ν, μ and Colin de Verdiére

THE ONLY SOLUTION

We shall have to evolve

problem-solvers galore–

for each problem they solve

creates ten problems more!

–Piet Hein

So far, we've focused on applying the Strong Arnol'd Property to study maximum nullity over the entire set of symmetric matrices with a given graph.

However, the Strong Arnol'd Property is really about inertia. Because congruence not only preserves the rank but also the inertia of A.

Also, the Strong Arnol'd Property is a local property, so we can apply the SAP in settings other than $\mathcal{S}(G)$, e.g., all the symmetric matrices with a given sign-pattern, weighted Laplacians, etc.

SAP historically and other settings

1. The positive semidefinite matrices with a given graph G; $\mathcal{S}^+(G)$.

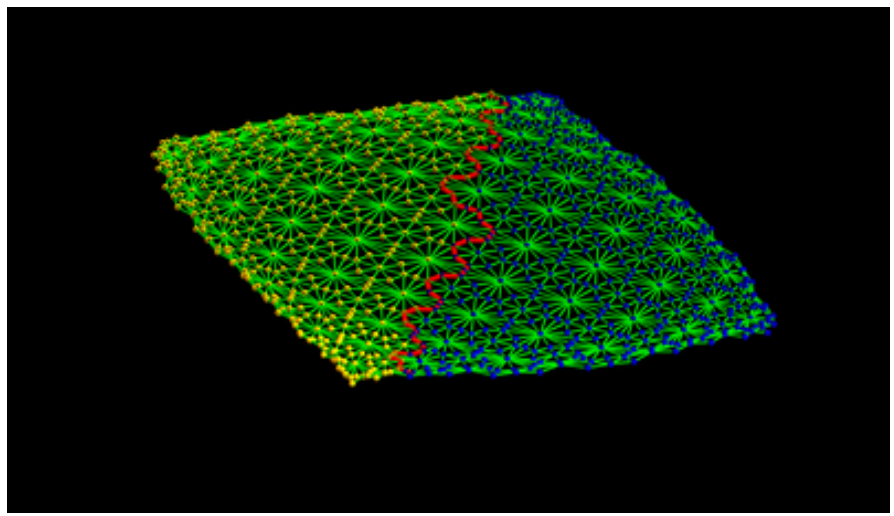
2. The generalized Laplacian matrices with graph G; $\mathcal{S}_L(G)$.
   All matrices $A \in \mathcal{S}(G)$ with $A\mathbf{1} = \mathbf{0}$, and $a_{ij} \leq 0$ for all $i \neq j$.
   These can be thought of as discrete versions of the Laplace operator.

3. The discrete Schrödinger matrices with G; $\ddot{\mathcal{S}}(G)$.
   All matrices in $A \in \mathcal{S}(G)$ such that $a_{ij} \leq 0$ for all $i \neq j$.
   So $A \in \ddot{\mathcal{S}}(G)$ if and only if $A = D + B$ for some $B \in \mathcal{S}_L(G)$ and some diagonal matrix D.
   These can be thought of as generalized Laplacian matrices with potential d_i at vertex i.

4. The Colin de Verdière matrices with graph G: $\mathsf{CdV}(G)$.
   These are the generalized Laplacian matrices with $n_- = 1$; i.e., $A \in \ddot{\mathcal{S}}_L(G)$ with exactly one negative eigenvalue.

Colin de Verdiére's seminal work, 1990s

Colin de Verdiére is a differential geometer interested in a deeper understanding of nodal domains (those maximal connected regions where an eigenfunction has constant sign) Remember the images from Lecture 1?.



CdV and History

Specifically, Colin de Verdiére studied Laplace-Beltrami and continuous Schrödinger operators on Riemannian manifolds, where he was confronted with the irritating reality: the multiplicity of a given eigenvalue can fluctuate erratically under small geometric deformations.

This led to development of structural constraints that guarantee an eigenvalue's multiplicity remains *stable* under small perturbations.

The key tool was Arnol'd's work on eigenvalues in symplectic geometry, which used the transversal intersection of manifolds (now known as the Strong Arnol'd Property).

When presented the Emile Picard Award, the French Math Society commendation translated to:

> *His work established an unexpected, deep unification between three traditionally separate fields: differential geometry (continuous Schrödinger operators), topology (knot theory and graph embeddings), and combinatorics (matrix algebra and graph minors).*

Académie des sciences. (n.d.). Les prix de l'Académie des sciences. French Academy of Sciences. https://www.academie-sciences.fr

A glimpse oF some of CdV's results

His motivating question:

> *What is the maximum possible multiplicity of the second smallest eigenvalue of a graph's Laplacian, subject to an algebraic stability condition that rigorously mimics smooth geometric deformations?*

Why the second-smallest eigenvalue?

Motivated by lots of intriguing work bubbling up in the discrete setting that has been used in numerical methods and theoretical computer science.

The second smallest eigenvalue of the Laplacian matrix of a graph G is known as the algebraic connectivity of G, and is denoted by $\alpha(G)$.

- $\alpha(G)$ was introduced by Fielder in 1973, as a quantitative measure of graph connectivity. Fiedler proved:
  - $\alpha(G) \geq 0$ if and only if G is connected.
  - $\alpha(G)$ is a lower bound on both the vertex and edge connectivity of G.
  - Insertion of an edge does not decrease algebraic connectivity.
- Fiedler (1974): Established fundamental properties of the eigenvector corresponding to $\alpha(G)$. Such a vector is now known as a *Fiedler vector*. In particular:

Theorem

Let $\mathbf{f}$ *be a Fiedler vector of a connected graph* G*. Let* $V^+ = \{i : f_i > 0\}$ *and* $V^- = \{i : f_j < 0\}$*. Then both* $G[V^+]$ *and* $G[V^-]$ *are connected subgraphs of* G*.*

- Isoperimetric number (1980's)
  The isoperimetric number $i(G)$ of the graph is defined as:

$$i(G) = \min_{S \subseteq V, 0 < |S| \leq \frac{|V|}{2}} \frac{|\partial S|}{|S|}$$

  where $\partial S = \{(u, v) \in E : u \in S, v \notin S\}$ is the edge boundary of the vertex subset S.

Theorem (Cheeger's inequality for Graphs (Mohar 1989))

Let $G = (V, E)$ be a finite, connected, undirected graph with largest vertex degree $\Delta = \max_{v \in V} \deg(v)$ and Laplacian matrix L. Then the following inequalities hold:

$$\frac{\alpha}{2} \leq i(G) \leq \sqrt{\alpha(2\Delta - \alpha)}$$

See, also, the work of Alon-Milman (1986), Alon (1986).

Recent applications

- More recent uses, see Spielman's work on spectral partitioning (2007)
- In multi-agent systems where $a(G)$ directly determines the asymptotic convergence speed of consensus algorithms;
- As a metric to evaluate network's resiliency against node or link failures.

Back to CdV

> *What is the maximum possible multiplicity of the second eigenvalue of a graph's Laplacian, subject to an algebraic stability condition that rigorously mimics smooth geometric deformations?*

Colin de Verdiére was led to define, $\mu(G)$, the maximum nullity of a CdV matrix with graph G that has the strong Arnol'd Property.

He established what we've been calling the Superpattern theorem, the Decontraction theorem, and the minor monotonicity of $\mu(G)$ (that is, if G is a minor of H then $\mu(G) \leq \mu(H)$).

These all follow from our treatment of the SAP for $\mathcal{S}(G)$ and the fact that the SAP is a local property.

The major consequences of CdV's work are:

(a) $\mu(G) \leq 1 \iff G$ is a disjoint union of paths.

(b) $\mu(G) \leq 2 \iff G$ is an outerplanar graph.

(c) $\mu(G) \leq 3 \iff G$ is a planar graph.

(d) If $\overline{G}$ is a linear forest (union of paths), then $\mu(G) \geq n - 3$.

(e) If $\overline{G}$ is outerplanar, then $\mu(G) \geq n - 4$.

(f) If $\overline{G}$ is planar, then $\mu(G) \geq n - 5$.

Note for (a), just show $\mu(K_{1,3}) = 2 = \mu(C_3)$, and that μ of a disconnected graph is the maximum μ of its components.

For (b), it is known that G is outerplanar if and only if it avoids K_4 and $K_{2,3}$ as minors. To prove $\Rightarrow$ of (b), show that $\mu(K_4) = 3$ and $\mu(K_{2,3}) = 3$. For the other direction, argue by induction on the number of vertices, the minimum degree of a vertex in an outerplanar graph is at most two, and augmenting with a vertex of degree at most two can't increase μ from 2 to 3.

For (c), show $\mu(K_5) = 4 = \mu(K_{3,3})$. We see the converse in a more general setting in just a bit.

Theorem (Lovász–Schrijver)

$\mu(G) \leq 4$ remained an open mystery until Lovász and Schrijver proved that $\mu(G) \leq 4$ if and only if G is linklessly embeddable in $\mathbb{R}^3$.

A graph is linklessly embeddable if it can be represented in three-space such that no two disjoint cycles are linked together like in a chain.
Because $\mu(G) \leq 3$ corresponds precisely to planar graphs, it is intimately related to the four color theorem.
Indeed, Colin de Verdière conjectured that for any graph G:

$$\chi(G) \leq \mu(G) + 1$$

The conjecture remains open for $\mu(G) \geq 5$. We'll discuss more, shortly.

Lovàsz, Shrijver, van der Holst work

How much of CdV's work really relied upon the constraints of being a CdV matrix?

LSV decided to look at the class $\mathcal{S}^+(G)$. The maximum nullity of a matrix in $\mathcal{S}^+(G)$ like $M(G)$ it is not well behaved with respect to subraphs and minors.

LSV defined $\nu(G)$ to be the maximum nullity of a matrix in $\mathcal{S}^+(G)$ with the SAP. (This was later generalized by Fallat and Hogben to $\xi(G)$)¿

Our big theorems about the SAP imply:

- $\nu(G) \leq M_+(G)$
- ν is minor monontone; that is, if H contains G as a minor that $\nu(G) \leq \nu(H)$.

Survey of known results and techniques related to ν.

Theorem

If G is a tree, then $M_+(G) = 1$

Proof. The Laplacian matrix of $G \implies M_+(G) \leq 1$.

We now show $M_+(G) \leq 1$. Let $A \in \mathcal{S}_+(G)$ of largest nullity.

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$\exists$ signing D and a scalar s s.t. $sI + DAD \leq 0$.

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$\exists$ signing D and a scalar s s.t. $sI + DAD \leq 0$.

Hence, $-(sI + DAD)$ is a nonnegative matrix whose largest eigenvalue ρ has multiplicity the nullity of A.

Since the graph of $sI + DAD$ is connected, the Perron-Frobenius theorem implies that the multiplicity of ρ is 1. Thus, the nullity is 1. $\square$

Survey of known results and techniques related to ν and $M_+(G)$.

For connected G on n vertices:

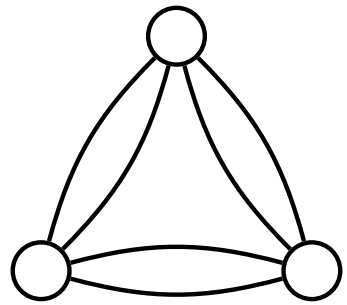
(a) $1 \leq \nu(G) \leq M_+(G) \leq n-1$.

(b) $\nu(G) = 1$ if and only if G is a tree. Already seen G tree $\implies$ $M_+(G)$, and hence $\nu(G) = 1$. For converse, $\nu(C_3) = 1$. By Supergraph Theorem $\nu(G) = 1 \to G$ is acyclic.

(c) $\nu(G) = n-1$ if and only if $G = K_n$.

Theorem (van der Holst)

For connected G, $\nu(G) \leq 2$ if and only if G is K_4- and C_3^2-minor free.



The multigraph C_3^2

Basic idea of proof.

Know $\nu(K_4) = 3$. Also, if G is obtained from C_3^2 by subdividing one of each duplicate edge, then $\nu(G) = 3$.
So Supergraph theorem gives $\Rightarrow$.

For converse:

If G is K_4- and C_3^2-minor free, then $\nu(G) \leq 2$,

we argue by contradiction.

Take a minimal counterexample.

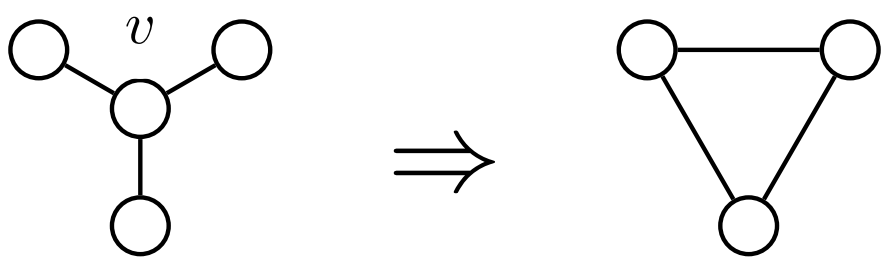
Show minimality $\implies$ at least 4 vertices and 2-connected.

Show minimality and no C_3^2-minor $\implies$ 3-connected.

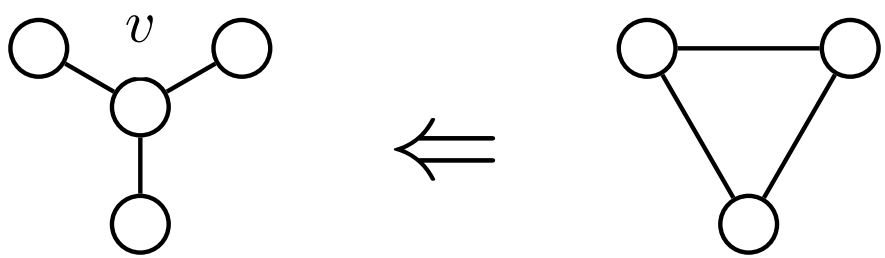
If G is 3-connected with at least 4 vertices, then G has K_4 as a minor; a contradiction.

van der Holst also gave a characterization of G with $\nu(G) \leq 3$. Central to the proof of this are two graph transformations that arose originally in the analysis of electrical circuits.

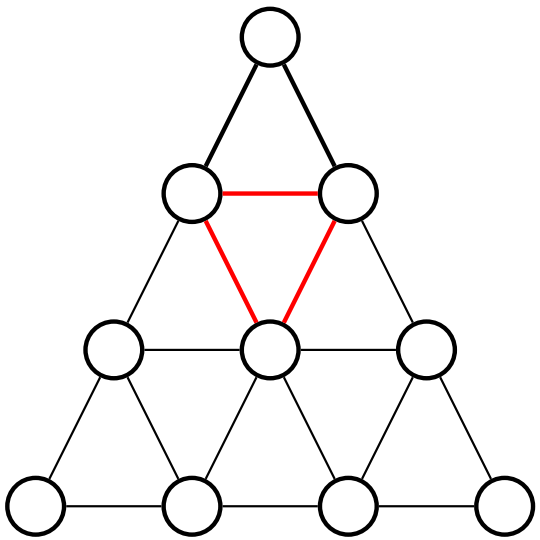
A graph H is a $Y\Delta$-transformation of G at v provided v is a vertex of G of degree 3, and H is obtained from G by removing v and inserting edges joining any pair of the neighbors of v.

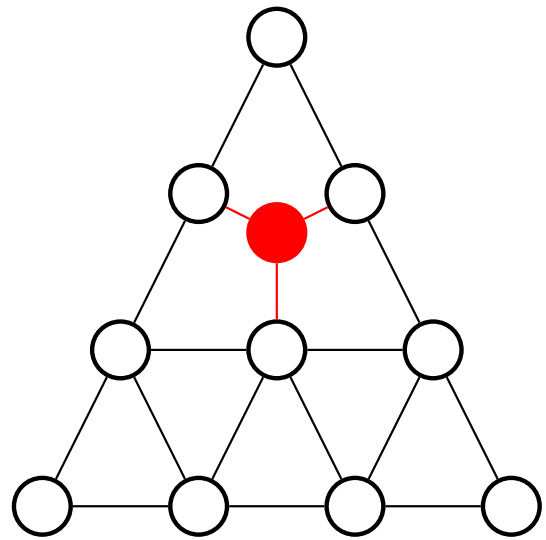


A graph H is a ΔY-**transformation of** G provided H is obtained by removing the edges of a K_3 of G, and inserting a new vertex that is joined to each of the vertices on the K_3.



An example within a larger graph.





Linear algebraic significance of ΔY and $Y\Delta$ transformations

If $A \in \mathcal{S}(G)$ has an induced Y at vertex v, then there is a matrix B in $\mathcal{S}(H)$ with the same nullity as A where H is the graph obtained from G by $Y\Delta$ transform at vertex v.
WLOG, $v = 1$ and $N(v) = 2, 3, 4$.
So A has the form:

$$A = \left[\begin{array}{cccc|c} a & b & c & d & \mathbf{0}^\top \\ b & e & 0 & 0 & \mathbf{w}^\top \\ c & 0 & f & 0 & \mathbf{x}^\top \\ d & 0 & 0 & g & \mathbf{y}^\top \\ \hline \mathbf{0} & \mathbf{w} & \mathbf{x} & \mathbf{y} & A(\{1,2,3,4\}) \end{array}\right]$$

$a \neq 0$ since A is positive semidefinite and $b \neq 0$.

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Let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -b/a & 1 & 0 & 0 \\ -c/a & 0 & 1 & 0 \\ -d/a & 0 & 0 & 1 \end{bmatrix} \oplus I.$$

$$A = \left[\begin{array}{cccc|c} a & b & c & d & \mathbf{0}^\top \\ b & e & 0 & 0 & \mathbf{w}^\top \\ c & 0 & f & 0 & \mathbf{x}^\top \\ d & 0 & 0 & g & \mathbf{y}^\top \\ \hline \mathbf{0} & \mathbf{w} & \mathbf{x} & \mathbf{y} & A(\{1,2,3,4\}) \end{array}\right]$$

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Then Q is invertible, QAQ^T is the direct sum of $[a]$ and a matrix whose graph is H.

More holds:

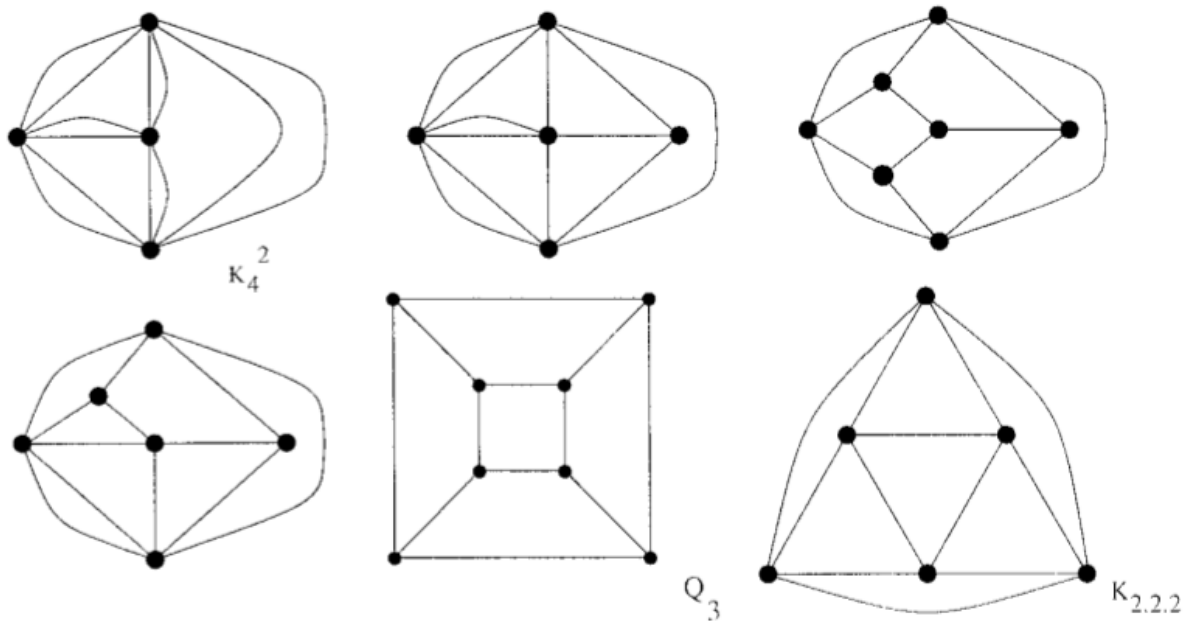
Theorem

If H is obtained from G by a $Y\Delta$ transformation, then $\nu(H) \geq \nu(G)$.

Note: this need not be tight: take $H = K_3$ and $G = K_{1,3}$, then $\nu(H) = 2$ and $\nu(G) = 1$.

Theorem (van der Holst)

$\nu(G) \leq 3$ if and only if G has no K_5-minor and no minor isomorphic to a graph in the $K_{2,4}$-family.



$K_{2,4}$-family

Properties of eigenvectors corresponding to extreme eigenvalues.

Let A be an $n \times n$ matrix and let λ be the largest or smallest eigenvalue of A.

Since $\pm(A - \lambda I)$ is similar to A and has the same eigenvectors of A, in studying the eigenvectors of A corresponding to an extreme eigenvalue, can study the nullvectors of a related positive semidefinite matrix B.

Proposition (Nullvectors extend for PSD matrices)

Let A be an $n \times n$ positive definite matrix and α be a subset of row indices. If $A[\alpha]\mathbf{w} = \mathbf{0}$, then $A\left[\hat{\mathbf{w}}\right] = \mathbf{0}$, where $\hat{\mathbf{w}}$ agrees with $\mathbf{w}$ on entries in α and is zero elsewhere.

Proof.

A is positive definite $\implies$ $A = B^\top B$ for some $n \times n$ matrix B.

Since $A[\alpha]\mathbf{w} = O$, $\mathbf{w}^\top A[\alpha]\mathbf{w} = 0$.

This implies that

$$\begin{bmatrix}\mathbf{w}^\top & \mathbf{0}^\top\end{bmatrix} B^\top B \begin{bmatrix}\mathbf{w} \\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{w}^\top & \mathbf{0}^\top\end{bmatrix} A \begin{bmatrix}\mathbf{w} \\ \mathbf{0}\end{bmatrix} = \mathbf{w}^\top A[\alpha]\mathbf{w} = 0.$$



Consequences of null vectors extend.

Given a nullvector $\mathbf{w}$ of A we partition the vertices of V of the graph of A as $V_+(\mathbf{w})$, $V_0(\mathbf{w})$ $V_-(\mathbf{w})$ by the sign of corresponding entry of $\mathbf{w}$.

Theorem

Let $A \in \mathcal{S}(G)$ be positive semidefinite of nullity $k > 0$, and $\mathbf{w}$ be a nonzero nullvector of A.
Then the subgraph of H of G induced on $\overline{V_0(\mathbf{w})}$ has at most k connected components.

Proof.

Let ℓ be the number of connected components of H.
The restriction of $\mathbf{w}$ onto the vertices of a connected component of H is a null vector for corresponding principal submatrix.
Hence each principal submatrix corresponding to a connected component of H is singular.
This means that the nullity of $A(V_0(\mathbf{w}))$ is at least ℓ. By row containment $k \geq \ell$. □

Theorem

Let A be a positive semidefinite matrix and $\mathbf{w}$ be a nonzero nullvector of A. Then no vertex of V_0 is adjacent to exactly one vertex in a connected component of $G(V_0(\mathbf{w}))$.

Proof.

If $i \in V_0$ and j is the neighbor of i in the connected component $V[\beta]$ of $\overline{V_0(\mathbf{w}))}$, then A has the form

$$\left[\begin{array}{ccccc|c} & & A[\beta] & & & * \\ \hline 0 & \cdots & 0 & a_{ij} & & \\ & & * & & & A(\beta) \end{array}\right]$$

$A[\beta]$ is has nowhere zero nullvector $\mathbf{w}[\beta]$. But no nullvector extension is violated.

□

See also Positive Definite Zero Forcing.

Theorem

Let A be a positive definite with graph G, having the SAP and let $\mathbf{w}$ be a nonzero nullvector of A, then $G(V_0(\mathbf{w}))$ is a connected graph.

Proof.

We prove the contrapositive. Assume that $G(V_0(\mathbf{w}))$ is disconnected.

Then A has nonzero null vectors $\mathbf{v}$ and $\mathbf{w}$ such that the support of $\mathbf{v}$ is the vertex set of a connected component of $G[V_0(\mathbf{w})]$, and the support of $\mathbf{w}$ is that for another connected component of $G(V_0(\mathbf{w}))$.

Then $X = \mathbf{u}\mathbf{w}^\top + \mathbf{w}\mathbf{u}^T$ is a nonzero symmetric matrix satisfying $A \circ X = O$, and $AX = O$. Therefore, A does not have the SAP.

□

Corollary

If A is PSD and has the SAP then for each nonempty proper subset α of row indices of A, at least one of $A[\alpha]$ and $A(\alpha)$ is invertible.

Recent work on nodal Domain: McKenzie and Urschel

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix representing the generalized Laplacian of a signed graph. For an eigenvector $x \in \mathbb{R}^n$, the vertex set $V = \{1, \ldots, n\}$ is partitioned into positive, negative, and zero coordinates.

A nodal domain is defined as a maximal connected component in the subgraph induced by either:

- The positive support $V^+ = \{i \in V : x_i > 0\}$ using only positive edges $(A_{ij} < 0)$.
- The negative support $V^- = \{i \in V : x_i < 0\}$ using only negative edges $(A_{ij} > 0)$.

Courant's Nodal Domain Theorem for operators on manifolds states that the k-th eigenfunction has at most k nodal domains.

This fails for arbitrary discrete graphs due to non-uniqueness of eigenbases. But they have proven the following (using their own version of the SAP).

Theorem (McKenzie, Urschel)

Let A be an $n \times n$ real symmetric matrix, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its eigenvalues. There exists an orthonormal basis of eigenvectors $x^{(1)}, \ldots, x^{(n)}$ such that for each $k \in \{1, \ldots, n\}$, the eigenvector $x^{(k)}$ decomposes the underlying signed graph into at most k signed nodal domains.

Fallat, Gupta and Lin (2024, 2026) have initiated the study for the strong spectral property of Generalized Laplacians. Here the strong property needs to take into account that the row sums are zero; that is, the perturbations are symmetric with row sum 0.

They've established the Superpattern, Proximity, Bifurcation theorems. They use this to completely solve the inverse eigenvalue problem in this setting.

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