The Strong Interlacing Property

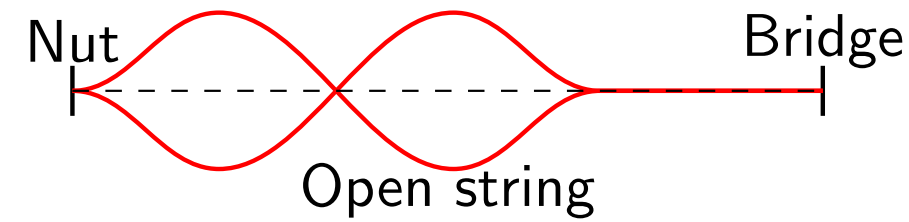
One of the pleasures of looking at the world through mathematical eyes is that you can see certain patterns that would otherwise be hidden.

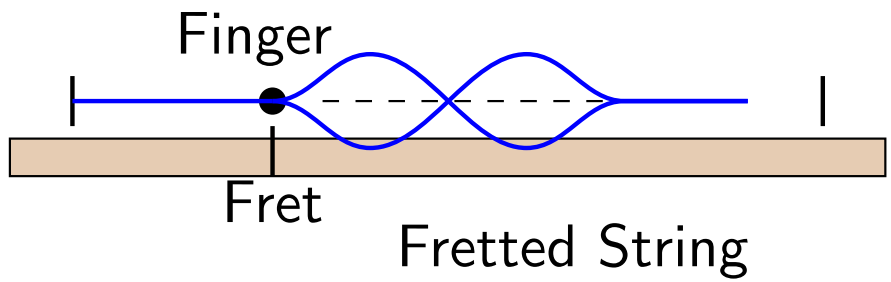
–Steven Strogatz

Sometimes it is of interest to study the combined properties of a matrix A and a principal submatrix B.

For example, the vibrations of an (open) cello string and a shorter string resulting from pressing the string on a fret.

Mathematically, we are looking at the spectra of a symmetric matrix A that models the vibrations of the open string and one of its trailing principal submatrices.





Nullity pairs

Today we will be looking at an $n \times n$ matrix A with $n \geq 2$, and a principal submatrix B.

We focus just on their nullities, and restrict to the case that $B = A(r)$ for some index r.

The nullity pair of (A, r) is the ordered pair

$$\text{nul}(A, r) := (\text{nul } A, \text{nul} A(r)).$$

Examples:

For each r, $\text{nul}(J_n, r) = (n-1, n-2)$.

For the adjacency matrix, A, of the star on n-vertices with center 1, $\text{nul}(A, \text{center}) = (n-2, n-1)$ and $\text{nul}(A, \text{leaf})$ is $(n-2, n-3)$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

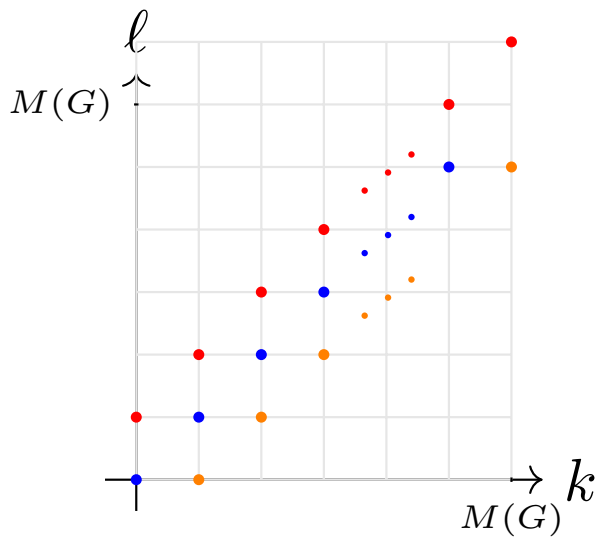
Basic properties of nullity pairs

For $n \times n$ symmetric matrix $A = \begin{bmatrix} a_{rr} & \mathbf{u}^\top \\ \mathbf{u} & A(r) \end{bmatrix}$ (over any field!) and any index $r \in \{1, \ldots, n\}$, we have the following.

- $\operatorname{nul} A - 1 \leq \operatorname{nul} A(r) \leq \operatorname{nul} A + 1$
- $\operatorname{nul} A(r) = \operatorname{nul} A + 1$ if and only if $\mathbf{u}$ is not in the column space of $A(r)$.
- $\operatorname{nul} A(r) = \operatorname{nul} A$ if $\mathbf{u}$ is in the column space of $A(r)$ but the r-th column of A is not in the column space of the other columns of A.
- $\operatorname{nul} A(r) = \operatorname{nul} A - 1$ if and only if the r-th column of A is in the span of the other columns of A.
- if $\operatorname{nul} A(r) \leq \operatorname{nul} A$, then there is a unique t such that $\operatorname{nul}(A(r, r)) = \operatorname{nul}(A + tE_{rr}) - 1$.

Consequently, we have the following constraints and relations on the nullity pair $\text{nul}(A, r) = (k, \ell)$ for $A \in \mathcal{S}(G)$ and r a fixed index.

- $|k - \ell| \leq 1$
- (G, r) realizes the nullity pair (k, k) if and only if G realizes the nullity pair $(k + 1, k)$.
- $k \leq M(G)$ the maximum multiplicity among matrices in $S(G)$
- $\ell \leq M(G(r))$ the maximum multiplicity among matrices in $\mathcal{S}(G(r))$.



Possible nullity pairs achievable by graph G.

Trivial nullity pairs $(0,0)$,$(1,0)$, $(0,1)$

Each graph G and each r realizes the nullity pair $(0,0)$; emphasize the main diagonal.

So each graph G and each r realizes the nullity pair $(1,0)$ w.r.t. r.

What about $(0,1)$?
Need $A \in \mathcal{S}(G)$ with A invertible and $\operatorname{nul} A(r) = 1$.

If r is an isolated vertex of G, then the r-nullity pair $(0,1)$ is not achievable.

If r is not isolated, then take A so that

- $A(r)$ is the Laplacian matrix on one of the connected components of $G(r)$ that is joined to r,
- an invertible matrix on remaining components of $G(r)$, and
- $a_{vw} \in \{0,1\}$ for $v \neq w$ and a_{vv} sufficiently large.

Such A has nullity pair $(0,1)$.

(G,r) where r is not an isolated vertex does allows the nullity pair $(0,1)$.

Southwest

Proposition (Southwest rule)

If the rooted graph (G, r) *allows nullity pair* $(k+1, \ell+1)$ *with* $k, \ell \geq 0$*, then* (G, r) *allows that nullity pair* (k, ℓ)*.*

Idea of proof.
We've already handled, $k = 0$ or $\ell = 0$.

Otherwise, take $A \in \mathcal{S}(G)$ and r and index with $\text{nul}(A, r) = (k+1, \ell+1)$.

Argue there is a nonzero nullvector $\mathbf{u}$ of A whose r-th coordinate is 0. Suppose the j-th coordinate of $\mathbf{u}$ is nonzero.

Show that $\text{nul}(A + E_{jj}, r) = (k, \ell)$.



The Strong Nullity Interlacing Property

The Strong Arnol'd property is very useful in the study of maximum nullity, $M(G)$, of matrices with graph G, and yields many interesting results related to planarity.

E.g., G if planar if and only if $\mu(G) \leq 3$.
Or if G is non-planar, then $\mathcal{S}(G)$ contains a matrix with the SAP and nullity at least 3.

Given this, it is natural to ask for an analogous strong property for nullity pairs.

This places us in the rooted graph world. A rooted graph (G, r) is a graph with an identified vertex r called its root. Subgraphs of (G, r) are required to have r as their root.

Strong property for nullity pairs

For simplicity, let's assume that $r = 1$.

Let $A \in \mathcal{S}(G)$. What transformations preserve the 1-nullity pair (and symmetry) of A?

Congruence: $A \mapsto Q^\top AQ$ preserves the nullity of A, but not necessarily of $A(1)$. But if we restrict our Q to be of the form

$$Q = \begin{bmatrix} a & \mathbf{0}^\top \\ \mathbf{q} & \hat{Q} \end{bmatrix},$$

then $Q^\top AQ$ is congruent to A and $(Q^\top AQ)(1) = \hat{Q}^\top A(1)\hat{Q}$ is congruent to $A(1)$. Hence, $A \mapsto Q^\top AQ$ preserves the nullity pair with respect to $r = 1$.

Let T_n be the set of the set of $n \times n$ matrices of the form

$$\begin{bmatrix} s & \mathbf{0} \\ \mathbf{t} & T \end{bmatrix}$$

and let $T_{n,\epsilon}$ be the subset restricted to those matrices of spectral radius less than ϵ.

For $\epsilon < 1$, if $R \in \mathbf{T}_{n,\epsilon}$, then $(I + R)$ is invertible and $(I + R)^\top A(I + R)$ and A have the same nullity pair with respect to 1.

This leads us to define a map $f : T_{n,\epsilon} \to \mathrm{Sym}(n)$ by

$$f(R) = (I + R)^\top A(I + R).$$

The derivative at $R = O$ of this map is $Df_O = R^\top A + AR$.

Thus, the map defined by

$$\mathfrak{f} \colon T_{n,1} \times \langle \mathcal{S}(G) \rangle \to \mathrm{Sym}(n)$$

by

$$\mathfrak{f}(R, B) = (I + R)^\top A(I + R) + B$$

is locally surjective at O if and only if

$$\{R^\top A + AR : R \in T_n\} + \langle \mathcal{S}(G) \rangle = \mathrm{Sym}(n). \tag{1}$$

This gives our definition of the strong nullity interlacing property SNIP:

$(A, 1)$ has the SNIP provided (1) holds.

Complementary definition of SNIP

One can show (exercise) that $\{R^\top A + AR : R \in T_n\}^\perp$ equals

$$\{X \in \mathrm{Sym}(n) : (AX)[\{2,\dots,n\},\{1,\dots,n\}] = O\}.$$

Thus (A, r) has the SNIP if and only if $X = O$ is the only symmetric matrix such that

- $A \circ X = O$,
- $I \circ X = O$ and
- $(AX)(\{r\},\{1,\dots,n\}] = O$.

Note the resemblance to the SAP: A has the SAP if and only if $X = O$ is the only symmetric matrix such that

- $A \circ X = O$,
- $I \circ X = O$ and
- $AX = O$.

Examples

1. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ has nullity pair $(1, 2)$ w.r.t $r = 1$.

   $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ certifies that $(A, 1)$ does not have the SNIP.

   Note that A does have the SAP.

2. A has nullity pair is $(1, 0)$ w.r.t 3.

   $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{bmatrix}$, $(AX)(3, :] = O_{2\times 3}$ requires $x = 0$.

   So, $(A, 3)$ has the SNIP.

Basic properties of SNIP

Property 1. For each real number t, (A, r) has the SNIP if and only if $(A + tE_{rr}, r)$ has the SNIP.

Property 2. If (A, r) has the SNIP, then both A and $A(r)$ have the SAP.

Proof. Without loss $A = \begin{bmatrix} a_{11} & \mathbf{u}^\top \\ \mathbf{u} & A(1) \end{bmatrix}$ and $(A, 1)$ has SNIP.

By definition A has the SAP.

Suppose Y is in the SAP Spoiler space of $A(1)$. Consider $X = [0] \oplus Y$.
Note

$$AX = \begin{bmatrix} 0 & \mathbf{u}^\top Y \\ \mathbf{0} & A(1)Y \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{u}^\top Y \\ \mathbf{0} & O \end{bmatrix}.$$

Since $(A, 1)$ has the SNIP, $Y = O$.

Thus $A(1)$ has the SAP. □

Converse is not true in general. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, both A and $A(1)$ have the SAP.

But $(A, 1)$ does not have the SNIP; use $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Property 3. [SNIP in terms of SAP]

(i) If $\operatorname{nul} A(r) = \operatorname{nul} A + 1$, then (A, r) has the SNIP if and only if $A(r)$ has the SAP.

(ii) if $\operatorname{nul} A(r) = \operatorname{nul} A - 1$, then (A, r) has the SNIP if and only if A has the SAP.

(iii) if $\operatorname{nul} A(r) = \operatorname{nul} A$, then (A, r) has the SNIP if and only if $A + tE_{rr}$ has the SAP, where t is the unique real number such that $\operatorname{nul}(A + tE_{rr}) = \operatorname{nul} A(r) + 1$.

Advanced Properties of SNIP

Theorem (SNIP is an open property)

If (A, r) has the SNIP, then for each symmetric matrix B sufficiently close to A has the SNIP w.r.t. r.

Theorem (Supergraph theorem for SNIP)

Let $A \in \mathcal{S}(G)$ and let H be a supergraph of G. If (A, r) has the SNIP, then there exists a matrix $B \in \mathcal{S}(H)$ such that A and B have the same nullity pair with respect to r and (B, r) has the SNIP.

Theorem (Direct sum theorem)

If G is disconnected, $A \in \mathcal{S}(G)$, then (A, r) has the SNIP if and only if the submatrix corresponding to the connected component containing r has the SNIP with respect to r and each submatrix corresponding to the other connected components is invertible.

Theorem (Extended Superpattern Theorem)

If (A, r) has the SNIP, $A \in \mathcal{S}(G)$ and H is a graph that contains G as an induced subgraph, then there exists a matrix $B \in \mathcal{S}(H)$ such that (B, r) has the SNIP and the same nullity pair as A w.r.t. r.

Theorem (Proximity Theorem)

Let (A, r) have the SNIP with $A \in \mathcal{S}(G)$. Then there exists an $\epsilon > 0$ such that if there is a symmetric matrix B with $\|A - B\| < \epsilon$, then there is a matrix in $\mathcal{S}(G)$ with the same nullity pair w.r.t. to r as B.

Bifurcation Results

Theorem (East of the Diagonal theorem)

$\mathcal{S}(G)$ allows a matrix with the SNIP and nullity pair (k, k) w.r.t. r if and only if it allows a matrix with the SNIP and nullity pair $(k+1, k)$ w.r.t. r.

Proof. SNIP-ness w.r.t. r in these cases depends only on SAP-ness of $A + tE_{rr}$, where t is the unique value where

$$\text{nul}\,(A + tE_{rr}) = 1 + \text{nul}(A(r)).$$

□

Bifurcation Results

Theorem (West theorem)

If $\mathcal{S}(G)$ contains a matrix with the SNIP and nullity pair $(k+1, k+1)$ w.r.t. r, then $\mathcal{S}(G)$ contains a matrix B with the SNIP and nullity pair $(k, k+1)$ w.r.t. r.

Proof. WLOG, $r = 1$. There is an orthogonal matrix Q such that

$$([1] \oplus Q^\top) A ([1] \oplus Q) = \begin{bmatrix} D & O \\ O & O_{k+1} \end{bmatrix},$$

where D is an invertible diagonal matrix. Adding ϵE_{nn} to this matrix gives a matrix with nullity pair (k, k) w.r.t. $r = 1$. The Proximity theorem implies there is a matrix $C \in \mathcal{S}(G)$ with the SNIP and nullity pair (k, k) with respect to $r = 1$. Now apply the East of the Diagonal theorem. $\square$

Theorem (South theorem)

If $\mathcal{S}(G)$ contains a matrix A with the SNIP and nullity pair $(k, k+1)$ w.r.t. r, then $\mathcal{S}(G)$ contains a matrix B with the SNIP and nullity pair (k, k) w.r.t. r.

Proof. As before, wlog $r = 1$ and A has the form $\begin{bmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{bmatrix}$. Since $\operatorname{nul} C > \operatorname{nul} A$, $\mathbf{b}$ is not in the column space of C. This guarantees a $\mathbf{v}$ in nullspace of C with $\mathbf{v}^\top \mathbf{b} \neq 0$.

Now pick $\epsilon > 0$, and consider $M = \begin{bmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C + \epsilon \mathbf{b}\mathbf{b}^\top \end{bmatrix}$.

Note that

$$(C + \epsilon \mathbf{b}\mathbf{b}^\top)\mathbf{v} = \epsilon(\mathbf{b}^\top \mathbf{v})\mathbf{b}.$$

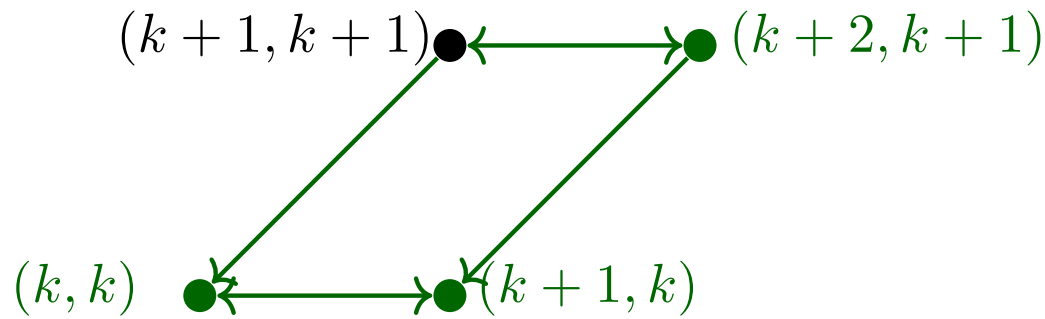
So $\mathbf{b}$ is in the column space of $C + \epsilon \mathbf{b}\mathbf{b}^\top$ and $\operatorname{nul}(M(1)) \leq \operatorname{nul} M$.

For ϵ sufficiently small, $k - 1 \leq \operatorname{nul} M \leq \operatorname{nul} A = k$, and $k \leq \operatorname{nul} M(1) \leq \operatorname{nul} A(1) = k + 1$.

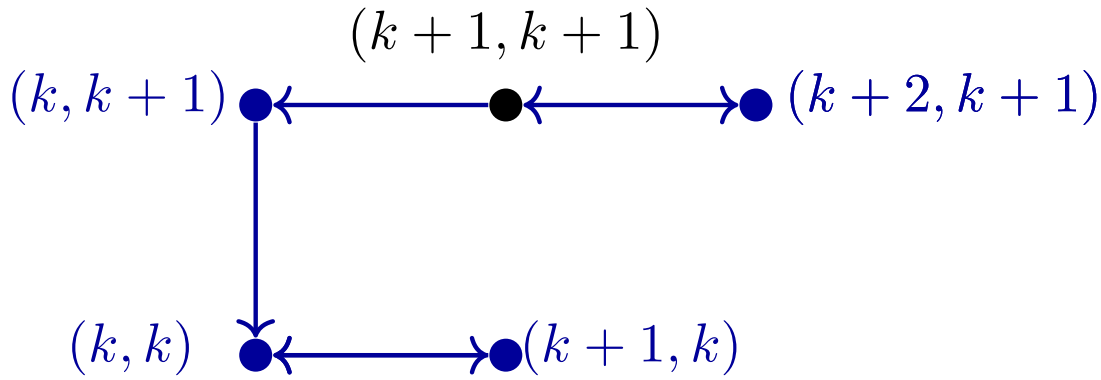
Therefore, $\operatorname{nul}(M, 1) = (k, k)$.

Now finish by applying the Proximity theorem. □

Cartoon of implications of the SNIP



Implications of $\mathcal{S}(G)$ allowimg a matrix
with the SNIP and nullity pair $(k+1,k+1)$ w.r.t. r.



Implications of $\mathcal{S}(G)$ allowing a matrix
with nullity pair $(k+1,k+1)$ w.r.t. r.

Decontraction

We need to discuss minors of rooted graphs.

Let (H, r) be a rooted graph.

A minor of (H, r) is a rooted graph (G, r) obtained from H by a sequence of the following operations:

- Removal of an edge;
- Removal of an isolated vertex other than the root r;
- Contraction along an edge ij where the neighborhoods of i and j are disjoint. If r is one of i or j, then the root of the contraction is the vertex that is obtained by identifying vertex i and j, and is named r.

Decontraction

Theorem (Decontraction theorem)

Let (G, r) be a minor of the rooted graph (H, r). If there is a matrix A in $\mathcal{S}(G)$ with the SNIP w.r.t. r, then there is a matrix in $\mathcal{S}(H)$ with the SNIP and having the same nullity pair as that of A w.r.t to r.

In particular, this implies

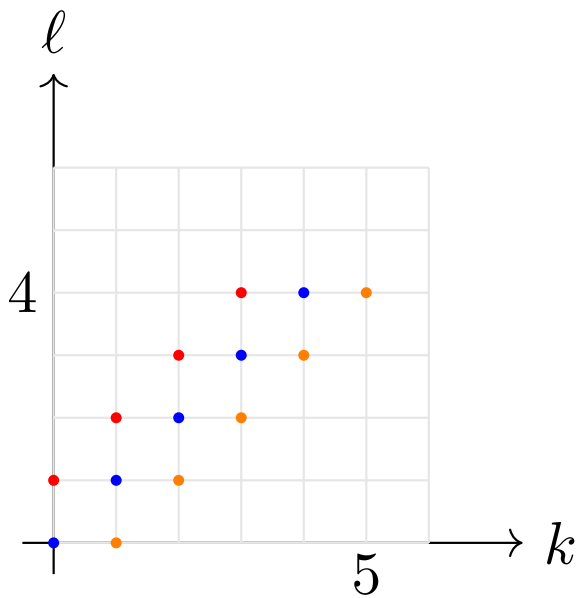
- If the graph (H, r) realizes nullity (k, ℓ), then (H, r) has a minimal rooted minor (G, r) that realizes the nullity pair (k, ℓ) with the SNIP.
- For fixed (k, ℓ) there exists a finite list $\mathcal{L}$ of r-rooted graphs such that (G, r) allows a matrix with nullity pair (k, ℓ) and the SNIP if and only if no graph in $\mathcal{L}$ is a r-rooted minor of G.

Consequences

Nullity pairs for K_n and $n \geq 2$.

Without loss, $r = 1$. Every matrix in $\mathcal{S}(G)$ has the SNIP w.r.t. r.
$\text{nul}(J_n, 1) = (n-1, n-2)$.

By the West and South theorems, K_n allows a matrix with the SNIP and nullity pair (k, ℓ) for each k and ℓ with $|k - \ell| \leq 1$ and $0 \leq k \leq n-2$ and for $(n-1, n-2)$.
We also get the pairs $(k+1, k)$ $k = 0, \ldots, n-1$ by the East of the diagonal theorem.



The realizable nullity pairs for K_6.

Each can be realized with SNIP.

Consequences: nullity pairs $(1,1)$ and $(2,1)$

Nullity pairs $(1,1)$ and $(2,1)$

Note:

(a) If G is a generalized star and r its center, then G does not allow the nullity pair $(2,1)$.

(b) (K_3, r) allows the nullity pair $(2,1)$ with the SNIP for any r.

(c) $(K_{1,3}, \text{leaf})$ allows the nullity pair $(2,1)$ with the SNIP.

$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ has nullity pair $(2,1)$ and the SNIP w.r.t. 1.

If last 3 rows of $A = \begin{bmatrix} 0 & a & b & 0 \\ a & 0 & c & 0 \\ b & c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are each $\mathbf{0}^\top$, then $a+b=0$, $a+c=0$ and $b+c=0$. This requires $a=b=c=0$.

Consequences

Theorem (Char'n (G, r) that allow nullity pair $(2, 1)$)

Let (G, r) be a connected, rooted graph. Then the following are equivalent.

(i) *(G, r) allows the nullity pair $(2, 1)$ with the SNIP;*

(ii) *(G, r) allows the nullity pair $(2, 1)$;*

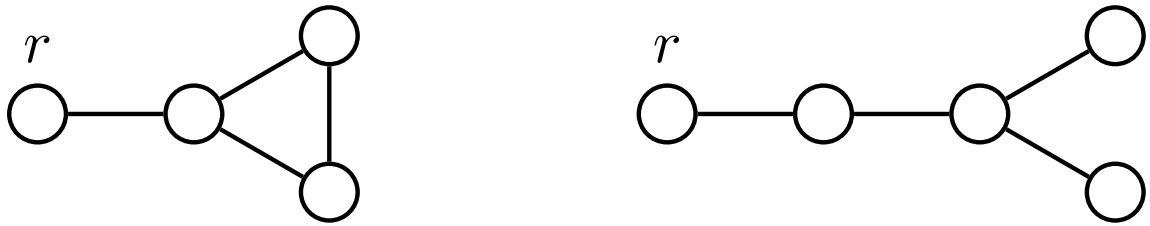
(iii) *G is not a path, and (G, r) is not a generalized star with r at its center.*

(iv) *(G, r) contains either (K_3, r) or $(K_{1,3}, \text{leaf})$ as a minor;*

Proof.

(i) $\implies$ (ii) is clear.
(ii) $\implies$ (iii) previous observations and $M(\text{path}) = 1$.
(iii) $\implies$ (iv). Done if G is not a tree. (iii) implies either G has at least two vertices of degree 3 or one vertex of degree more than 3. So (G, r) has $(K_{1,3}, r)$ as a minor.
(iv) $\implies$ (i) follows from the Decontraction theorem. $\square$

Nullity pair $(1, 2)$



Rooted graphs that allow
a matrix with the SNIP and nullity pair $(1, 2)$

One can show that both the rooted graphs $(\text{Paw}, \text{leaf})$ and $(S_{2,1,1}, \text{leaf}_2)$ above allow a matrix with the SNIP, nullity pair $(1, 2)$.

Additionally, no proper rooted minor of these has this property; that is, these are minor minimal rooted graphs for nullity pair $(1, 2)$.

Rooted graphs that avoid these minors

So which rooted graphs avoid these a minors?

A Yam graph is a connected rooted graph (G, r) such that

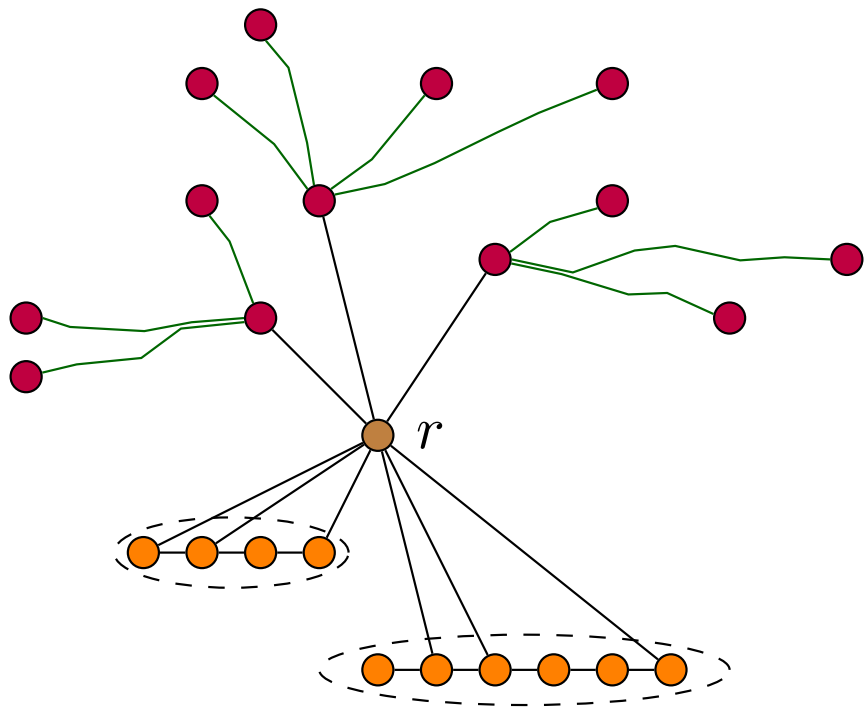
- Each connected component of $G(r)$ is either a generalized star or a path; and
- r is only adjacent to the center vertex of each component of $G(i)$ that is a generalized star.

Rooted graphs that avoid these minors

So which rooted graphs avoid these a minors?

A Yam graph is a connected rooted graph (G, r) such that

- Each connected component of $G(r)$ is either a generalized star or a path; and
- r is only adjacent to the center vertex of each component of $G(i)$ that is a generalized star.



An example of yam graph [ACFHLS].

Yam graphs and nullity pair $(1, 2)$

Lemma

No yam graph allows nullity pair $(1, 2)$ with the SNIP.

Sketch of proof.

Suppose that the graph of A is a Yam graph, then $\text{nul}(A, r) = (1, 2)$ and A has the SAP.

Then the r-th column of A is not in the span of the other columns of A. As the r-column of A is nonzero, this implies that every nullvector of A has r-th coordinate equal to 0.

WLOG, $r = 1$ and A has the form

$$A = \begin{bmatrix} a & \mathbf{b}_1^\top & \cdots & \mathbf{b}_k^\top \\ \mathbf{b_1} & B_1 & & \\ \vdots & & \ddots & \\ \mathbf{b_k} & & & B_k \end{bmatrix}$$

By non-disjoint support of nullvectors SAP theorem, exactly one B_i is singular. Without loss B_1 is singular. So $\text{nul}(B_1) \geq 2$.

This requires that the graph of B_1 is a generalized star.

The Parter-Wiener theorem implies that (B_1, center) has nullity pair $(\text{nul}(B_1), (B_1) + 1)$. This also contradicts the nullvectors SAP theorem.

Therefore, we conclude that if the rooted graph of A is a Yam graph, then A does not have the SNIP. □

Characterizaton for allows $(2, 1)$ with SNIP

Theorem

Let (G, r) be a connected rooted graph. Then the following are equivalent.

(a) *(G, r) contains (Paw, leaf) or $(S(2, 1, 1), r)$ as a minor;*

(b) *(G, r) allows the nullity pair $(2, 1)$ with the SNIP.*

(c) *G is not a Yam graph.*

Sketch of proof.

(a) $\implies$ (b) is the Decontraction theorem.

$\neg$(c) $\implies$ $\neg$(b) is the last lemma.

For $\neg$ (a) $\implies$ $\neg$ (c).

Assume (G, r) is connected and avoids the two minors.

Each cycle of G must contain vertex r.

Thus, each component of $G(r)$ is a tree.

If C is a component of $G(r)$ that is a tree and not a path, then avoidance of $S(2, 1, 1), \text{leaf}_2)$ as a minor requires C to be a generalized star and r to be adjacent to only the center of C.

Characterization for allows $(1, 2)$

Not too difficult to prove the next result.

Theorem

Let (G, r) be a connected rooted graph. Then (G, v) allows the nullity pair $(1, 2)$ if and only if G is not a Yam graph for which $G(r)$ has exactly one component.

Note the general strategy here.

0. Want to characterize graphs (or rooted graphs) with a given property (e.g. allows nullity pair $(1, 2)$)
1. Start by characterizing such graphs that allow a matrix with the desired property and the strong property. Easier to do, as one can use minors.
2. Use the characterization in 1. to give structure for graphs that one needs to look at; and identify which of these have the property without the strong assumption.

Minor monotone parameter $\xi\xi$

Recall that $M(G)$, the maximum nullity of a matrix, with graph G is the original motivation for the strong properties.

We saw that $M(G)$ is a poorly behaved parameter; it doesn't cooperate with subgraphs or minors.

The parameter $\xi(G)$ which is the largest nullity of a matrix with graph G and the SAP is a much nicer parameter.

In particular, ξ is minor monotone; that is, if G is a minor of H, then $\xi(G) \leq \xi(H)$.

We now discuss a minor monotone parameter associated with nullity pairs of (G, r).

Paraphrasing Tom Lehrer: Excise, excise. Let no one else's work evade your eyes!

By the East of the Diagonal, South and West theorems, the study of nullity pairs for (G, r) under the assumption of the SNIP reduces to finding the (k, ℓ) with $\ell \in \{k, k+1\}$ for which (G, r) allows (k, ℓ) as a nullity pair with a matrix having the SNIP.

We define $\xi\xi(G, r)$ to be the maximum of $k + \ell$ such that $k \leq \ell$, $\mathcal{S}(G)$ has a matrix with nullity pair (k, ℓ) and the SNIP w.r.t. r.

Pronounced: "the excise of (G, r)."

Examples:
$\xi\xi(K_n, r) = 2n - 4$ (with nullity pair $(n-2, n-2)$)
$\xi\xi(P_n, \text{pendant}) = 1$
$\xi\xi(K_{1,n}, \text{center}) = 1$ (for $n \geq 3$).

The Decontraction and Supergraph theorems for the SNIP imply that $\xi\xi$ is minor monotone; that is, if (G, r) is a minor of (H, r), then $\xi\xi(G, r) \leq \xi\xi(H, r)$.

Minimal minors for $\xi\xi \geq k$ where $k \in \{0, 1, 2, 3, 4, 5\}$ [AFHLS]

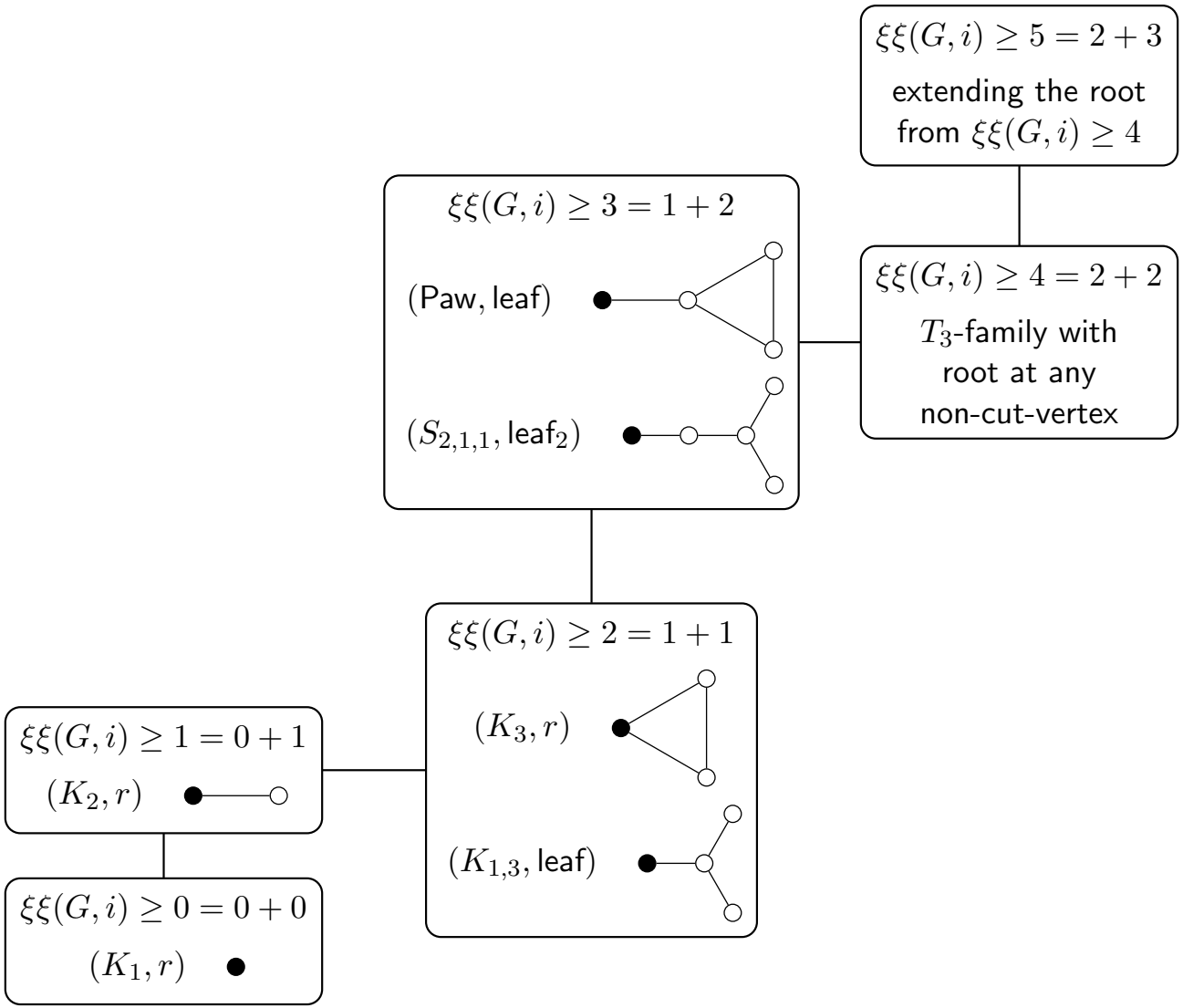


Figure: Minimal minors for $\xi\xi$ [AFHLS].

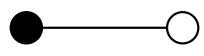
Unpacking $\xi\xi \geq 0$; $\xi\xi \geq 1$

$$\xi\xi(G, i) \geq 0 = 0 + 0$$

(K_1, r) 

This just says that $\xi\xi(G) \geq 0$ if and only if G has at least one vertex; i.e. if and only if G has (K_1, r) as a minor. In this case, $\xi\xi$ is achieved by nullity pair $(0, 0)$.

$$\xi\xi(G, i) \geq 1 = 0 + 1$$

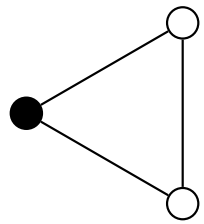
(K_2, r) 

This says $\xi\xi(G, r) \geq 1$ if and only if G contains at least one edge.

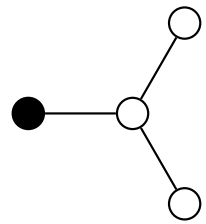
Unpacking $\xi\xi \geq 2$

$\xi\xi(G, i) \geq 2 = 1 + 1$

(K_3, r)



$(K_{1,3}, \text{leaf})$

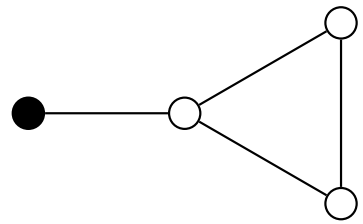


So, a connected graph (G, r) has $\xi\xi \geq 2$ if and only if it contains a cycle, or is not a generalized star with r as its center.

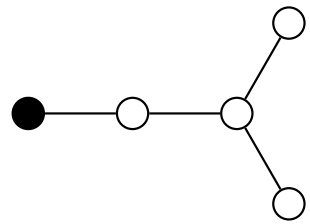
Unpacking $\xi\xi \geq 3$

$\xi\xi(G, i) \geq 3 = 1 + 2$

(Paw, leaf)



$(S_{2,1,1}, \text{leaf}_2)$



Thus, the connected (G, r) has $\xi\xi$ at least 3 if and only if it is not a yam graph.

Unpacking $\xi\xi \geq 4$

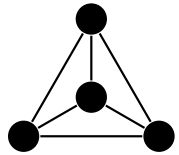
Hogben and van der Holst [HH] have characterized the minimal minors for $\xi(G) = 3$.

That is, they have identified the family of graphs $\mathcal{F}$ such that $\xi(G) \geq 3$ if and only if G contains a graph in $\mathcal{F}$ as a minor. Additionally, any proper minor G of a graph in $\mathcal{F}$ has $\xi(G) \leq 2$.

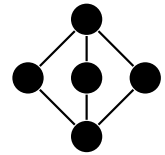
These are known as the T_3-family of graphs.

Not surprisingly they arise in the study of graph (G, r) with $\xi\xi(G, r) \geq 4$, as either $\xi\xi(G)$ comes from $(2, 2)$, and hence $\xi(G) \geq 3$ or form $(2, 3)$ and $\xi(G) \geq \xi(G(r)) \geq 3$.

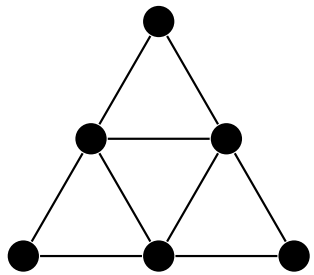
The T_3-family



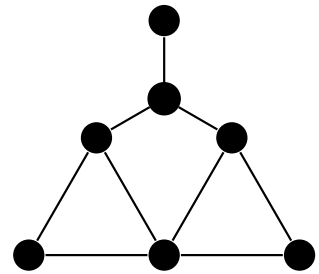
K_4



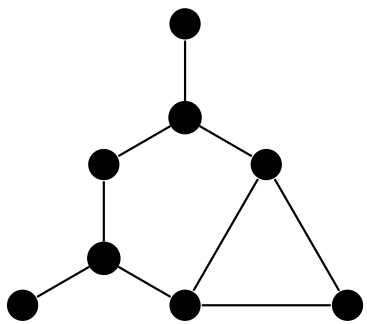
$K_{2,3}$



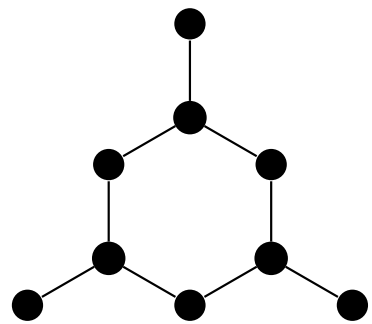
T



$T^{(1)}$



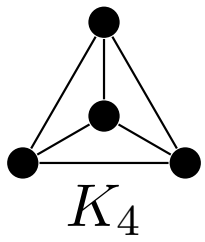
$T^{(2)}$

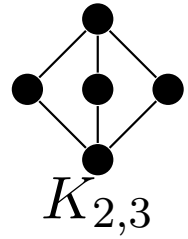


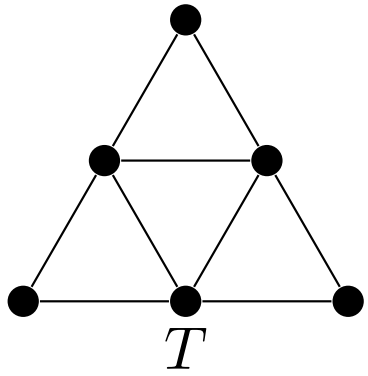
$T^{(3)}$

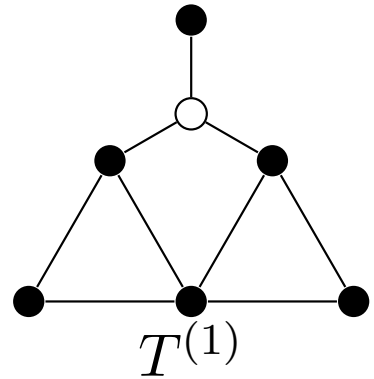
The rooted T_3-family

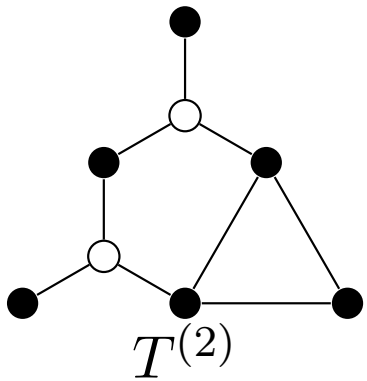
The T_3-family

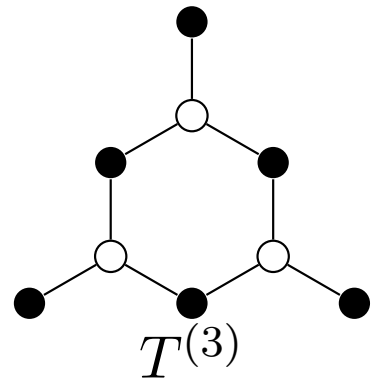












The T_3-family with all possible roots marked as filled vertices.

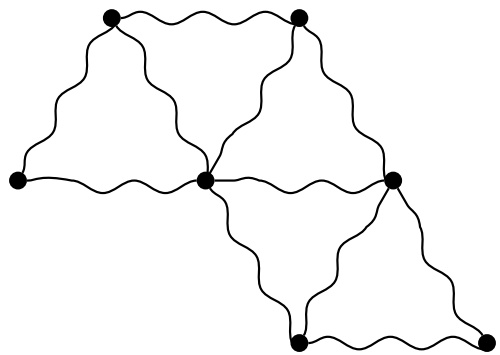
Unpack $\xi\xi \geq 4$

$$\xi\xi(G, i) \geq 4 = 2 + 2$$

T_3-family with root at any non-cut-vertex

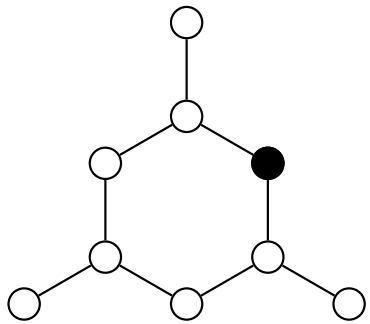
This means $\xi\xi(G, r) \geq 4$ if and only if G contains one of the graphs in the T_3-family as a minor, where r is not a cut-vertex.

Consequently, if G is two-connected, then $\xi\xi(G, r) < 4$ if and only if T is a 2-tree.

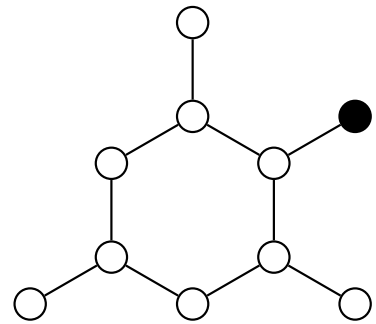


Extending roots

If (G, r) is a rooted graph, then the extension of (G, r) is obtained from G by attaching a pendant vertex to r and then moving r to the new pendant vertex.



(G, r)



(Extended G, r)

Theorem

For each $k \in \{0, 1, 2\}$, the minimal minors of $\xi\xi \geq 2k + 1$ are precisely the root extensions of the minimal minors for $\xi\xi \geq 2k$.

Open problem: Is this true for all k?

Density theorem

Theorem

Let (G, r) be a rooted graph $\xi\xi(G) = k + \ell$ for some $k \leq \ell \leq k + 1$. Then

(a) $|E(G)| \geq \binom{k+2}{2} - 1$ *if $k + \ell$ is even; and*

(b) $|E(G(r))| \geq \binom{k+2}{2} - 1$, *if $k + \ell$ is odd.*

Theorem (Nordhaus-Gaddum type bound for $\xi\xi$)

Let (G, r) be a rooted graph on n vertices. Then

$$\xi\xi(G, r) + \xi\xi(\overline{G}, r) \leq 2\sqrt{2}n.$$

Proof is an exercise.

Question: How good are the above inequalities?

References

- [FHLS] A. Abiad, M. Flagg, H. Hall, J. Lin and B. Shader. A strong nullity parameter for rooted graphs, under review, 2026.
- [BCFHLS], A. Abiad et al., The inverse nullity pair problem and the strong nullity interlacing property, Linear Algebra Appl. **699** (2024), 539–568.
- [HH] L. Hogben and H. van der Holst, Forbidden minors for the class of graphs G with $\xi(G) \leq 2$, Linear Algebra Appl. **423** (2007), no. 1, 42–52.

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