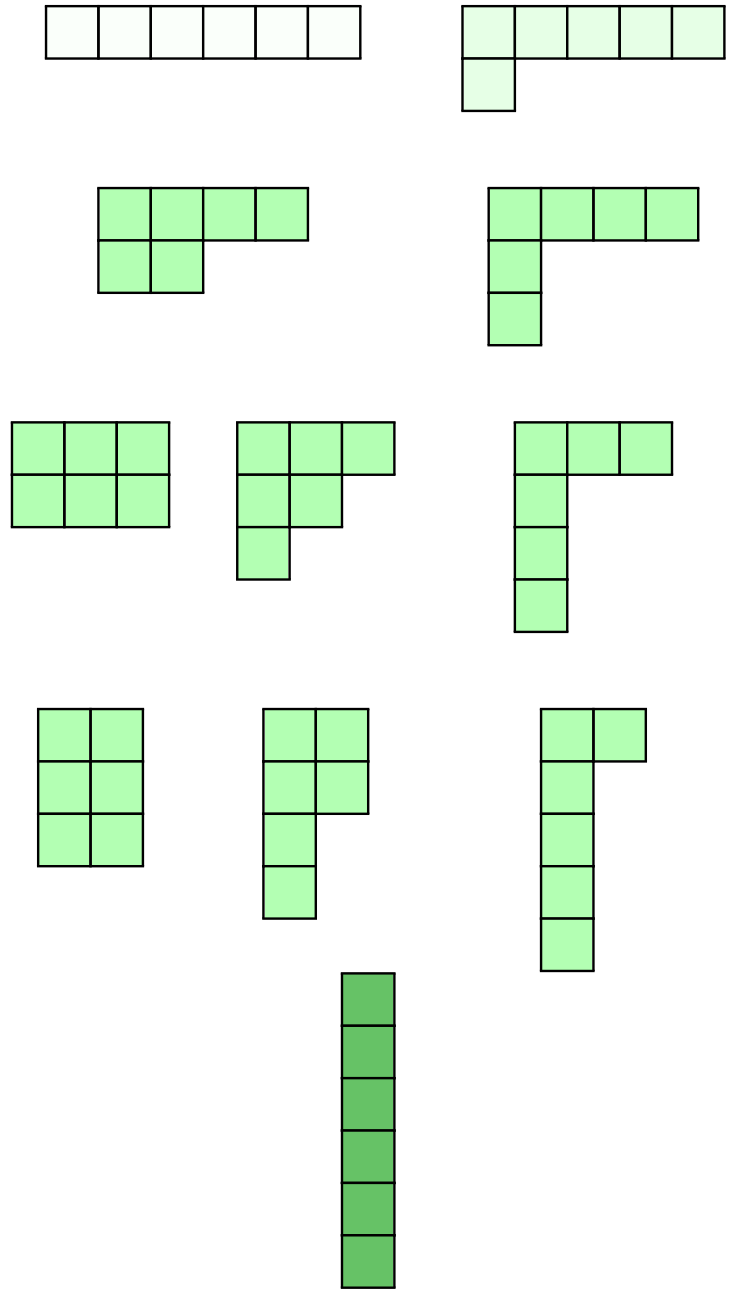
Multiplicities and Minimum Number of Distinct Eigenvalues

Multiplicity lists as partitions



Given G on n vertices, characterize all

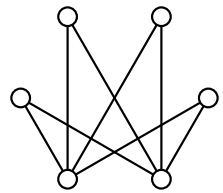
$$\mathbf{m} = (m_1, m_2, \ldots, m_s),\ m_1 \geq \ldots \geq m_s$$

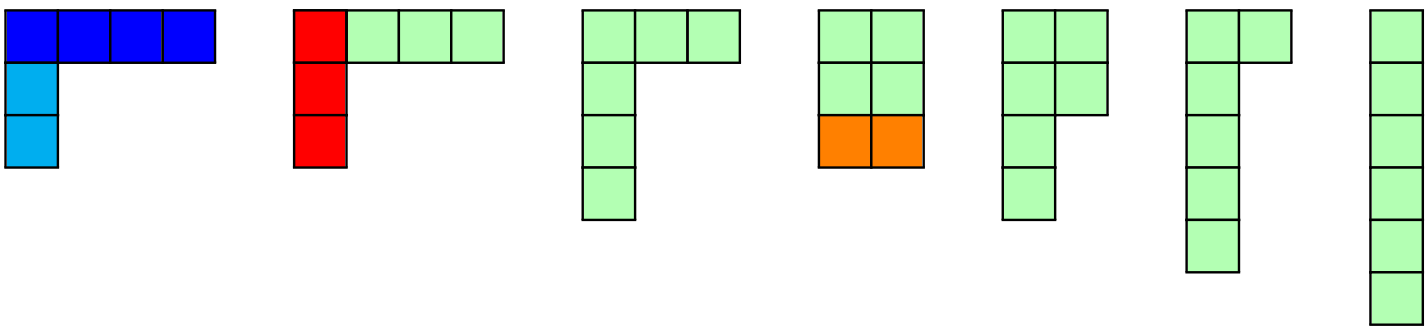
realisable for G.

- $\mathbf{m}$ is a partition of n
- $\mathbf{m}(A) = (n) \Leftrightarrow A = Q(\lambda I_n)Q^\top \Leftrightarrow$
  $\Leftrightarrow A = \lambda I_n$
- G connected $\Longrightarrow\ s \geq 2$

Parameters

IMPG: Find all $\mathbf{m} = (m_1, \ldots, m_s)$, realisable for G.





$$M(G) = \max\{m_1 : \mathbf{m} \text{ realisable for } G\}$$

$$\mathrm{mr}(G) = n - M(G)$$

$$\mathrm{Mm}(G) = \max\{m_s : \mathbf{m} \text{ realisable for } G\}$$

$$q(G) = \min\{s : \mathbf{m} \text{ realisable for } G\}$$

Anything goes

$$\mathbf{m} = (m_1, \ldots, m_k)$$

is spectrally arbitrary for G, if the multiset

$$\{\lambda_1^{(m_1)}, \ldots, \lambda_k^{(m_k)}\}$$

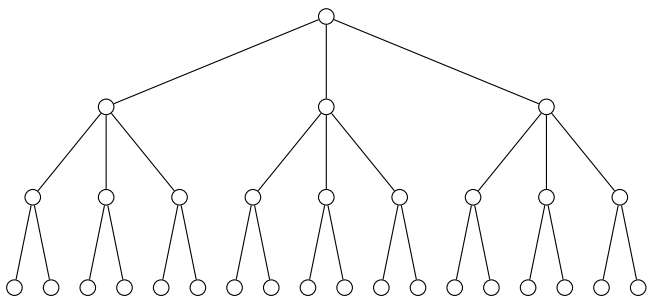
is realizable for G for any real numbers

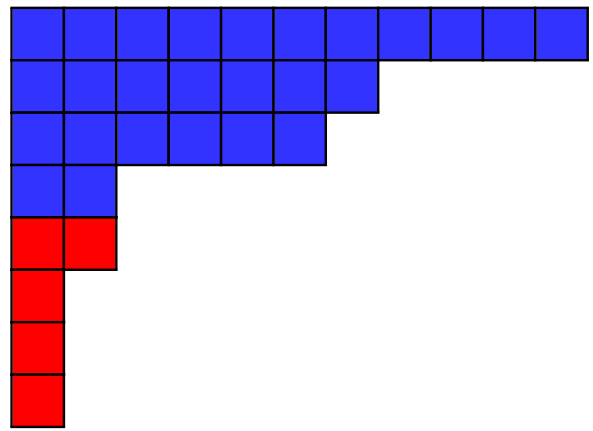
$$\lambda_1 < \cdots < \lambda_k.$$

Murphy's Law

Quite a reasonable Conjecture (Buckley, Corliss, Johnson, Araúz Lombardía, Saiago, '16): If $(m_1, m_2, \ldots, m_s)$ is realisable for a tree T, then $(m_1, \ldots, m_j - 1, \ldots, m_s, 1)$ is also realisable for T.

Fallat, Hall, Levene, Meyer, Nasserasr, Oblak, Š., '24





$$V = 31, M = 11, q \geq 7$$

Blue multiplicities force the red ones.

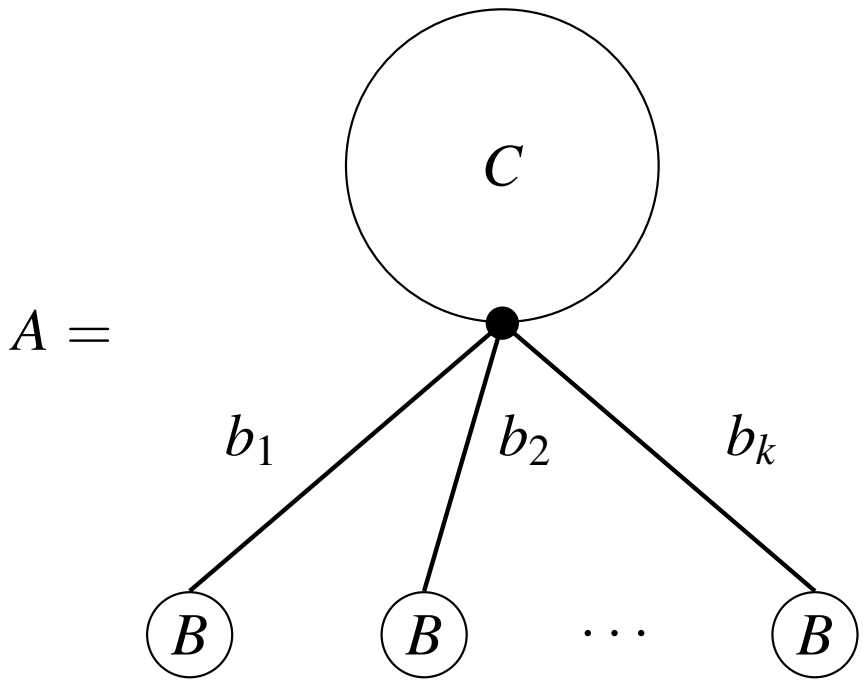
Multiplicites can rule the world

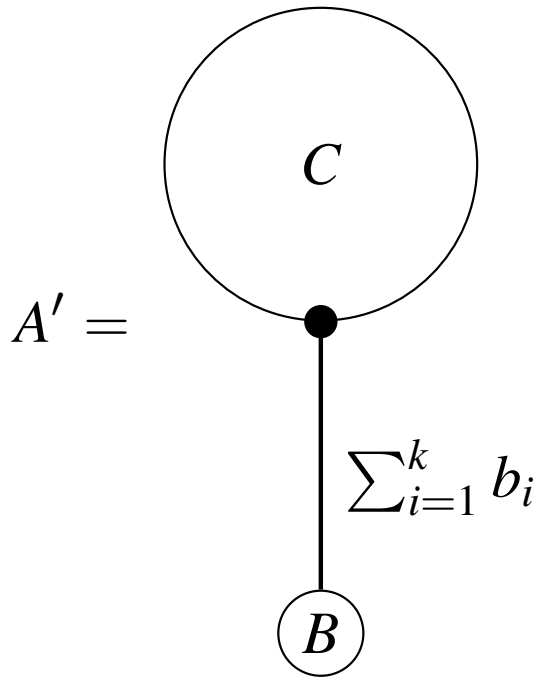
Fallat, Hall, Levene, Meyer, Nasserasr, Oblak, Š., '24
T_8- smallest lush hedge of height $H = 8$. $V(T_8) = 7654$

$$\mathbf{m} = \begin{pmatrix} 2734, 1640, 1514, 505, 492, 168, 164, 163, \\ 54, 54, 54, 18, 18, 18, 18, 6, 6, 6, 6, 2, 2, 2, 2, 2, \\ 1, 1, 1, 1, 1, 1 \end{pmatrix}$$

- $\mathbf{m}$ can be realized in T_8,
- red multiplicities $\Rightarrow$ black multiplicities
- red and blue multiplicities $\Rightarrow$ the assignment of eigenvalues to $\mathbf{m}$ is unique up to shifting and scaling.

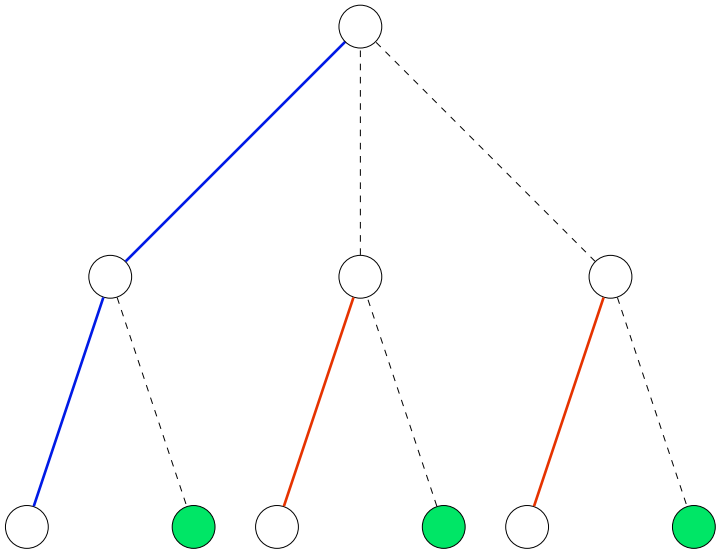
Increasing multiplicities: Collapsing and Branching





(Johnson, Saiago, 2008): $\sigma(A) = \sigma(A') \cup (k-1)\sigma(B)$

Path-to-Hedge Construction, small example



$$\sigma(A) = \sigma\left(\begin{array}{c} a_3 \\ b_3 \\ a_2 \\ b_2 \\ a_1 \end{array}\right) \cup 2\sigma\left(\begin{array}{c} a_2 \\ b_2 \\ a_1 \end{array}\right) \cup 3\sigma\left(a_1\right)$$

Invertible Subtrees Lemma

Fallat et. al 2024

T a tree, T_1 and T_2 subtrees of T

- T_1 and T_2 are independent in T if $u \in V(T_1)$ and $v \in V(T_2)$ implies $\{u, v\} \notin E(T)$.

Assuming:

- $T_1, \ldots, T_c$ are mutually independent subtrees of T
- $A \in \mathcal{S}(T)$ such that $A[T_i]$, $i = 1, \ldots, c$, are all invertible,

then

$$\text{nullity}(A) \leq P(T \setminus (T_1 \cup \cdots \cup T_c)).$$

Invertible Subtrees Lemma, corollaries

$A \in \mathcal{S}(T)$

- $T_1, \ldots, T_c$ are mutually independent subtrees of T
- $\text{mult}(\lambda, A) > P\left(T \setminus (T_1 \cup \cdots \cup T_c)\right)$

then $\lambda \in \text{spec}(A[T_i])$ for at least one $i \in \{1, \ldots, c\}$.

- T be a tree
- Q a path in a minimal path cover of T.

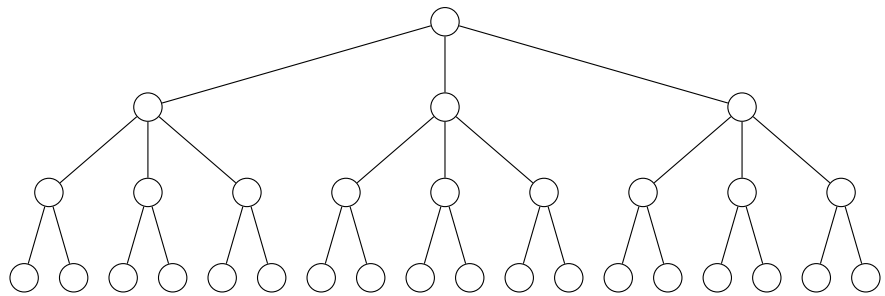
$A \in S(T)$ and $\text{mult}(\lambda, A) = P(T) \Rightarrow \lambda \in \sigma(A[Q])$

Let's start with some multiplicities

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	11	7	6	2	1

.

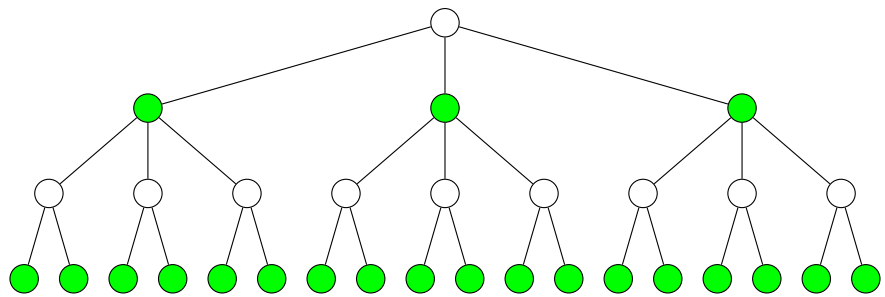
$A =$



Green eigenvalue

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	11	7	6	2	1

.

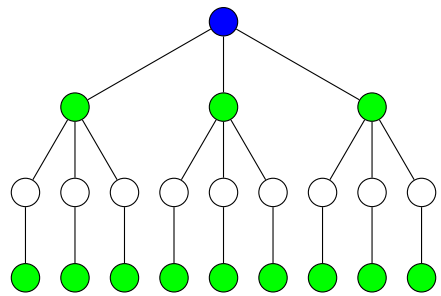
$A =$ 

Blue Eigenvalue

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	2	7	6	2	1

.

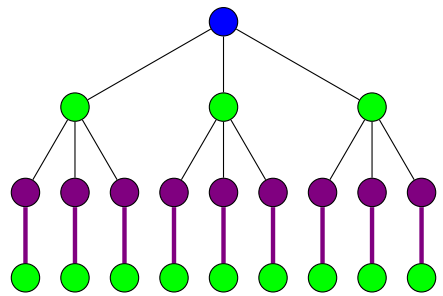
$A^{(1)} =$



Blue and Red make Purple

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	2	7	6	2	1

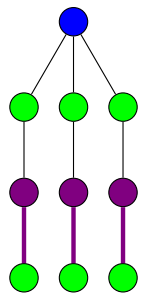
.

$A^{(1)} =$ 

Down to $P(T) = 2$

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	2	1	0	2	1

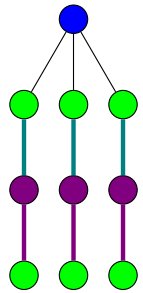
.

$A^{(2)} =$ 

Teal and finally Orange

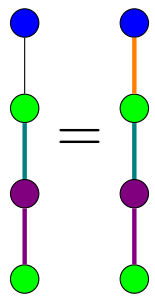
λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	2	1	0	2	1

.

$A^{(2)} =$ 

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	0	1	0	0	1

.

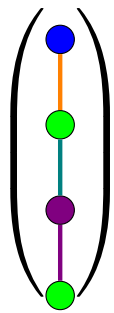
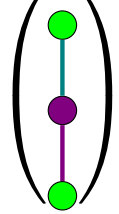
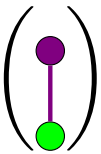
$A^{(3)} =$  $=$

Nonlinear Constraint on Eigenvalues

If $A \in \mathcal{S}(T)$ satisfies:

λ	α_1	α_2	β_2	β_3	β_4
$\mathrm{mult}(\lambda, A)$	11	7	6	2	1

Then:

$$\sigma(A) = \sigma$$  $$\cup\, 2\sigma$$  $$\cup\, 6\sigma$$  $$\cup\, 9\sigma\,(\,)$$

λ	δ_1	δ_2	δ_3
$\mathrm{mult}(\lambda, A)$	2	1	1

$$\alpha_2 + \beta_2 = \beta_3 + \delta_1$$
$$\alpha_1 + \beta_3 + \delta_1 = \beta_4 + \delta_2 + \delta_3.$$
$$(\beta_2 - \beta_4)(\beta_2 - \delta_2)(\beta_2 - \delta_3) = (\beta_2 - \alpha_1)(\beta_2 - \beta_3)(\beta_2 - \delta_1).$$

Let T be a lush hedge.....

There exists $\mathbf{m}_0 = \{m_1, m_2, n_2, n_3, n_4\}$ and $A \in S(T)$ with multiplicity list $\mathbf{m}$ that contains $\mathbf{m}_0$ so that:

1. Every such A can be repeatedly collapsed to $B \in S(P_n)$.
2. B is uniquely defined by $\alpha_1, \alpha_2, \beta_2, \beta_3, \beta_4$: five eigenvalues of A with multiplicities $\mathbf{m}_0$.
3. *Every* such matrix A can be determined from B by branching.
4. $\sigma(A)$ can be computed just from B (or equvalently from $\alpha_1, \alpha_2, \beta_2, \beta_3, \beta_4$) and T.

Paths and Coincidences

High multiplicities of $\alpha_1, \alpha_2, \beta_2, \beta_3, \beta_4$ are achieved by carefully choosing $B \in S(P_n)$. Let C_k denote the trailing submatrix of P_n on k vertices. Then B is constructed so that:

$$\alpha_1 \in \sigma(C_1) \cap \sigma(C_3) \cap \sigma(C_5) \ldots$$
$$\alpha_2 \in \sigma(C_2) \cap \sigma(C_4) \cap \sigma(C_6) \ldots$$
$$\beta_2 \in \sigma(C_2) \cap \sigma(C_5) \cap \sigma(C_8) \ldots$$
$$\beta_3 \in \sigma(C_3) \cap \sigma(C_6) \cap \sigma(C_9) \ldots$$
$$\beta_4 \in \sigma(C_4) \cap \sigma(C_7) \cap \sigma(C_{10}) \ldots$$

We want more

$$\lambda \in (\sigma(C_3) \cap \sigma(C_7)) \setminus \Lambda$$

Okay, but we have to choose α_i and β_i so that

$$-\alpha_1\beta_2+\alpha_2\beta_2+2\alpha_1\beta_3+\alpha_1\beta_4+\alpha_2\beta_4-2\alpha_1\alpha_2+\beta_2^2-2\beta_3\beta_2-\beta_4^2=0$$

Very very greedy!

Demanding:

$$\lambda_{3,7} \in (\sigma(C_3) \cap \sigma(C_7)) \setminus \Lambda$$
$$\lambda_{4,8} \in (\sigma(C_4) \cap \sigma(C_8)) \setminus \Lambda$$
$$\lambda_{4,9} \in (\sigma(C_4) \cap \sigma(C_9)) \setminus \Lambda$$

results in a unique solution (up to shifting and scaling)!

Strong Properties make the world nice again

(Fallat, Hall, Lin Shader, '22): If $\mathbf{m}$ is is realisable for G with the SMP, then any refinement of $\mathbf{m}$ is realisable for G.

The number of distinct eigenvalues

For $A \in \mathcal{S}(G)$ we define $q(A)$ to be the number of distinct eigenvalues of A, or the degree of the minimal polynomial of A.

The minimum number of distinct eigenvalues of a graph G is

$$q(G) = \min\{q(A); A \in \mathcal{S}(G)\},$$

the minimum length k of all ordered multiplicity lists $(m_1, m_2, \ldots, m_k)$ of G.

We already know how to identify

- graphs G with $q(G) = 1$, and
- graphs G with $q(G) = |G|$.
- graphs G with $q(G) = |G| - 1$.

Unique shortest path

Combinatorial lower bound for $q(G)$

Theorem (Fonseca, 2010; Ahmadi et. al, 2013)

If a connected graph G contains vertices u, v at distance d, and the path of length d from u to v is unique, then $q(G) \geq d + 1$.

Proof.

If G contains vertices u, v at distance d, and the path of length d from u to v is unique then:

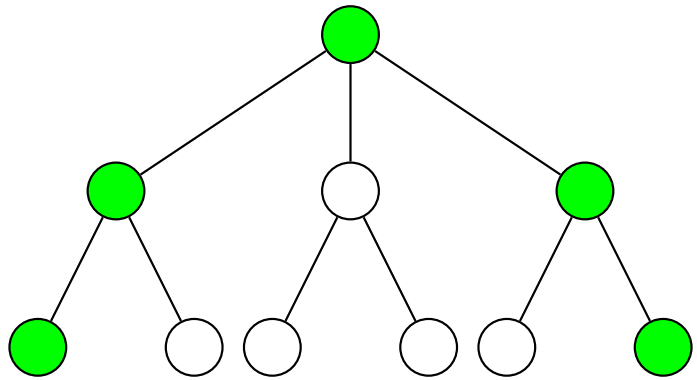
- $A^i_{u,v} = 0$ for $i = 1, \ldots, d-1$, and
- $A^d_{u,v} \neq 0$.

Hence, $I_n, A, \ldots, A^d$ are linearly independent.

The number of distinct eigenvalues of A is equal to the degree of the minimal polynomial of A.

□

Example



- $\mathrm{diam}(T) = 4 \Rightarrow q(T) \geq 5$
- It turns out $q(T) = 5$.

Trees

- If T is a tree, then $q(T) \geq \mathrm{diam}(T) + 1$.
- Bariolli-Fallat (2004), Kim, Shader (2009):There exist trees with $q(T) > \mathrm{diam}(T) + 1$.

$q(G)$ and the strong properties

Theorem (Barrett, Fallat, Hogben, Lin, Shader, 2017)

If a matrix $A \in \mathcal{S}(G)$ has the SMP, and H is a subgraph of G, then

$$q(H) \leq |H| - |G| + q(A).$$

Corollary

If a matrix $A \in \mathcal{S}(G)$ has the SMP, and H is a supergraph of G, on the same set of vertices, then $q(H) \leq q(A)$.

$$q_M(G) = \min\{q(A); A \in \mathcal{S}(G) \text{ and } A \text{ has the SMP}\}$$

$q(G) = 2$ and orthogonal matrices

$q(G) = 2$ if and and only if $\mathcal{S}(G)$ contains an orthogonal matrix.

Necessary conditions on the structure of G with $q(G) = 2$

If $q(G) = 2$, then

- Ahmadi, Alinaghipour, Cavers, Fallat, Meagher, Nasserasr, '13
  - G has no pendant vertices,
  - G has no cut edges,
  - any two non-adjacent vertices have at least two common neighbors,
- Barrett, Fallat, Furst, Kenter, Nasserasr, Rooney, Tait, van der Holst,'23

$$E(G) \geq \begin{cases} 2n-4, & n = |V(G))| \text{ even}, \\ 2n-3, & n \text{ odd}. \end{cases}$$

Nordhaus–Gaddum conjecture for q

For any graph G, the inequality

$$q(G) + q(G^c) \leq |G| + 2 \tag{1}$$

holds if and only if G is not one of the four graphs P_4, P_5, P_5^c, and $T_{2,2}$.

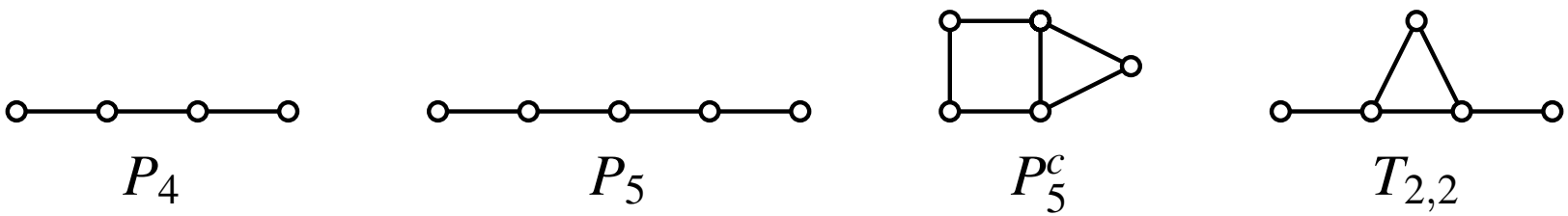


Figure: In all four cases, we have $q(G) + q(G^c) = 8$.

Sometimes $q(G^c) = 2$

Theorem (Levene, Oblak, Š., 2019)

Let G be a 2-colourable graph which admits a 2-colouring with m black vertices and n white vertices, where $|G| = m + n \geq 3$. If G does not contain

$$K_{m_1,n_1} \cup K_{m_2,n_2}$$

as a spanning subgraph, where m_1, m_2, n_1, n_2 are nonnegative integers satisfying $m = m_1 + m_2$, $n = n_1 + n_2$ and either $m_1 m_2 \neq 0$ or $n_1 n_2 \neq 0$, then $q(G^c) = 2$.

$q(T^c)$ for T a tree is resolved

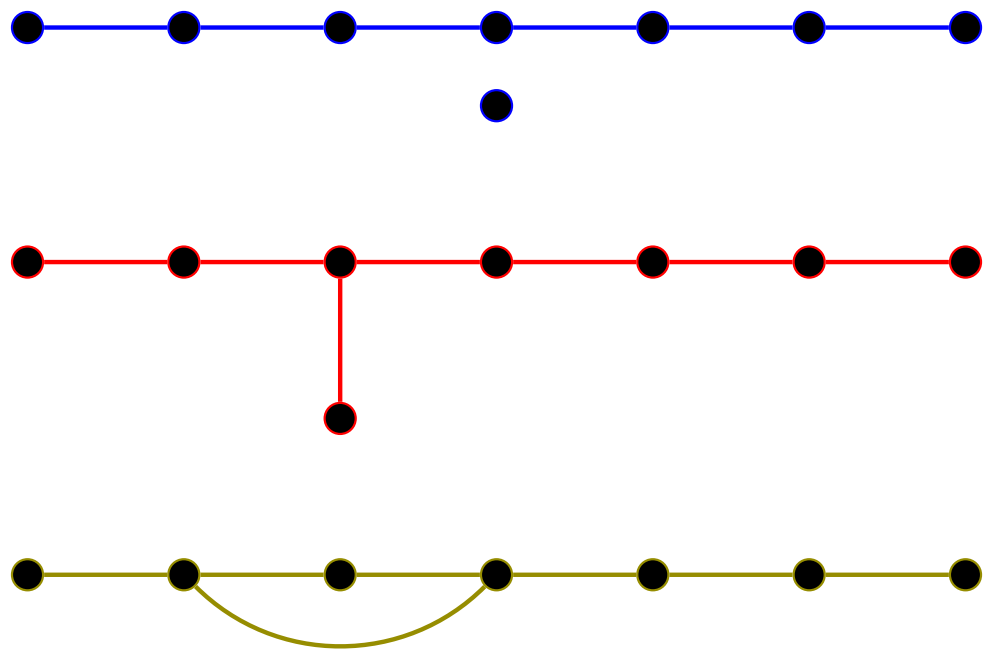
If T is a tree, then

$$q(T^c) = \begin{cases} 4, & \text{for } T = P_4, \\ 3, & \text{for } T \neq P_4 \text{ and either } T = S^k_{m,1},\ k \in \{2,3\},\ m \geq 1 \\ & \text{or } T = W(2, l, (1, d)),\ l \geq 0,\ d \geq 1, \\ 2, & \text{otherwise.} \end{cases}$$

Recall: Graphs with $q(G) = |G| - 1$

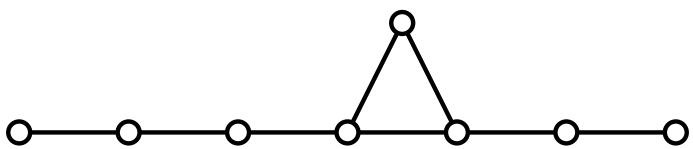
Barrett, Fallat, Hall, Hogben, Lin, Shader (2017): Suppose G is a graph on n vertices. Then $q(G) = n - 1$ if and only if G is either

- the union of a path and an isolated vertex, or
- a path with one leaf attached to an interior vertex, or
- a path with an extra edge joining two vertices of distance 2.

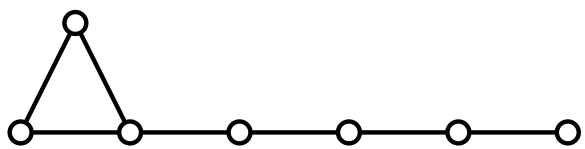


Complements of graphs G with $q(G) = |G| - 1$

- $q((P_{n-1} \cup \{n\})^c) = \begin{cases} 2, & n = 2 \text{ or } n \geq 6, \\ 3, & 3 \leq n \leq 5. \end{cases}$
- $q_S(T^c_{m,n}) = 2$ for any $m, n \geq 1$ with $m + n \geq 6$.
- $q(T^c_{m,n}) = \begin{cases} 1, & (m,n) = (1,1), \\ 4, & (m,n) = (2,2), \\ 3, & (m,n) \in \{(2,1),(3,1),(4,1),(3,2)\}, \\ 2, & \text{otherwise.} \end{cases}$



$T_{4,3}$



$T_{1,5}$

Nordhaus–Gaddum conjecture for q holds for several families

1. bipartite graphs and their complements, including trees;
2. graphs with minimum rank at most three and their complements; and
3. joins $G \vee H$ of two graphs with G and H connected, and such that the number of vertices of G and H differ by at most 2.
4. disjoint unions $G \cup H$ of two graphs with G^c and H^c connected, and such that the number of vertices of G and H differ by at most 2.
5. unicyclic graphs
6. graphs with at most 7 vertices

Complementary submatrices of orthogonal matrices

Theorem (Fiedler, 2009)

Any two complementary submatrices of an orthogonal (unitary) matrix have the same singular values different from one, even with the same multiplicities.

Proof.

Let

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

be an orthogonal matrix, where U_{ii} is an $n_i \times n_i$ matrix for $i = 1, 2$. Then:

$$I_{n_1} - U_{11}U_{11}^\top = U_{12}U_{12}^\top$$
$$I_{n_2} - U_{22}^\top U_{22}^\top = U_{12}^\top U_{12}$$

Therefore, U_{11} and U_{22} have the same singular values different from one, even with multiplicities. □

Complementary submatrices of symmetric orthogonal matrices

Theorem (Levene, Oblak, Š., 2022)

Let $X = \begin{pmatrix} A & B \\ B^\top & -C \end{pmatrix}$ *be a symmetric orthogonal matrix. Then:*

- *Matrices* A *and* C *have the same eigenvalues different from* 1 *and* -1*, even with the same multiplicities.*
- *Let* $A = \oplus_{i=1}^{k} A_i$*,* $C = \oplus_{j=1}^{\ell} C_j$*, and let* $B = (B_{ij})$ *be the block partition of* B *compatible with block-diagonal partitions of* A *and* C*. If* $B_{ij} \neq 0$*, then* A_i *and* C_j *have at least one common eigenvalue* $\lambda \in (-1, 1)$*.*

Proof.

$M = \begin{pmatrix} A & B \\ B^\top & -C \end{pmatrix}$ orthogonal, U_A and U_C orthogonal matrices that diagonalize A and C.

$$M' := (U_A \oplus U_C)^\top M (U_A \oplus U_C) = \begin{pmatrix} D_A & U_A^\top B U_C \\ U_C^\top B^\top U_A & -D_C \end{pmatrix}$$

- Any row or column of M' with ± 1 on the diagonal has every other entry equal to zero.
- Deleting all such rows and columns we get an orthogonal matrix M''.

$$M'' := \begin{pmatrix} D'_A & F \\ F^\top & -D'_C \end{pmatrix}$$

- $F^\top F = I - D'^2_C$, $FF^\top = I - D'^2_A \Rightarrow F$ is square and invertible.
- $D'_A F - F D'_C = 0 \Rightarrow F^{-1} D'_A F = D'_C$ and $\sigma(D'_A) = \sigma(D'_C)$.

Proof.

- $A = A_1 \oplus \cdots \oplus A_k \in S(G)$, $U = \oplus_{i=1}^{k} U_i$ orthogonal with $U^\top A U = D_A$ diagonal.
- $C = C_1 \oplus \cdots \oplus C_l \in S(H)$, $V = \oplus_{i=j}^{l} V_j$ orthogonal with $V^\top C V = D_C$ diagonal.

B is composed of blocks B_{ij} of the form:

$$U_i S_{ij} V_j^\top$$

$S_{ij} \neq 0 \iff A_i$ and C_j share an eigenvalue $\lambda \in (-1, 1)$.

Pattern of $B_{i,j}$

- $S_{ij} \neq 0$, U_i can be chosen "generically" $\mathscr{G} \Rightarrow B_{ij}$ nowhere zero.
- $S_{ij} \neq 0$, V_j can be chosen "generically" $\mathscr{G} \Rightarrow B_{ij}$ nowhere zero.

We will return to the discussion of "genericity". For now, notice that demanding that the pattern of $B_{i,j}$ is nowhere zero might be easiest to prove and this corresponds to join of two graphs.

Complete graphs are generically realizable. $\mathscr{G}$

Multiplicity matrices for graphs

Let G be a graph with connected components $G_1, \ldots, G_k$.
A multiplicity matrix for G is a matrix

$$V = \left[\text{mult}(\lambda_i, \sigma(A_j))\right]_{1 \le i \le r,\ 1 \le j \le k}$$

where

- $A_1 \in S(G_1), \ldots, A_k \in S(G_k)$ and
- $A = A_1 \oplus \cdots \oplus A_k \in S(G)$ has $\sigma(A) \subseteq \{\lambda_1 < \lambda_2 < \cdots < \lambda_r\}$.

	K_1	$\cup$	K_3	$\cup$	K_5
λ_1	0		0		1
λ_2	1		1		3
λ_3	0		2		0
λ_4	0		0		1

Compatible pairs of matrices

- Write $\widetilde{V}$ for a matrix V with its first and last rows deleted.
- V and W are compatible if they have the same number of rows, and
  1. $\widetilde{V}$ and $\widetilde{W}$ have the same row sums;
     $U^\top AU = D \oplus I_{t_A} \oplus -I_{s_A}$, $V^\top CV = -D \oplus I_{t_C} \oplus -I_{s_C}$)
  2. no column of $\widetilde{V}$ is orthogonal to a column of $\widetilde{W}$.

     $S_{ij} \neq 0 \iff A_i$ and B_j share eigenvalue $\lambda \in (-1, 1)$.

$$V = \begin{matrix} K_7 \cup K_2 \\ \begin{pmatrix} 2 & 0 \\ \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \\ 1 & 1 \end{pmatrix} \end{matrix} \qquad W = \begin{matrix} K_3 \cup K_2 \cup K_1 \\ \begin{pmatrix} 0 & 1 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

$q(G \vee H) = 2$

- If $q(G \vee H) = 2$, then G and H have compatible multiplicity matrices.
- Let G and H be unions of complete graphs.

  $q(G \vee H) = 2 \iff$ there exist compatible multiplicity matrices V, W for G, H.

$q\left((K_7 \cup K_2) \vee (K_3 \cup K_2 \cup K_1)\right) = 2$, proof

$$K_7 \cup K_2 \qquad \vee \qquad K_3 \cup K_2 \cup K_1$$

$$V = \begin{pmatrix} 2 & 0 \\ \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \\ 1 & 1 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & 1 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}$$

A game for every graph

Compatible multiplicity matrices for tK_1 and $2K_2$

t	V for tK_1	W for $2K_2$
1	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	
2	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
4	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$
≥ 5		

Solving a combinatorial question

$\iota(G)$... number of isolated vertices of G

Theorem

$G = \bigcup_{i\in[k]} K_{m_i}$, $H = \bigcup_{j\in[\ell]} K_{n_j}$, $k \leq \ell$. *We have* $q(G \vee H) = 2$ *if and only if one of the following is true:*

1. $\iota(G) = 0$, $\iota(H) = 0$ *and* $\ell \leq |G|$,
2. $\iota(G) > 0$ *and* $k + \ell \leq |G| + \iota(G)$, *or*
3. $\iota(G) = 0$, $\iota(H) > 0$ *and either* $k + \ell \leq |G|$, *or* $2k \leq \ell \leq |G|$, *or* $\ell \leq 2k \leq |H|$.

Otherwise, $q(G \vee H) = 3$.

$\iota(G) = 0, \iota(H) > 0$

Necessary and Sufficient Conditions:

$k + \ell \leq |G|$, or $2k \leq \ell \leq |G|$, or $\ell \leq 2k \leq |H|$

$2K_2 \vee 4K_1$

$2k = 4, \ell = 4, |G| = 4$

$q = 2$

$2K_2 \vee 3K_1$

$k = 2, \ell = 3, |G| = 4, |H| = 3$

$q = 3$

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