tau^t\mathbf{w}.$$

$\lambda \approx 1.07$, $\mu \approx .275$, $\tau \approx .44$.

Thus, if $c \neq 0$, then $\mathbf{p}(t)$ is asymptotically $c\lambda^t\mathbf{u}$.

So we can tell that the population grows exponentially and that the distribution will stabilize according to $\mathbf{u}$.

Inverse problem

What percent of the adult trees can be sustainably harvested?

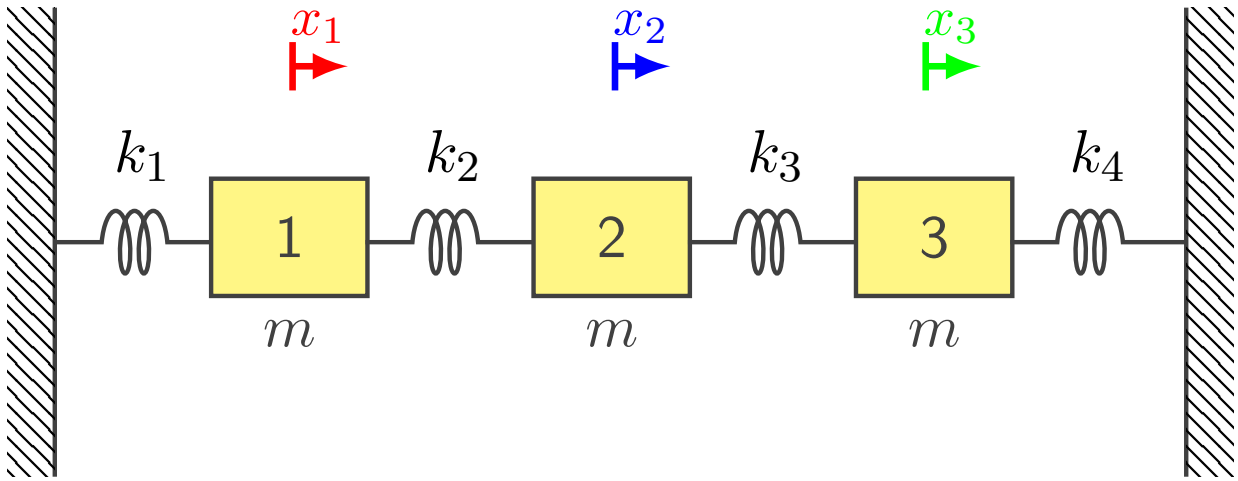
Asking to find the largest h so that

$$L_h := L - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h \end{bmatrix}$$

has its largest eigenvalue of modulus 1.

$h = .1$ or 10% works.

Example: Coupled spring mass system



Three blocks of equal mass m on a frictionless table are connected by four springs with Hooke constants k_1, k_2, k_3, and k_4. $x_i(t)$ are the displacements of the masses from rest at time t.

Forward problem: Can we describe the motion of the masses as a function of t given initial displacements and velocities?

Governing equations

Newton's and Hooke's Laws imply that the motion is governed by

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \frac{1}{m} \begin{bmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

is the stiffness matrix of the system $\ddot{\mathbf{x}} = \frac{-1}{m} K\mathbf{x}$.

(see [G] for the general theory of vibrating systems.)

Solutions: forward problem

K is symmetric, and its negative is the shift of an irreducible, nonnegative matrix. So each eigenvalue of $(-1/m)K$ is real and nonpositive.

$\frac{-1}{m}K$ has eigenpairs $(-\lambda, \mathbf{u})$, $(-\mu, \mathbf{v})$, $(-\tau, \mathbf{w})$ with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ linearly independent and λ, μ, $\tau > 0$.

The general solution is:

$$\begin{aligned}\mathbf{x}(t) = & A\cos(\sqrt{\lambda}t)\mathbf{u} + B\sin(\sqrt{\lambda}t)\mathbf{u} \\ & + C\cos(\sqrt{\mu}t)\mathbf{v} + D\sin(\sqrt{\mu}t)\mathbf{v} \\ & + E\cos(\sqrt{\tau}t)\mathbf{w} + F\sin(\sqrt{\tau}t)\mathbf{w}.\end{aligned}$$

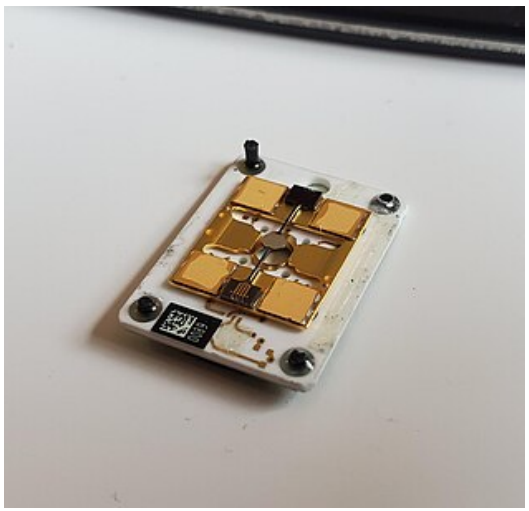
One can determine $\mathbf{x}(t)$ given initial conditions, maximum displacement, period, and fundamental angular frequencies $\sqrt{\lambda}$, $\sqrt{\mu}$ and $\sqrt{\tau}$.

Inverse Problem: Avoidance of resonance

The coupled spring mass system might be part of a larger system, e.g., an accelerometer within a smartphone.

To avoid conflict, one may want to design the system to avoid resonance with other components in the system.

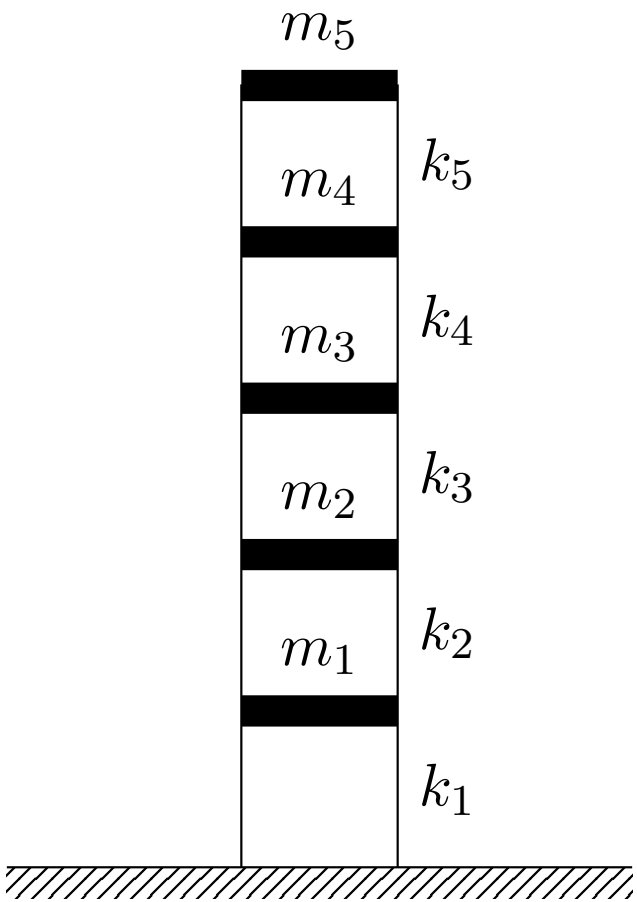
This requires designing a K whose eigenvalues avoid certain regions of the real line.



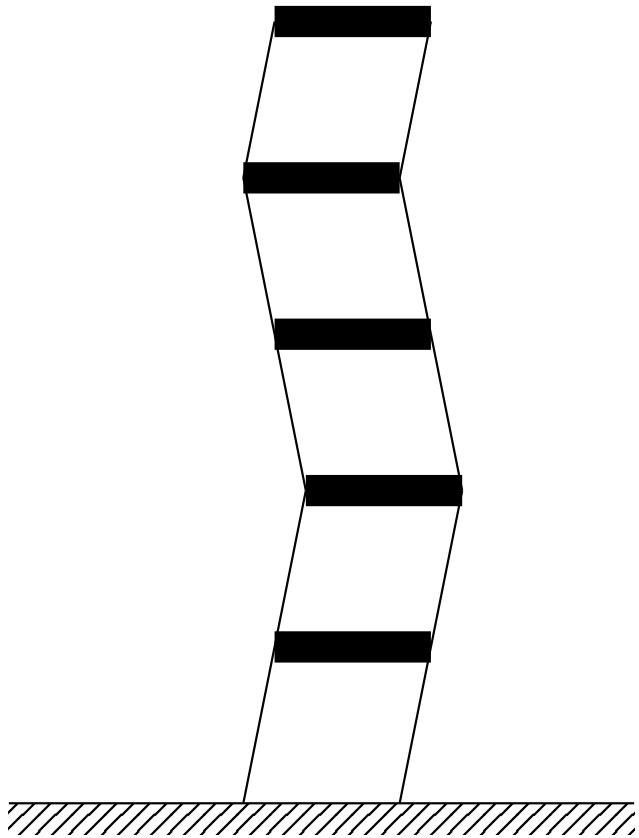
Accelerometer, gyroscope, and temperature sensor.

Image from [MEMS]

Inverse Problem: Tuned Mass Damper



5-story building
at rest



5-story building with
horizontal movements

Images from [HLS]

Model equations

Physics shows motion governed by:

$$M\ddot{\mathbf{x}} = -K\mathbf{x}$$

where

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & m_4 & \\ & & & & m_5 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & -k_3 & k_3 + k_4 & -k_4 & \\ & & -k_4 & k_4 + k_5 & -k_5 \\ & & & -k_5 & k_5 \end{bmatrix}$$

$m_i =$ the mass of the ith floor,

$k_i =$ stiffness of column i,

$$x(t) \in \mathbb{R}^5$$

is the vector of lateral displacements of the n stories at time t.

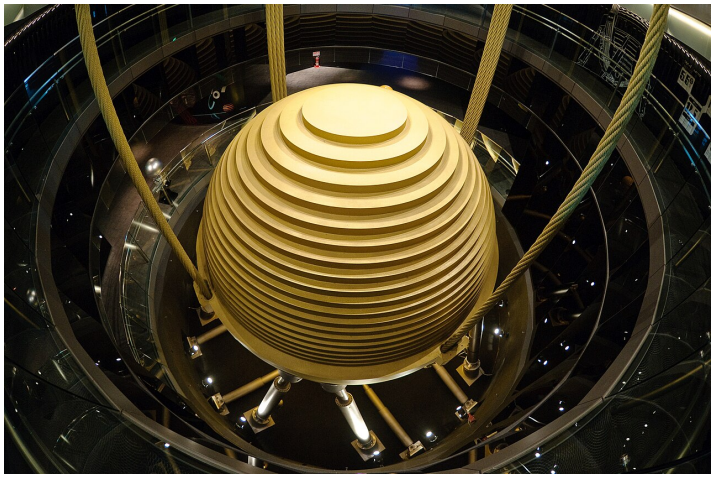
Forward problem

Fundamental angular frequencies correspond to $2\pi\sqrt{|\lambda|}$ where λ is an eigenvalue of $M^{-1}K$.

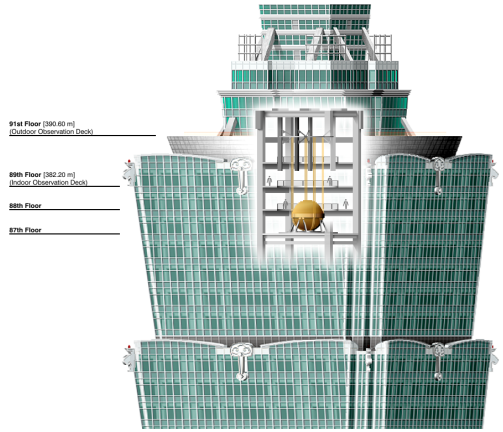
The structure of the building is reflected in the structure of $M^{-1}K$.

IEP-G problem arises as an engineering design problem. Find masses and stiffnesses to achieve a desired set of fundamental frequencies.

Tuned massed dampers in skyscrapers: Tapei 101



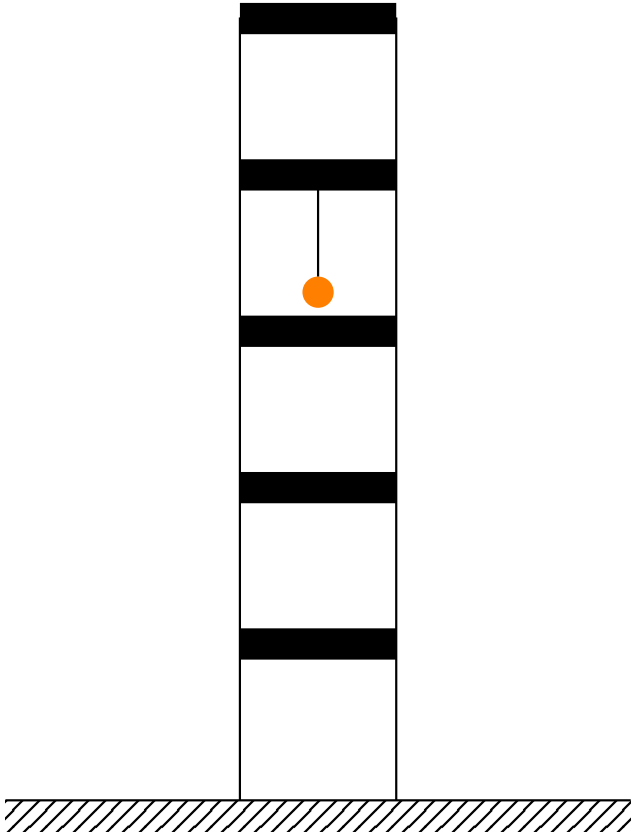
Tapei 101 TMD



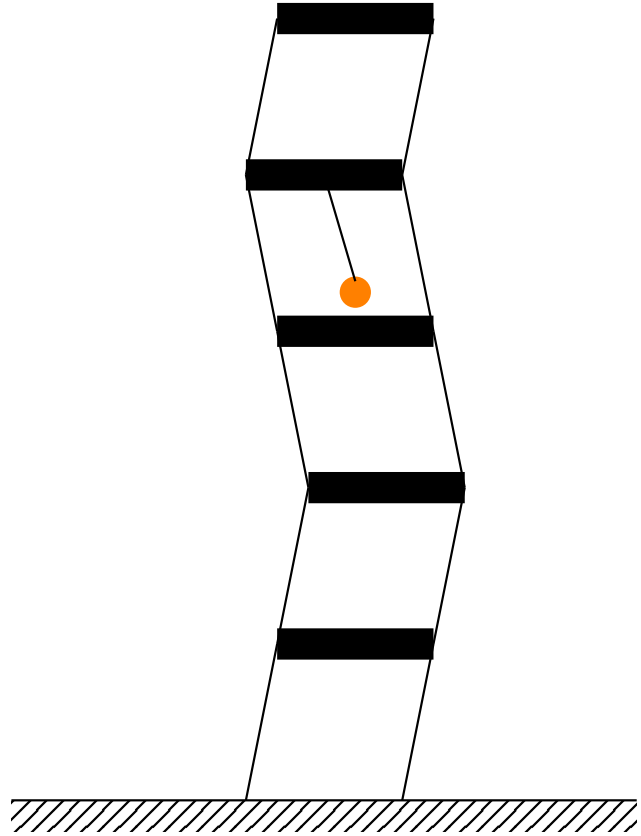
Tapei 101 structure

The 5.5 meter diameter, 660-metric-ton steel sphere is suspended by eight cables spanning 4 floors!

Images from [Tapei], [Tapei2]



Tuned pendulum
at rest



Tuned pendulum during movement

Pendulum "tuned" so that some of the frequencies of the building plus damper match the bothersome frequencies due to wind, ground vibration, and earthquakes.

Images from [HLS]

Take-aways from the toy problems

- Many real-world problems are described by matrix equations ($\mathbf{p}(t) = L^t\mathbf{p}(0)$ or $\ddot{\mathbf{x}} = \frac{-1}{m}K\mathbf{x}$.)
- Eigenvalues and eigenvectors are often the key ingredients in the general solutions of these equations.
- The structure of the underlying real-world problem is manifested in the structure of the governing matrices (e.g., Leslie matrices are entrywise nonnegative, the matrix K in spring-mass systems is symmetric and diagonally dominant).
- Inverse eigenvalue problems related to matrices with a given structure naturally arise.

General framework for inverse eigenvalue problem

Let $\mathcal{F}$ be a family of $n \times n$ matrices (typically, real matrices) and Λ be a multiset of n numbers.

A specific Inverse Eigenvalue Problem for $\mathcal{F}$ (IEP-$\mathcal{F}$) is:

Given Λ and $\mathcal{F}$ does there exist $A \in \mathcal{F}$ having eigenvalues Λ?

The general Inverse Eigenvalue Problem for $\mathcal{F}$ (a.k.a. IEP-$\mathcal{F}$) is:

Determine which Λ are the mutliset of eigenvalues of at least one matrix in $\mathcal{F}$.

Families of interest

- The set of all 5×5 companion matrices,

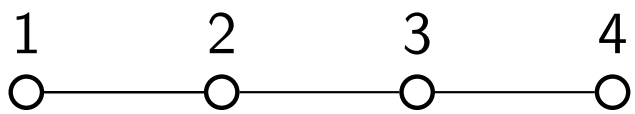
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -c_4 & -c_3 & -c_2 & -c_1 & -c_0 \end{bmatrix} \quad c_0, \ldots, c_4 \in \mathbb{R}.$$

- The set of all non-negative matrices with a given zero-nonzero pattern,

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ f & g & h & i \\ j & k & \ell & m \end{bmatrix} \quad a, \ldots, m > 0.$$

- The set $\mathcal{S}(G)$ of all symmetric matrices with graph G,

$$\mathcal{S}(G) = \left\{ \begin{bmatrix} a & b & 0 & 0 \\ b & c & d & 0 \\ 0 & d & e & f \\ 0 & 0 & f & g \end{bmatrix} : \quad b, d, f \neq 0 \right\}$$

$G =$ 

- The set of all matrices with the sign pattern S

$$S = \begin{bmatrix} + & + & 0 \\ 0 & + & + \\ - & 0 & + \end{bmatrix},$$

is denoted and defined by

$$\mathcal{Q}(S) = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & c & d \\ -e & 0 & f \end{bmatrix} : a, b, c, d, e, f > 0 \right\}$$

IEPs of interest

Leads to

- Nonnegative Eigenvalue Problem (NIEP)
- Inverse Eigenvalue Problem for Graphs (IEP-G)
- Nonsymmetric Inverse Eigenvalue Problem (nIEP)
- Inverse Eigenvalue Problem for Sign Patterns (IEP-$\mathcal{Q}$).

Prep work: change in nullity

Let A be an $n \times n$ symmetric matrix of nullity nul A.

How are nul A and nul $A(1)$ related?

$$A = \left[\begin{array}{c|c} a_{11} & \mathbf{c}^\top \\ \hline \mathbf{c} & A(1) \end{array}\right]$$

- Case: $\mathbf{c}$ not in CS$(A(1))$
  Then

$$\begin{aligned} \text{rank } A &= 1 + \text{rank} \left(\left[\frac{\mathbf{c}^\top}{A(1)} \right] \right) \\ &= 1 + 1 + \text{rank } A(1). \end{aligned}$$

  By the rank + nullity theorem:

$$\text{nul } A(1) = 1 + \text{nul } A \text{ when } \mathbf{c} \notin \text{CS}(A(1)).$$

$$A = \left[\begin{array}{c|c} a_{11} & \mathbf{c}^\top \\ \hline \mathbf{c} & A(1) \end{array}\right]$$

- Case: $\mathbf{c}$ in CS$(A(1))$
  So $\mathbf{c} = A(1)\mathbf{d}$ for some $\mathbf{d}$. Row and column operations give

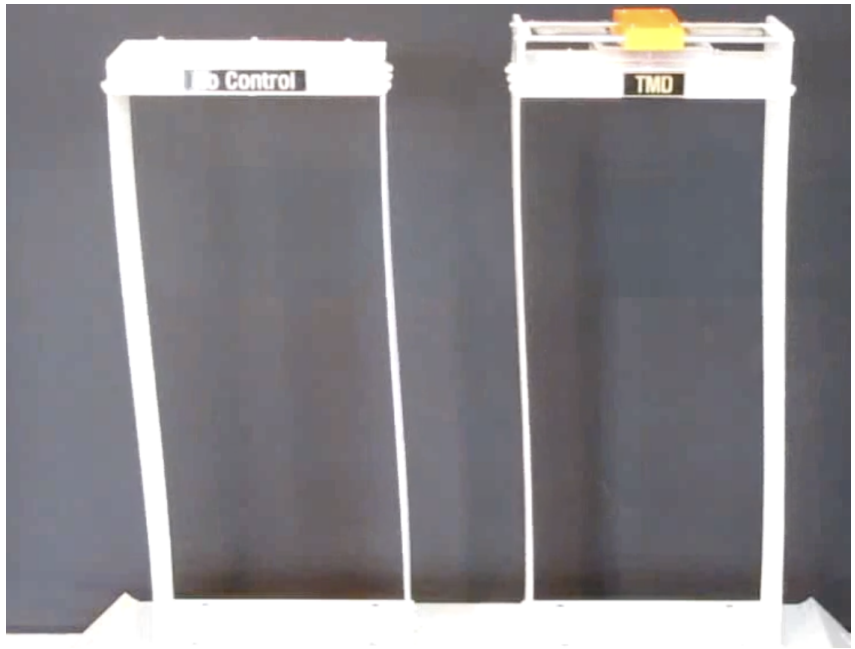
$$\text{rank } A = \text{rank}\left(\left[\begin{array}{c|c} a_{11} - \mathbf{d}^\top A(1)\mathbf{d} & \mathbf{0}^\top \\ \hline \mathbf{0} & A(1) \end{array}\right]\right)$$

So

nul $A(1) =$ nul A if $\mathbf{c} = A(1)\mathbf{d}$ and $a_{11} \neq \mathbf{d}^\top A(1)\mathbf{d}$;

nul $A(1) =$ nul$(A) - 1$, if $\mathbf{c} = A(1)\mathbf{d}$ and $a_{11} = \mathbf{d}^\top A(1)\mathbf{d}$.

Demo 1: Tuned Mass Damper in a Building

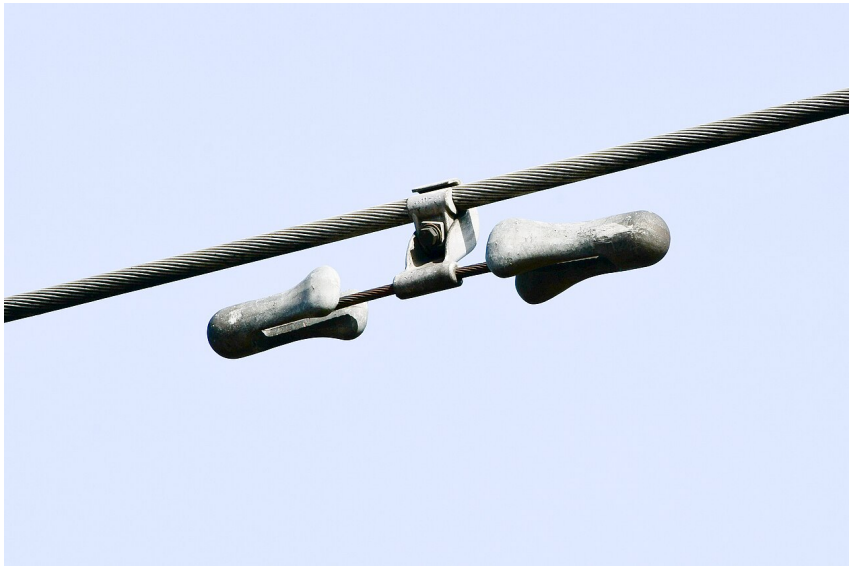


Tuned Mass Damper

Show video

Video from [TMD]

Demo 2: Stockbridge Damper

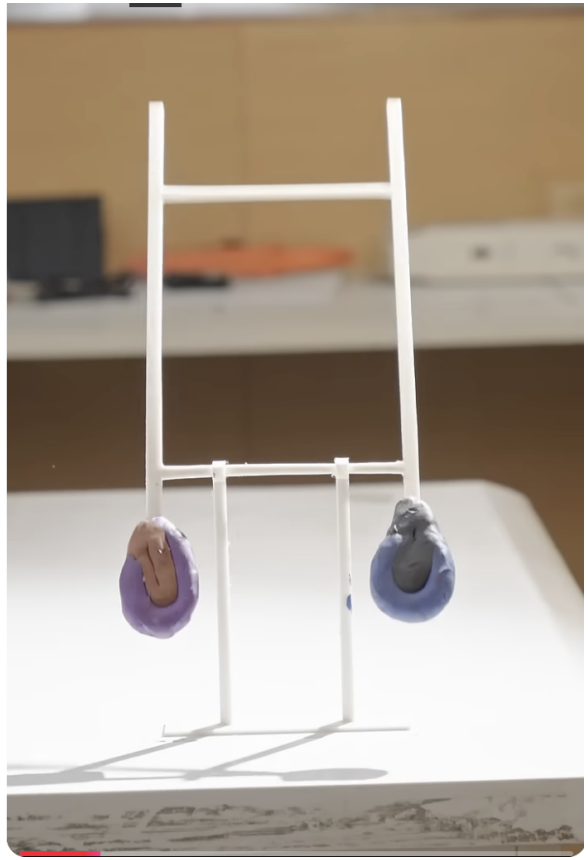


Stockbridge Damper

Show video

Video from [SD]

Demo 3: Simple demo

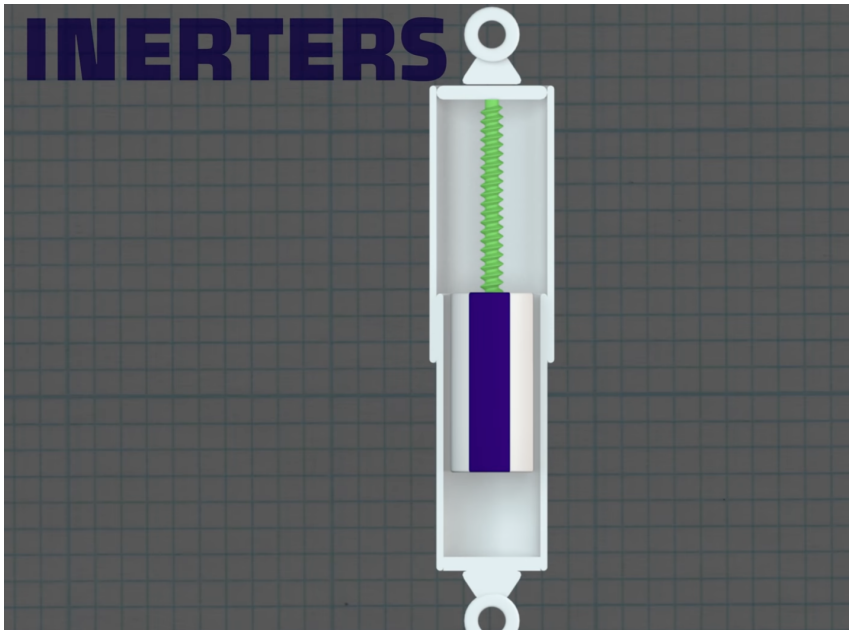


Swinging tuned damper

Show video

Video from [D]

McLaren's J-damper system



Inerter system

Show video

Video from [I]

References

[G] G. M. L. Gladwell, *Inverse problems in vibration*, second edition, Solid Mechanics and its Applications, 119, Kluwer Acad. Publ., Dordrecht, 2004.

[FHLS] S. M. Fallat et al., The inverse eigenvalue problem of a graph, zero forcing, and related parameters, Notices Amer. Math. Soc. **67** (2020), no. 2, 257–261.

[HLS]L. Hogben, J. C.-H. Lin and B. L. Shader, *Inverse problems and zero forcing for graphs*, Mathematical Surveys and Monographs, 270, Amer. Math. Soc., Providence, RI, [2022].

Photo credits (via Wikimedia Commons)

[FLS] E. Franchini, A. Lemmo, and S. Sbaragli. The role of text comprehension in the process of mathematization and modeling. Didattica della matematica. Dalla ricerca alle pratiche d'aula, 2017:38–63.

[S]
File:Stockbridge damper on 150kV power line Mcompanrheeze-Eindhoven Oost.jpg. (2021, November 7).

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[I]
https://www.youtube.com/watch?v=t58qjcNwEbo&t=277s