Strong spectral properties for nonsymmetric patterns

nSSP

- nSSP examples and theorems
- connect to previous lecture on Jacobian methods...
- examples of patterns that require the nSSP
- characterization of patterns that allow the nSSP
- insufficiency of nSSP for spectrally arbitrary patterns

nSMP

- definition, theorems, and examples
- patterns that require the nSMP
- patterns that allow the nSMP
- implications for an allow sequence

nSSP

A real $n \times n$ matrix A has the non-symmetric strong multiplicity property (nSMP) if

$$A \circ X = O \qquad AX^T = X^T A$$

$$\Downarrow$$

$$X = O$$

Note: no condition $X \circ I = O$.

Note: A has no zero entries $\Rightarrow$ A has the nSSP

nSSP

A real $n \times n$ matrix A has the non-symmetric strong multiplicity property (nSMP) if

$$A \circ X = O \qquad AX^T = X^T A$$

$$\Downarrow$$

$$X = O$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & x_{32} & x_{33} \end{bmatrix}$$

$$O = AX^T - X^T A = \begin{bmatrix} 0 & x_{22} & x_{32} + x_{33} \\ -x_{22} - x_{32} & 0 & x_{33} - x_{22} \\ -x_{33} & 0 & 0 \end{bmatrix} \Rightarrow X = O$$

A has the nSSP.

nSSP setup

[Fallat, Hall, Lin, Shader, 2022]

Let $\langle Z(\mathcal{A})\rangle = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} = 0 \text{ if } \mathcal{A}_{ij} = 0\}$

$$\mathfrak{g} : \mathcal{B}(O, \epsilon_1) \times \langle Z(\mathcal{A})\rangle \to \mathbb{R}^{n\times n}$$
$$\mathfrak{g}(L, B) = (I + L)^{-1}(A + B)(I + L).$$

Range of $d\mathfrak{g}$ is $\langle Z(\mathcal{A})\rangle + \{-LA + AL \mid L \in \mathbb{R}^{n\times n}\}$.

Orthogonal complement of $\langle Z(\mathcal{A})\rangle$ is $\{X \in \mathbb{R}^{n\times n} \mid A \circ X = O\}$

Orthogonal complement of $\{-LA + AL \mid L \in \mathbb{R}^{n\times n}\}$ is $\{X \mid AX^T = X^T A\}$ since

$$\begin{aligned} O &= \langle -LA + AL, X\rangle \\ &= \operatorname{tr}(X^T(-LA + AL)) = \operatorname{tr}(-X^T LA + X^T AL) \\ &= \operatorname{tr}(-LAX^T + LX^T A) \\ &= \langle X^T A - AX^T, L^T\rangle \end{aligned}$$

nSSP setup

Thus A has the nSSP $\implies$ $\mathfrak{g}$ is locally surjective.

$\implies$ $\exists \epsilon > 0$ such that $\forall M \in \mathbb{R}^{n \times n}$ within ϵ of A, $\exists B' \in \langle Q(\mathcal{A}) \rangle$ and $L' \in \mathbb{R}^{n \times n}$ such that

$$\mathfrak{g}(L', B') = (I + L')^{-1}(A + B')(I + L') = M$$

Let $A' = A + B'$. A' is similar to M. M sufficiently close to A implies $||B'||$ is small and $A' \in Q(\mathcal{A})$.

Theorem (Proximity theorem for nSSP)

Let A be a matrix with the nSSP. Then $\exists\ \epsilon > 0$ such that if Λ is the spectrum of a matrix within ϵ of A, then Λ is the spectrum of a matrix in $Q(A)$.

Other strong theorems for nSSP

Theorem (nSSP is an open property)

If A has the nSSP, then there exists $\epsilon > 0$ such that every matrix within ϵ of A has the nSSP.

Theorem (Superpattern Theorem for the nSSP)

Suppose A has digraph $D(A)$ and $D(A)$ is a subdigraph of H. If A has the nSSP, then there exists a matrix B with $D(B) = H$ that has the same spectrum as A.

Superpattern theorem for nSSP

Given $A \in Q(\mathcal{A})$ and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$.

$$\mathfrak{f} : \mathcal{B}(O, \epsilon_1) \times \langle Z(\mathcal{A}) \rangle \to \mathbb{R}^{n \times n}$$
$$\mathfrak{f}(L, B) = (I + L)^{-1} A (I + L) + B.$$

A has the nSSP $\implies$ $d\mathfrak{f}$ is locally surjective.

$\implies$ $\forall M \in Q(\hat{\mathcal{A}})$ close to A (e.g. $M = A + \delta S$), $\exists\ L', B'$ with

$$\mathfrak{f}(L', B') = (I + L')^{-1} A (I + L') + B' = M$$

Let $A' = M - B'$. Then

A' is similar to A.

$A' \in Q(\hat{\mathcal{A}})$.

And A' has the nSSP.

nSSP

A real $n \times n$ matrix A has the non-symmetric strong spectral property (nSSP) if

$$A \circ X = O \qquad AX^T = X^T A$$

$$\Downarrow$$

$$X = O$$

If A is a matrix with a zero diagonal, then A does not have the nSSP. In particular, take $X = I$. Then $AX^T = X^T A$ and $A \circ X = O$.

Note: In this case $D(A)$ has no loops (or 1-cycles).

Recall

Theorem (Cavers et.al. 2013)

Let A be a nonderogatory matrix. Then A allows a full-rank Jacobian if and only if A has the non-symmetric strong spectral property.

The above was observed for nonderogatory **nilpotent** matrices in [Garnett, Shader. 2012]

Jacobian method slightly weaker than nSSP

[Fallat, Hall, Lin, Shader, 2022]

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

A has the nSSP since $X \circ A = O \implies X = O$.
However, $A^2 = O \implies A$ is derogatory $\implies A$ does not allow a full-rank Jacobian

nSSP open property implies that there is a non-derogatory nilpotent matrix N nearby (perturb a Jordan form) with nSSP and hence N has a full-rank Jacobian.

A pattern can be verified to be spectrally arbitrary via nSSP if and only if the pattern can be verified to be spectrally arbitrary via the nilpotent-Jacobian method.

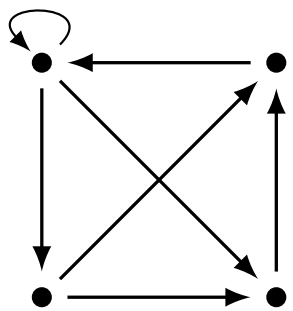
Digraph structure

Theorem (Cavers, Li, VM, 2026+)

If $n \times n$ A has the nSSP, then $D(A)$ has a composite k-cycle for each k, $1 \leq k \leq n$.

$$A = \begin{bmatrix} 3 & 0 & -4 & -5 \\ -6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$D(A)$



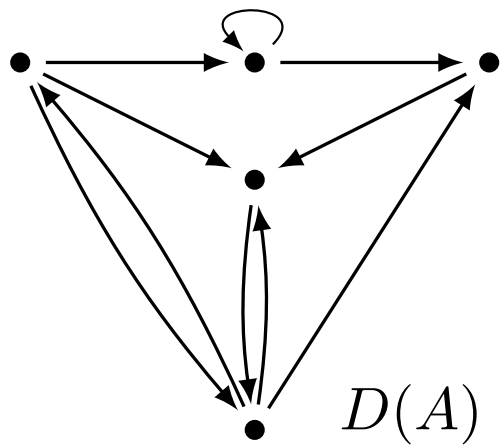
$D(A)$ has no composite 2-cycles $\Rightarrow$ A does not have the nSSP.

Theorem (Cavers, Li, V, 2026+)

If $n \times n$ A has the nSSP, then $D(A)$ has a composite k-cycle for each k, $1 \leq k \leq n$.

composite cycle condition not sufficient

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



$D(A)$ has composite cycles of all sizes, but A does not have nSSP

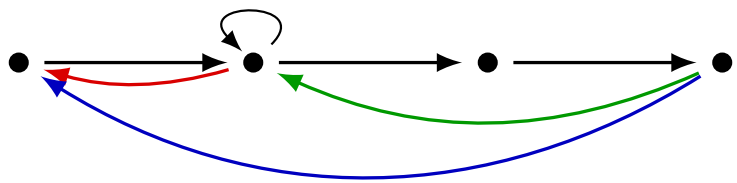
$$A \circ X = O \qquad AX^T = X^T A \qquad X \neq O$$

A cycle condition sufficient for Hessenberg matrices

Theorem (Koljančić, Oblak, 2026+)

If A is proper (lower) Hessenberg and $D(A)$ has at least one simple k-cycle for each k, then A has the nSSP.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ b & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d & c & 0 & 0 \end{bmatrix}$$



$D(A)$

A cycle condition sufficient for Hessenberg matrices

Theorem (Koljančić, Oblak, 2026+)

If A is proper (lower) Hessenberg and $D(A)$ has at least one simple k-cycle for each k, then A has the nSSP.

- Suppose $X \circ A = O$ and $AX^T - X^T A = O$.
- A has at least one nonzero entry on the diagonal and at least one nonzero entry on each subdiagonal
- A is nonderogatory, so $\exists \mathbf{c}$, $X^T = c_0 I + c_1 A + \cdots + c_{n-1} A^{n-1}$
- $A_{n,1} \neq 0$ so $(X^T)_{1,n} = 0$. $(A^k)_{1,n} \neq 0$ only for $k = n-1$.
$\Rightarrow c_{n-1} = 0$.
- Repeat. $X^T = c_0 I + c_1 A + \cdots + c_j A^j$. A^j has all nonzero entries on the jth superdiagonal, but smaller powers have zeros there.
- X^T has a zero entry on the jth superdiagonal if $D(A)$ has a j-cycle.
$\Rightarrow c_j = 0$.
- Now $X^T = c_0 I$ but A a nonzero diagonal entry
- so X^T has a zero diagonal entry. $X = O$

Reducible patterns

Theorem (Arav, Hall, van der Holst, Li, et al. 2024)

If A is block upper triangular with square diagonal blocks, then A has the nSSP if and only if each diagonal block has the nSSP and the spectra of the diagonal blocks are disjoint.

Corollary

A diagonal matrix with distinct nonzero diagonal entries has the nSSP.

Characterization of patterns that allow the nSSP.

Theorem (Cavers, Li, VM 2026+)

Let $\mathcal{A}$ be an $n \times n$ pattern and $B \in Q(\mathcal{A})$ be a matrix with variables in its nonzero positions. Let

$$c_B(z) = z^n + c_1(B)z^{n-1} + c_2(B)z^{n-2} + \cdots + c_n(B)$$

$\mathcal{A}$ allows the nSSP $\iff$ $\{c_1(B), c_2(B), \ldots, c_n(B)\}$ is algebraically independent set.

Consequences of algebraic independence: examples

Theorem (Cavers, Li, VM 2026+)

Let $B \in Q(\mathcal{A})$ be a matrix with variables in its nonzero positions and

$$c_B(z) = z^n + c_1(B)z^{n-1} + c_2(B)z^{n-2} + \cdots + c_n(B)$$

$\mathcal{A}$ allows the nSSP $\iff$ $\{c_1(B), c_2(B), \ldots, c_n(B)\}$ is algebraically independent.

$$\mathcal{B} = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & 0 & * \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b_{12} & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & b_{34} \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix},$$

$$c_B(z) = z^4 - b_{44}z^3 - (b_{12}b_{21} + b_{23}b_{32})z^2 + b_{44}(b_{12}b_{21} + b_{23}b_{32})z - b_{12}b_{23}b_{34}b_{41}.$$

Since $c_3(B) = c_1(B)c_2(B)$, the pattern $\mathcal{B}$ does not allow the nSSP

Consequences of algebraic independence: examples

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Let $B \in Q(\mathcal{A})$ be a matrix with variables in its nonzero positions and

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$\mathcal{A}$ allows the nSSP $\iff$ $\{c_1(B), c_2(B), \ldots, c_n(B)\}$ is algebraically independent.

$$\mathcal{B} = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & 0 & * \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ b_{21} & 0 & 1 & 0 \\ 0 & b_{32} & 0 & 1 \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix},$$

$$c_B(z) = z^4 - b_{44}z^3 - (b_{21} + b_{32})z^2 + b_{44}(b_{21} + b_{32})z - b_{23}b_{41}.$$

Since $c_3(B) = c_1(B)c_2(B)$, the pattern $\mathcal{B}$ does not allow the nSSP

Consequences of algebraic independence

Theorem (Cavers, Li, VM 2026+)

Let $B \in Q(\mathcal{A})$ be a matrix with variables in its nonzero positions and

$$c_B(z) = z^n + c_1(B)z^{n-1} + c_2(B)z^{n-2} + \cdots + c_n(B)$$

$\mathcal{A}$ allows the nSSP $\iff$ $\{c_1(B), c_2(B), \ldots, c_n(B)\}$ is algebraically independent.

Corollary (Cavers, Li, VM 2026+)

Reducible pattern $\mathcal{A}$ allows the nSSP if and only if each of its irreducible components allows the nSSP.

Recall that a reducible matrix has the nSSP if and only if each of its irreducible blocks has the nSSP and the blocks do not have overlapping spectra. The above theorem implies that a pattern that allows the nSSP can not be combinatorially-singular, the restriction of distinct spectra can be mitigated by scaling blocks as necessary.

Consequences of independence

By applying a diagonal similarity, we may scale $n-1$ nonzero entries (on a spanning tree of the digraph of $\mathcal{A}$) of any realization in $Q(\mathcal{A})$ to have magnitude 1.

Corollary (Cavers, Li, VM 2026+)

If $\mathcal{A}$ is an irreducible $n \times n$ pattern that allows the nSSP, then $\mathcal{A}$ has at least $2n-1$ nonzero entries.

Corollary (Cavers, Li, VM 2026+)

Suppose that an $n \times n$ sign pattern matrix $\mathcal{A}$ allows the nSSP. Then the digraph of $\mathcal{A}$ has a composite cycle of length k for each k, $1 \leq k \leq n$.

Insufficiency of nSSP for spectrally arbitrary patterns

$$P = \begin{bmatrix} + & + & 0 & 0 & 0 \\ 0 & 0 & + & + & + \\ - & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & - \end{bmatrix}$$

is spectrally arbitrary [Cavers, Fischer, V 2020], but

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

is only nilpotent matrix (up to scaling) with pattern P, but A does not have the nSSP (or nSMP).

Combinatorial constraint on requiring the nSSP

Theorem (Koljančić, Oblak, 2026+)

If every matrix A with zero-nonzero pattern P has the nSSP, then $D(A) = D(P)$ must have a 2-cycle.

Combinatorial constraint on requiring the nSSP

Theorem (Koljančić, Oblak, 2026+)

If every matrix A with zero-nonzero pattern P has the nSSP, then $D(A) = D(P)$ must have a 2-cycle.

Let A be the adjacency matrix of $D(P)$.
Let $X^T = I - A$.

$D(P)$ has no 2-cycles $\implies A \circ X = A \circ (I - A^T) = O$

X^T is a polynomial in $A \implies AX^T = X^T A$.

$X \neq O \implies A$ does not have the nSSP.

strong multiplicity property for not-necessarily-symmetric matrices

A real $n \times n$ matrix A has the non-symmetric strong multiplicity property (nSMP) if

$$A \circ X = O \qquad AX^T = X^T A \qquad \operatorname{tr}(X^T A^k) = 0 \quad (0 \leq k \leq n)$$

$$\Downarrow$$

$$X = O$$

Note: If A has the nSSP then A has the nSMP.

nSMP

A real matrix A has the non-symmetric strong multiplicity property (nSMP) if

$$A \circ X = O \qquad AX^T = X^T A \qquad \mathrm{tr}(X^T A^k) = 0 \quad (0 \le k \le n)$$

$$\Downarrow$$

$$X = O$$

Let

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{bmatrix}$$

Matrices X satsifying $A \circ X = O$ and $AX^T = X^T A$ are the multiples of I. Thus A does not have the nSSP, but by the trace condition $\mathrm{tr}(X^T A^0) = 0$, A has the nSMP.

[Curtis, Garnett, Shader, V 2025]

Theorem (nSMP open property)

If A has the nSMP, then $\exists\ \epsilon > 0$ such that every matrix within ϵ of A has the nSMP.

Theorem (Multiplicity preservation property for nSMP)

Let A have the nSMP. Then $\exists\ \epsilon > 0$ such that if m_M is the multiplicity list for a matrix M within ϵ of A, then $\exists\ B \in \mathcal{Q}(A)$ with $m_B = m_M$. (In fact, M and B would have the same Jordan structure, including the same geometric multiplicites.)

Same Jordan Structure

Two matrices A and B have the same Jordan structure if $m_A = m_B$ and the corresponding eigenvalues have the same geometric multiplicities.

In particular, if $A = HJH^{-1}$, where $J(A) = J_1(A) \oplus \cdots \oplus J_s(A)$ is the Jordan canonical form of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_s$ (with possible repeats) then there is an ordering of the eigenvalues of B, $\mu_1, \ldots, \mu_s$ (with any corresponding repeats), such that $J(B) = J_1(B) \oplus \cdots \oplus J_s(B)$ with corresponding Jordan blocks the same size:

$$J_{\lambda_i}(A) = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix} \text{ then } \quad J_{\mu_i}(B) = \begin{bmatrix} \mu_i & 1 & 0 & \cdots & 0 \\ 0 & \mu_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \mu_i \end{bmatrix}$$

Theorem (Superpattern theorem for nSMP)

If A has the nSMP, and $D(A)$ is a subdigraph of D, then there exists a matrix B with $D(B) = D$ having the nSMP and the same Jordan structure as A.

Theorem (Bifurcation theorem for nSMP)

Let A have the nSMP and filtered multiplicity list $\mathbf{m}$. Then for each refinement of $\hat{\mathbf{m}}$ of $\mathbf{m}$, $\exists$ a matrix in $\mathcal{Q}(A)$ with filtered multiplicity list $\hat{\mathbf{m}}$

Example: A is an 8×8 with spectrum $\{3, 3, 5 \pm 2i, 5 \pm 2i, 1 \pm i\}$. The filtered multiplicity list of A is $(2; 2, 1)$. There are three refinements of this filtered multiplicity list, namely $(1, 1; 2, 1)$, $(2; 1, 1, 1)$, and $(1, 1; 1, 1, 1)$.

Bifurcate with Jordan form

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

Consider the matrix

$$M = H(J + \delta E_{kk})H^{-1} = A + \delta H E_{kk} H^{-1},$$

where E_{kk} is the matrix with 1 in the (k, k)-entry and zero elsewhere for an appropriate choice of k, aligning with J_λ.

$$\delta \text{ sufficiently small} \Rightarrow ||M - A|| < \epsilon$$

Bifurcation of a complex eigenvalue

Associate the following block with $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$:

$$C_{\lambda_1,\lambda_2} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Arrange the Jordan form of A to include the block structure:

$$\begin{bmatrix} C_{\lambda_1,\lambda_2} & I & O & \cdots & O \\ O & C_{\lambda_1,\lambda_2} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ \vdots & & \ddots & C_{\lambda_1,\lambda_2} & I \\ O & \cdots & \cdots & O & C_{\lambda_1,\lambda_2} \end{bmatrix}.$$

Bifurcate any repeated complex pair of eigenvalues with

$$M = H(J+\delta(E_{kk}+E_{k+1,k+1}))H^{-1} = A+\delta H(E_{kk}+E_{k+1,k+1})H^{-1}$$

$$\delta \text{ sufficiently small} \Rightarrow ||M - A|| < \epsilon$$

Example

$$A \circ X = O \qquad AX^T = X^T A \qquad \operatorname{tr}(X^T A^k) = 0$$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

has the nSMP and has eigenvalues 1,1,1,1.
Every superpattern of

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & 0 & 0 & + \\ - & 0 & 0 & + \end{bmatrix}$$

allows a repeated eigenvalue of multiplicity 4.

cycle digraphs

Theorem (Saha,Tilis, V, Van Tuyl, 2026+)

If the digraph of an $n \times n$ matrix A is an n-cycle, with any number of vertices having a loop, then A has the nSMP.

$$A = \begin{bmatrix} \circledast & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \circledast & 1 \\ d & 0 & \cdots & 0 & \circledast \end{bmatrix}$$

Exercise

Lemma

If A has the nSMP and B is obtained from A via permutation similarity, diagonal similarity, transposition, or nonzero scalar multiplication, then B has the nSMP.

nSMP setup

For each $\mathbf{d} = (d_0, \ldots, d_{n-1}) \in \mathbb{R}^n$ we define the polynomial $p_{\mathbf{d}}(x) = x + \sum_{i=0}^{n-1} d_i x^i$ and choose $\|\mathbf{d}\|$ to be small.

Lemma

Let $A \in \mathbb{R}^{n \times n}$ have

$$\operatorname{spec}(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_t^{(m_t)}\}$$

Then $\exists\ \epsilon > 0$ such that $\forall \mathbf{d} \in \mathbb{R}^n$ with $\|\mathbf{d}\| < \epsilon$ and $\forall\ L \in \mathcal{B}(O, \epsilon_1)$

$$M = (I + L)^{-1} p_{\mathbf{d}}(A)(I + L)$$

has

$$\operatorname{spec}(M) = \{\eta_1^{(m_1)}, \ldots, \eta_t^{(m_t)}\}$$

that satisfies $|\eta_i - \lambda_i| < \frac{1}{2}|\lambda_j - \lambda_\ell| \quad \forall i, j$ and ℓ with $j \neq \ell$. In particular, $\mathbf{m}(A) = \mathbf{m}(M)$. Moreover, $\operatorname{gm}_A(\lambda_i) = \operatorname{gm}_M(\eta_i)$

nSMP setup

$$\mathfrak{f} : \langle Z(\mathcal{A}) \rangle \times \mathcal{B}(O, \epsilon_1) \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$$

$$\mathfrak{g} : \langle Z(\mathcal{A}) \rangle \times \mathcal{B}(O, \epsilon_1) \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$$

given by

$$\mathfrak{f}(B, L, \mathbf{d}) = (I + L)^{-1} p_{\mathbf{d}}(A + B)(I + L),$$

$$\mathfrak{g}(B, L, \mathbf{d}) = (I + L)^{-1} p_{\mathbf{d}}(A)(I + L) + B$$

nSMP setup

Theorem (Curtis, Garnett, Shader, V 2025)

Let $A \in \mathbb{R}^{n\times n} \in Q(\mathcal{A})$. Let

$$\mathcal{T}_A := \{AL - LA : L \in \mathbb{R}^{n\times n}\} + \operatorname{span}\{A^k : k = 0, \ldots, n-1\}.$$

Then the following are equivalent:

(i) *A has the nSMP;*

(ii) $\operatorname{range}(D\mathfrak{f}_{(O,O,\mathbf{0})}) = \langle Z(\mathcal{A})\rangle + \mathcal{T}_A = \mathbb{R}^{n\times n}$;

(iii) $\operatorname{range}(D\mathfrak{g}_{(O,O,\mathbf{0})}) = \langle Z(\mathcal{A})\rangle + \mathcal{T}_A = \mathbb{R}^{n\times n}$.

(iv) $\langle Z(\mathcal{A})\rangle^{\perp} \cap \mathcal{T}_A^{\perp} = \{O\}$;

nSMP and $q(A)$

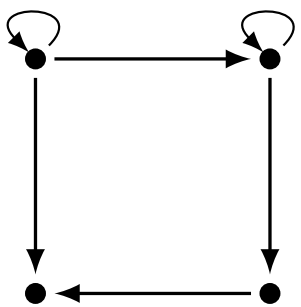
Theorem (Curtis, Garnett, Shader, V 2025)

If A has nSMP, then $q(S) \leq q(A)$ for every superpattern S of the sign pattern of A.

Example: Let

$$A = \begin{bmatrix} \frac{5}{3} & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & 0 & + \\ + & 0 & 0 & 0 \end{bmatrix}$$

$D(A)$



$\text{spec}(A) = \left\{1, 1, \frac{4}{3} \pm \frac{2\sqrt{7}}{3}\right\}$ $\qquad c_A(z) = z^4 - \frac{14}{3}z^3 + 5z^2 - \frac{4}{3}$. Further, if $X \circ A = O$, $AX^T - X^TA = O$ and $\mathrm{tr}(\mathrm{X^T A^2}) = 0$, then $X = O$. Hence A has nSMP. Since A has three distinct eigenvalues, $q(S) \leq 3$ for each superpattern S

Patterns that require the nSMP

Theorem (Saha, Tilis, et al. 2026)

If all the off-diagonal entries of a sign pattern $\mathcal{A}$ are nonzero, then $\mathcal{A}$ requires the nSMP.

Theorem (Saha, Tilis, et al. 2026)

If the digraph of an $n \times n$ sign pattern $\mathcal{A}$ is an n-cycle, with any number of vertices having a loop, then $\mathcal{A}$ requires the nSMP.

Star sign pattern

$$\mathcal{A} = \begin{bmatrix} \circledast & * & \cdots & \cdots & * \\ * & \circledast & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ * & \vdots & \ddots & \circledast & 0 \\ * & 0 & \cdots & 0 & \circledast \end{bmatrix}$$

Theorem (Saha, Tilis, et al. 2026)

Suppose $\mathcal{A}$ is an $n \times n$ star sign pattern for some $n \geq 2$. Then $\mathcal{A}$ requires the nSMP if and only if $D(\mathcal{A})$ does not have more than two non-centre vertices with loops of the same sign (including possibly a "sign" of zero) and in addition, if $D(\mathcal{A})$ has two non-centre vertices with no loop, then the incident 2-cycles have the same sign.

Corollary (Saha, Tilis, et al. 2026)

If $\mathcal{A}$ is an $n \times n$ star sign pattern and $n > 7$, then $\mathcal{A}$ does not require the nSMP.

Hessenbergs that require nSMP

Theorem (Saha, Tilis, et al. 2026)

If $\mathcal{A}$ is a proper Hessenberg pattern whose digraph has at least one k-cycle for each k, $2 \leq k \leq n$, then $\mathcal{A}$ requires the nSMP.

Requires distinct eigenvalues

Theorem (Saha, Tilis, VM, Van Tuyl, 2026)

If a sign pattern $\mathcal{A}$ requires distinct eigenvalues, then $\mathcal{A}$ requires the nSMP.

Converse of does not hold.

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ + & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{A}$ requires the nSMP. However, $\mathcal{A}$ does not require distinct eigenvalues since $A \in Q(\mathcal{A})$ and A has a repeated eigenvalue of -1.

Reducible patterns: allows nSMP

Theorem (Curtis, Garnett, Shader, et al. 2025)

$A = A_1 \oplus A_2$ has the nSMP if and only if both A_1 and A_2 have the nSMP and $\operatorname{spec}(A_1) \cap \operatorname{spec}(A_2) = \emptyset$.

Corollary (Curtis, Garnett, et al. 2025)

Suppose A_1 and A_2 have the nSMP and A_1 has full rank. If S_1 and S_2 are the sign patterns of A_1 and A_2, then $S_1 \oplus S_2$ allows the nSMP.

Corollary (Curtis, Garnett, et al. 2025)

If a sign pattern S is reducible, but there is a matrix $A \in Q(S)$ whose irreducible blocks have the nSMP, and do not share eigenvalues, then S allows the nSMP.

Allowing nSMP

Theorem (Saha, Tilis, VM, Van Tuyl, 2026)

A sign pattern $\mathcal{A}$ allows the nSMP $\Leftrightarrow$ $\mathcal{A}$ allows distinct eigenvalues

($\Rightarrow$): Suppose A has the nSMP. Then $\exists B \in Q(A)$ with distinct eigenvalues by Bifurcation theorem.

Allowing nSMP

Theorem (Saha,Tilis,V,Van Tuyl, 2026)

A sign pattern $\mathcal{A}$ allows the nSMP $\Leftrightarrow$ $\mathcal{A}$ allows distinct eigenvalues

($\Leftarrow$): Suppose A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let V be the nonsingular Vandermonde matrix $V_{ij} = \lambda_i^{j-1}$. Suppose $X^T A = A X^T$ and $\text{tr}(X^T A^k) = 0$ for $0 \leq k \leq n-1$.

Then $X^T = \sum_{k=0}^{n-1} c_k A^k$ since A is nonderogatory.

$$0 = \text{tr}(X^T A^k) = \text{tr}\left(\sum_{i=1}^{n} c_{i-1} A^{k+i-1}\right) = \sum_{i=1}^{n} c_{i-1} \text{tr}(A^{k+i-1}).$$

$$0 = \sum_{i=1}^{n} c_{i-1} \text{tr}(A^{j+i-2}) = \sum_{i=1}^{n} c_{i-1} \left(\sum_{m=1}^{n} \lambda_m^{j+i-2}\right) = \sum_{i=1}^{n} c_{i-1} (V^T V)_{ij}.$$

$$\mathbf{c}(V^T V) = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{0} \Rightarrow X^T = O$$

Digraph characterization for allowing nSMP

Lemma (Eschenbach, Johnson 1993)

An $n \times n$ sign pattern $\mathcal{A}$ allows distinct eigenvalues if and only if the largest composite cycle length in $D(\mathcal{A})$ is at least $(n-1)$.

Corollary

An $n \times n$ sign pattern $\mathcal{A}$ allows the nSMP if and only if the largest composite cycle length in $D(\mathcal{A})$ is at least $(n-1)$.

Corollary

An $n \times n$ star sign pattern $\mathcal{A}$ does not allow the nSMP if and only if $D(\mathcal{A})$ has less than $(n-3)$ non-centre loops.

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