SAP Connections

Mathematical science is in my opinion
an indivisible whole,
an organism whose vitality is conditioned
upon the connection of its parts.

–David Hilbert

Min Rank/Max Nullity

Recall: For a graph G,

minimum rank $\mathrm{mr}(G)$ / maximum nullity $M(G)$

$\mu(G)$ nullity for certain generalized Laplacian matrices with SAP.
$\nu(G)$ for positive semidefinite matrices with SAP.
$\xi(G)$ for real symmetric matrices with SAP.

SAP and Vectors

Orthogonal Representations of Graphs

An *orthogonal representation* of a graph is an assignment of vectors to the vertices so that adjacent vertices receive orthogonal vectors.

Such a representation is *faithful* if non-adjacent vertices have non-orthogonal vectors.

[Alternately, orthogonal vectors for non-adjacent vertices; equivalent problems in that one can think of representation of graph or its complement. One version is natural when compared to graph coloring; the other is more common for minimum rank.]

Gram Matrices

Every positive semidefinite matrix A can be factored as $A = B^T B$.

. . . and so thought of as a matrix of dot products of columns of B (a *Gram matrix*).

And PSD minimum rank is equivalent to smallest dimension for a faithful orthogonal representation.

SAP for representations

So there is a natural definition of SAP for vector representations:

An orthogonal representation $\{v_i\}$ has SAP if there is no symmetric X with zero diagonal and graph a subgraph of $\overline{G}$ and

$$\sum_j X_{ij} v_i = 0$$

for all j.

General Position

Vectors in $\mathbb{R}^d$ are in *general position* if any d of them are linearly independent.

Theorem [Lovász, Saks, and Schrijver (1989/2000)]: A graph with n vertices has a faithful general position orthogonal representation in $\mathbb{R}^d$ if and only if it is $(n-d)$-connected.

Local General Position

A representation is in *locally general position* if vectors representing the vertices nonadjacent to any vertex v are linearly independent.

Theorem: A graph with n vertices has a faithful locally general position orthogonal representation in $\mathbb{R}^d$ if and only if it is $(n-d)$-connected.

SAP

The SAP can be thought of as a "symmetrized" version of locally general position.

An orthogonal representation v_i has SAP if there is no symmetric X with zero diagonal and graph a subgraph of $\overline{G}$ and

$$\sum_j X_{ij} v_i = 0.$$

This means linear dependence between the non-neighbors of i, so every orthogonal representation of a graph G in locally general position has the SAP.

Connectivity and SAP

e.g. The Strong Arnold Property for 4-connected flat graphs by Schrijver and Sevenster (2017)

(A graph G is flat if it can be embedded injectively in 3-dimensional Euclidean space such that the image of any circuit is the boundary of some disk disjoint from the image of the remainder of the graph. AKA linklessly embeddable)

Extends similar results for 2-connected outerplanar graphs and 3-connected planar graphs.

Two Nus

SAP and "the other" Nu

For a CdV matrix M for a connected graph G, by Perron-Frobenius, the one negative eigenvalue has all-positive eigenvector.

WLOG assume eigenvalue is -1 and eigenvector is all-ones.

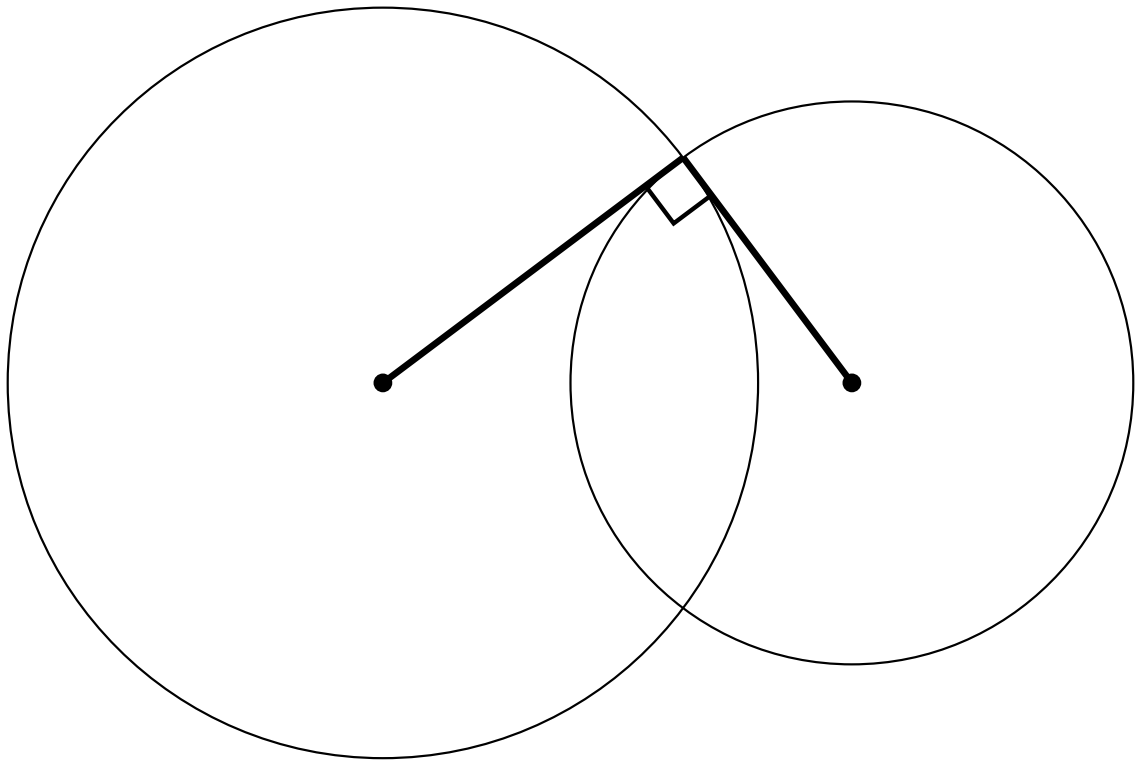
Then $M + J$ is positive semidefinite. Corresponds to representation by vectors with dot products 1 or less than 1 that have SAP for $\overline{G}$.

Kotlov, Lovász, and Vempala, 1997:
For every graph G on n vertices different from $\overline{K_2}$,

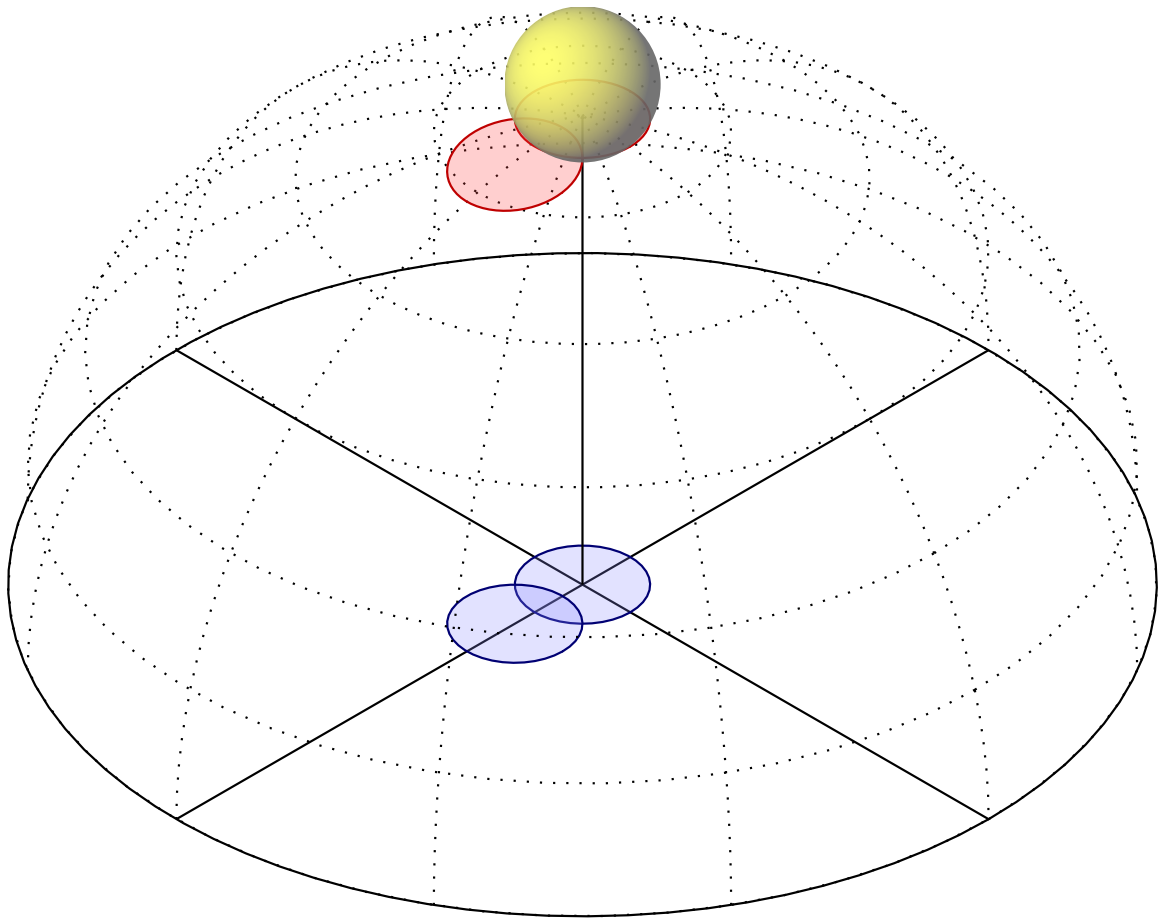
$$\nu(\overline{G}) = |G| - \mu(G) - 1.$$

Interlude

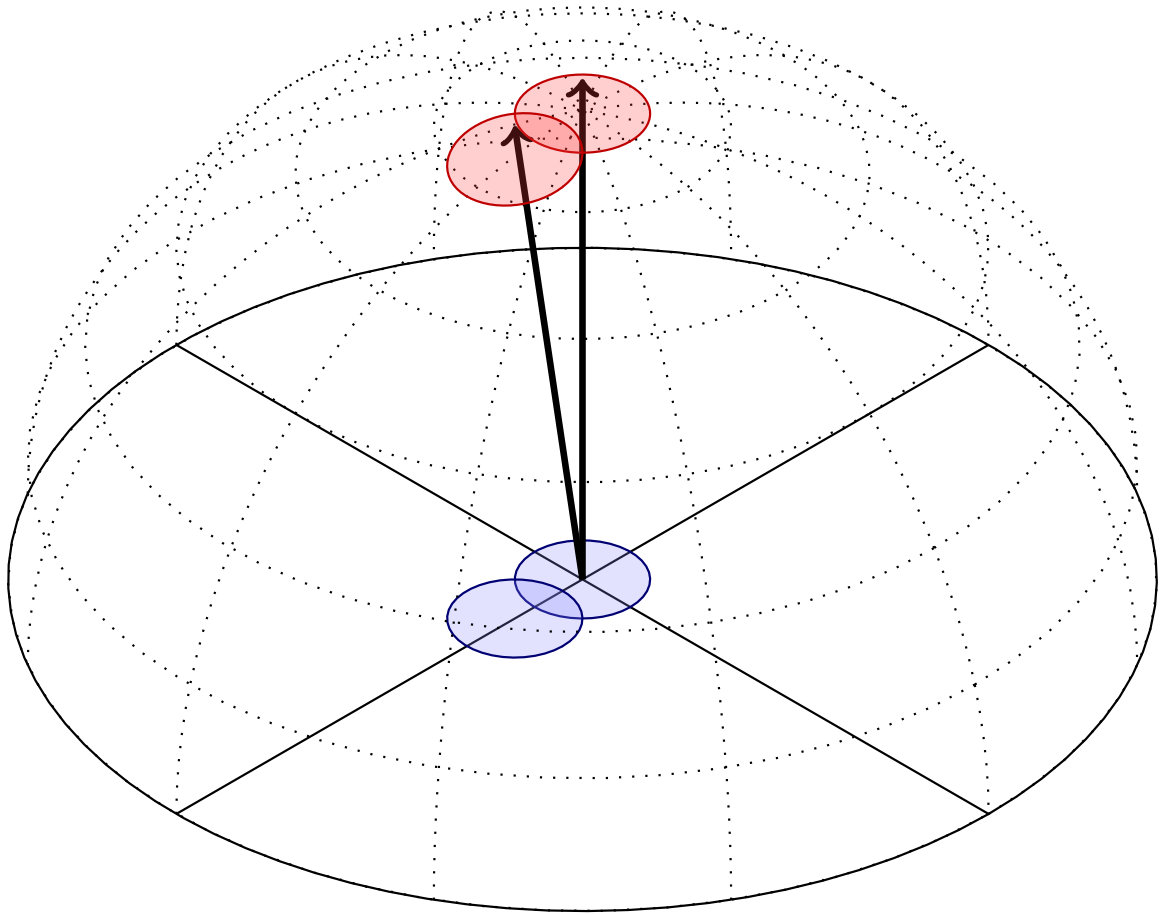
Orthogonal Circles/Spheres



Normal Orthogonal Sphere Representations



Spheres and Vectors



Stressed?

Kotlov, Lovász, and Vempala, 1997: Sphere representations.

Static stress on vectors:

$$\sum_j X_{ij}(v_j - v_i) = 0.$$

If no unit vectors, then equivalent to SAP.

Cauchy: In $\mathbf{R}^3$ all convex polytopes are stress free.

If $\overline{G}$ is planar, then $\mu(G) \geq |G| - 5$.

Graph Complement Conjecture:

$$\mu(G) + \mu(\overline{G}) \geq |G| - 2.$$

Laurent & Varvitsiotis (2014) *Positive semidefinite matrix completion, universal rigidity and the Strong Arnold Property*

Strong Vector Properties?

Dot Product Dimension

Smallest dimension for a vector representation such that $u_i \cdot u_j \geq 1$ if and only if ij is an edge.

Research Question: Strong property for dot product dimension?

Gram Dimension

The Gram dimension $\mathrm{gd}(G)$ of a graph is the smallest integer $k \geq 1$ such that, for every assignment of unit vectors to the nodes of the graph, there exists another assignment of unit vectors lying in $\mathbb{R}^k$, having the same inner products on the edges of the graph.

SAP and Gram Dimension

$\nu^{=}(G)$ was defined by van der Holst (2003) to require zero matrix entries for non-edges but not non-zero for edges.

Thus $\nu^{=}(G) \geq \nu(G)$ in general.

The result $\nu^{=}(G) \leq \mathrm{gd}(G)$ follows from work showing that $\nu^{=}(G)$ is the smallest k for certain assignments of unit vectors.

Research Question: Strong property for gram dimension?

SAP for Subspaces

Representations of Vectors

Hein van der Holst (2010) defined a version of the SAP for subspaces.

A *representation* of linearly independent column vectors $x_1, \ldots, x_r$ in $\mathbb{R}^n$ is a function $\phi : \{1, \ldots, n\} \to \mathbb{R}^r$ such that

$$\begin{bmatrix} \phi(1) & \ldots & \phi(n) \end{bmatrix}^T = \begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix}.$$

A representation of a linear subspace L of $\mathbb{R}^n$ is a representation of a basis of L.

SAP Equivalences I

An n-by-n matrix M has SAP for a graph G if and only if for all $x \in \ker(M)$ and every n-by-n symmetric matrix A, there is an n-by-n symmetric matrix B with graph G such that $x^T Ax = x^T Bx$.

SAP Equivalences II

... if and only if for a fixed basis $x_1, \ldots, x_r$ of $\ker(M)$ and every symmetric r-by-r matrix C, there is a symmetric n-by-n matrix B with graph G such that

$$C = \begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix}^T B \begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix}.$$

SAP Equivalences III

... if and only if, for a representation ϕ of the basis, the matrices of the form $\phi(i)\phi(i)^T$ and $\phi(i)\phi(j)^T + \phi(j)\phi(i)^T$ for edges ij span all symmetric r-by-r matrices.

SAP for Subspaces Definition(s)

1. Matrices of the form $\phi(i)\phi(i)^T$ and $\phi(i)\phi(j)^T + \phi(j)\phi(i)^T$ for edges ij span all symmetric r-by-r matrices for any representation of L.

2. If N is symmetric and $\phi(i)N\phi(j)^T = 0$ for all edges ij and all $i = j$, then $N = 0$.

Connection

Theorem: A matrix M has SAP for G if and only if $\ker(M)$ does.

Results (e.g.)

If $k \leq 2$, then a k-dimensional L fails to have SAP if and only if it contains two nonzero vectors with nonadjacent supports.

Sufficient conditions for 3 and 4.

New proofs that dSos with one negative eigenvalue for 2-connected outerplanar and 3-connected planar graphs have SAP.

Conjectured (at the time; now proved as mentioned on earlier slide): 4-connected with linkless embedding.

Research question: Subspace proof for above?

Nullspace Embeddings

A representation ϕ of a basis of the nullspace of M is a *nullspace embedding* of the graph G defined by M.

The span of such an embedding is the union of the subspaces spanned by each $\phi(i)$ and by $\phi(i)$ and $\phi(j)$ for each edge ij.

Hein van der Holst (1995) *A short proof of the planarity characterization of Colin de Verdière.*

Lovász & Schrijver (2016): *Nullspace embeddings for outerplanar graphs*.

Quadrics

M has the SAP if and only if $|G|$ is not contained in any homogeneous quadric $\{y \mid y^T N y = 0\}$ for some symmetric N.

If a homogeneous quadric Q contains a hyperplane, then Q is the union of one or two hyperplanes (as WLOG Q contains $\{x|x_1 = 0\}$, hence the quadratic form is $(a^T x)x_1$ for some a).

[Used in proof for 4-connected with linkless embedding.]

Zero Forcing

On a graph G, the conventional zero forcing game is a color-change game such that each vertex is colored blue or white initially, and then the color change rule (CCR) is applied repeatedly.

(CCR-Z) If i is a blue vertex and j is the only white neighbor of i, then j may turn blue.

If starting with an initial blue set and every vertex turns blue eventually, this set is called a zero forcing set (ZFS).

Maximum Nullity

The zero forcing number $Z(G)$ of a graph G is defined as the minimum cardinality of a ZFS.

In general, $M(G) \leq Z(G)$.

ZF for SAP

The SAP zero forcing game is coloring "non-edges" to be blue or white, instead of coloring vertices; also, a set of initial blue non-edges is called a zero forcing set if every non-edge turns blue eventually by repeated applications of the given color change rules.

For a graph G, the SAP zero forcing number is the minimum number of blue non-edges such that every non-edge will become blue by repeated applications of the color change rule for Z_{SAP}.

Theorem

Suppose G is a graph with $Z_{\mathsf{SAP}}(G) = 0$. Then every matrix in has the SAP.

Graph Classes

Any graph with diameter 2 and maximum degree at most 3.

If G has no isolated vertices and $\overline{G}$ is a forest.

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