On the number of distinct eigenvalues: nonsymmetric patterns

Outline

Counting the number of distinct eigenvalues allowed by a pattern.

- allow sequence for sign patterns
  - definition; distinguish from graph problem
  - digraph bounds
  - Jacobian and strong techniques
  - sequences ending in zeros
  - impossible sequences
  - cyclic sequences
- allow sequence for zero-nonzero patterns
  - star patterns characterization and the nSMP
  - derivative Jacobian technique
  - n-cycle patterns; result using nSMP
  - open problems

Counting number of distinct eigenvalues

$q(A)$ is the number of distinct eigenvalues of A.

$q(\mathcal{A})$ is the minimum $q(A)$ over all A with pattern $\mathcal{A}$.

$$\mathcal{B} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

$q(\mathcal{B}) = 1$ since $\mathcal{B}$ is potentially nilpotent (allows a nilpotent matrix).

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Facts:

$\mathcal{A}$ is potentially nilpotent $\Rightarrow$ $q(\mathcal{A}) = 1$

$\mathcal{A}$ is spectrally arbitrary $\Rightarrow$ $q(\mathcal{A}) = 1$

Extreme q values for sign patterns

Theorem

Given $\mathcal{A}$ is an $n \times n$ sign pattern. If $q(\mathcal{A}) = 1$ then either

- *$\mathcal{A}$ is potentially nilpotent OR*
- *one of $\mathcal{A}$ or $-\mathcal{A}$ is potentially stable.*

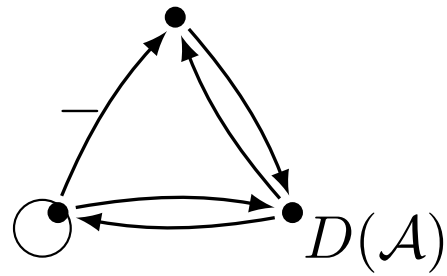
Open problem: what patterns $\mathcal{A}$ are potentially stable and have $q(\mathcal{A}) = 1$?

If $q(\mathcal{A}) = n$ then $\mathcal{A}$ requires n distinct eigenvalues

[Z. Li and L. Harris, Sign patterns that require all distinct eigenvalues. 2002]

Digraph considerations: $q(\mathcal{A}) = 1$

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ - & + & + \end{bmatrix}$$



Theorem

Let $\mathcal{A}$ be an $n \times n$ sign pattern with $q(\mathcal{A}) = 1$. Then $D(\mathcal{A})$ has a negative 2-cycle if any one of the following hold:

(i) *$D(\mathcal{A})$ has a 2-cycle;*

(ii) *$D(\mathcal{A})$ has ℓ loops, for some $\ell \in \{1, \ldots, n-1\}$; or*

(iii) *$D(\mathcal{A})$ has n loops, and two oppositely signed loops,*

In the example:

$$q(\mathcal{A}) \neq 1$$

More digraph considerations: $q(\mathcal{A}) = 1$

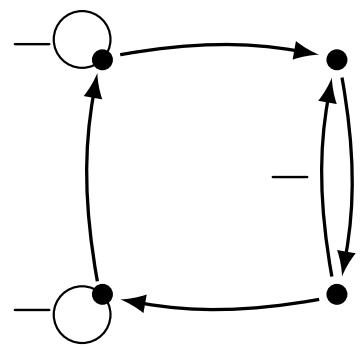
Theorem

For $n \geq 3$, suppose $q(\mathcal{A}) = 1$ and $D(\mathcal{A})$ has at least one negative loop and no positive loops. Then either $\mathcal{A}$ is a diagonal shift of a loopless potentially nilpotent pattern or both of the following hold:

(i) *$D(\mathcal{A})$ has a negative 3-cycle, or a loop and a negative 2-cycle that are vertex disjoint.*

(ii) *$D(\mathcal{A})$ has a positive 3-cycle, or a loop and a negative 2-cycle that share a vertex.*

$$\mathcal{A} = \begin{bmatrix} - & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & - & 0 & + \\ + & 0 & 0 & - \end{bmatrix}$$



$$A = \begin{bmatrix} -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 3 \\ 3 & 0 & 0 & -4 \end{bmatrix}$$

$$q(\mathcal{A}) \neq 1$$

More digraph considerations: $q(\mathcal{A}) = 1$

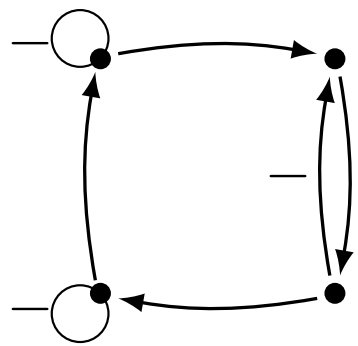
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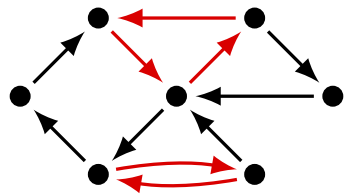


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$q(\mathcal{A}) \neq 1$ $\qquad$ $q(\mathcal{A}) = 2$

Composite cycles

A composite cycle of order k is a disjoint collection of cycles spanning k vertices.



A composite cycle of order 5

$c(\mathcal{A}) =$ maximum order of a composite cycle in $D(\mathcal{A})$
(maximum number of vertices in a collection of disjoint cycles)

Lemma (Eschenbach, Johnson, 1993)

If $\mathcal{A}$ be an $n \times n$ sign (or zero-nonzero) pattern, then

$$q(\mathcal{A}) \leq c(\mathcal{A}) + 1$$

Girth of a digraph

$m(\mathcal{A}) =$ minimum length of a cycle in $D(\mathcal{A})$
(girth of digraph)

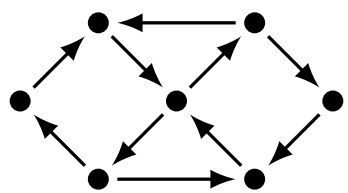
Theorem

Let $\mathcal{A}$ be an $n \times n$ sign (or zero-nonzero) pattern that is not potentially nilpotent. Then

$$q(\mathcal{A}) \geq m(\mathcal{A}).$$

Furthermore, if $c(\mathcal{A}) \leq n-1$, then

$$m(\mathcal{A}) + 1 \leq q(\mathcal{A}).$$



A digraph with cycles of length 3 and 4.
$4 \leq q(\mathcal{A}) \leq 5$

Requires distinct eigenvalues: $q(\mathcal{A}) = n$

Theorem (Eschenbach 1987; Eschenbach, Hall, Li, 1994)

If $\mathcal{A}$ is an $n \times n$ skew symmetric irreducible tridiagonal sign pattern matrix, then $q(\mathcal{A}) = n$ (In fact, $\mathcal{A}$ requires n distinct pure imaginary [possibly including zero] eigenvalues.)

Theorem (Eschenbach 1987; Eschenbach, Hall, Li, 1994)

If $\mathcal{A}$ is $n \times n$ and $D(\mathcal{A})$ is a directed n-cycle, then $q(\mathcal{A}) = n$.

Theorem (Li, Harris, 2002)

If $\mathcal{A}$ is $n \times n$, $D(\mathcal{A})$ has a simple odd cycle of length $k > 1$, and $\mathcal{A}$ has at most one zero diagonal entry, then $q(\mathcal{A}) \neq n$.

Superpatterns and $q(A)$

Theorem

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in \mathcal{A}$ has the non-symmetric strong spectral property. Then there exists a $B \in \hat{\mathcal{A}}$ with $q(B) = q(A)$.

Note: superpatterns $\hat{\mathcal{A}}$ with $q(\hat{\mathcal{A}}) \leq Q(A)$ are not guaranteed:

Example

$$\mathcal{B} = \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix} \qquad \hat{\mathcal{B}} = \begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix},$$

$$q(\mathcal{B}) = 1 \qquad \text{but} \qquad q(\hat{\mathcal{B}}) = 2.$$

(superpattern is not potentially nilpotent
nor equivalent to a potentially stable pattern)

Jacobian with deficient rank

Theorem

Let $\mathcal{A}$ be a loopless pattern and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. If $A \in \mathcal{A}$ allows a Jacobian with rank $n-1$ then there exists $B \in \hat{\mathcal{A}}$ such that $q(B) = q(A)$.

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ + & + & + & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 8 & 6 & 0 \end{bmatrix}$$

$q(A) = 2 \quad \Rightarrow \quad q(\mathcal{A}) = 2$ ($\mathcal{A}$ is not potentially nilpotent)

A allows a rank 3 Jacobian $\quad \Rightarrow \quad q(\hat{\mathcal{A}}) \leq 2$ for all superpatterns $\hat{\mathcal{A}}$ of $\mathcal{A}$

Allow sequence for number of distinct eigenvalues

Define a q-sequence:
$q_{seq}(\mathcal{A}) = \langle q_1(\mathcal{A}), q_2(\mathcal{A}), \ldots, q_n(\mathcal{A}) \rangle$
with $q_i(\mathcal{A}) = 1$ if $q(A) = i$ for some $A \in \mathcal{A}$, and $q_i(\mathcal{A}) = 0$ otherwise.

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ - & + & + \end{bmatrix}$$

$$q_{seq}(\mathcal{A}) = \langle 0, 1, 1 \rangle$$

Example

$$\mathcal{B} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

$q(\mathcal{B}) = 1$ since $\mathcal{B}$ is potentially nilpotent. $\Rightarrow q_1(\mathcal{B}) = 1.$

There is no $B \in \mathcal{B}$ such that $q(B) = 2.$ $\Rightarrow q_2(\mathcal{B}) = 0.$

There exists $B \in \mathcal{B}$ with $q(B) = 3.$ $\Rightarrow q_3(\mathcal{B}) = 1.$

Thus

$$q_{seq}(\mathcal{B}) = \langle 1, 0, 1 \rangle$$

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Thus

$$q_{seq}(\mathcal{B}) = \langle 1, 0, 1 \rangle$$

Lemma

$\mathcal{A}$ is spectrally arbitrary $\Rightarrow$ $q_{seq}(\mathcal{A}) = \langle 1, 1, \ldots, 1 \rangle$

Composite cycles and trailing zeros in an allow sequence

$c(\mathcal{A}) =$ maximum order of a composite cycle in $D(\mathcal{A})$

Lemma (Eschenbach, Johnson, 1993)

Let $\mathcal{A}$ be an $n \times n$ sign pattern and $c = c(\mathcal{A})$. If $c \leq n-2$ then

$$q_{c+1}(\mathcal{A}) = 1$$

and $q_k(\mathcal{A}) = 0$ for $c+2 \leq k \leq n$. Further,

$$q_n(\mathcal{A}) = 1 \quad \Longleftrightarrow \quad c \geq n-1$$

Note: Let $\mathcal{A}$ be an $n \times n$ sign pattern and $1 \leq t \leq n-1$. Then $q_{\text{seq}}(\mathcal{A})$ has exactly t trailing zeros $\quad \Longleftrightarrow \quad c(\mathcal{A}) = n-t-1$.

Condition for $q_2(\mathcal{A}) = 1$

Theorem

For $n \geq 2$, let $\mathcal{A}$ be an $n \times n$ sign pattern with $c(\mathcal{A}) \leq n-1$. If $A \in Q(\mathcal{A})$ has $q(A) = 2$ and the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero, and any one of the following conditions hold:

(i) *$D(\mathcal{A})$ has a 2-cycle;*

(ii) *$D(\mathcal{A})$ has between 1 and $c(\mathcal{A})-1$ loops; or*

(iii) *$D(\mathcal{A})$ has $c(\mathcal{A})$ loops, and two oppositely signed loops,*

then $D(\mathcal{A})$ has a negative 2-cycle.

One sufficient condition for $q_3 = 1$

Lemma

Suppose $\mathcal{A}$ is a 4×4 bipartite sign pattern such that $c(\mathcal{A}) = 2$ or $c(\mathcal{A}) = 4$ and $\mathcal{A}$ allows singularity. Then $q_3(\mathcal{A}) = 1$.

Strong method for allow sequence

Theorem

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in \mathcal{A}$ has the nonsymmetric strong spectral property. If A has a real eigenvalue with algebraic multiplicity at least two, then

- $q_k(\hat{\mathcal{A}}) = 1$ *for each* k *with* $q(A) \leq k \leq n$ *otherwise*
- $q_k(\hat{\mathcal{A}}) = 1$ *for* $q(A) \leq k \leq n$ *with* $k - q(A)$ *even.*

Corollary

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ has the nonsymmetric strong spectral property.

(i) *If* $q(A) = 1$, *then* $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, \ldots, 1 \rangle$.

(ii) *If* $q(A) = 2$, *then* $q_k(\hat{\mathcal{A}}) = 1$ *for all even* k.

Strong example

$$\mathcal{C} = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & + \\ - & - & \cdots & - & - \end{bmatrix}$$

$\mathcal{C}$ realizes characteristic polynomial $(z+1)^n$ with the strong spectral property

Thus $q_{seq}(\hat{\mathcal{C}}) = \langle 1, 1, \ldots, 1 \rangle$ for every superpattern $\hat{\mathcal{C}}$.
Note $\mathcal{C}$ is not potentially nilpotent nor spectrally arbitrary.

Patterns with full allow sequence

Theorem

If $\mathcal{A}$ is a full sign pattern (no zero entries) and $q(\mathcal{A}) = 1$, then $q_{\mathrm{seq}}(\mathcal{A}) = \langle 1, 1, \ldots, 1 \rangle$.

Theorem

If $q_{\mathrm{seq}}(\mathcal{A}) = \langle 1, 1, \ldots, 1 \rangle$ then $D(\mathcal{A})$ has a 2-cycle or a loop.

Full sequence example

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ - & 0 & + & + \\ 0 & + & 0 & + \end{bmatrix}, \qquad A = \begin{bmatrix} \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ -1 & 0 & \boxed{a} & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

with $a > 0$.

$$p_A(z) = (z-1)^3(z-a)$$

$a = 2$: A allows a Jacobian with rank $4 \Rightarrow q_k(\mathcal{A}) = 1$ for $2 \leq k \leq 4$.

$a = 1$: $q(A) = 1$ (A does not allow a Jacobian with rank 4 in this case).

$$q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 1 \rangle$$

For superpatterns $\hat{\mathcal{A}}$,

$$q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 0, 1, 1, 1 \rangle \quad \text{or} \quad q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, 1, 1 \rangle$$

Deficient rank Jacobian

Theorem

Let $\mathcal{A}$ be an $n \times n$ loopless sign pattern and $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ allows a Jacobian with rank $n-1$. If A has a real eigenvalue with algebraic multiplicity at least two, then $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$, otherwise, $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$ with $k - q(A)$ even.

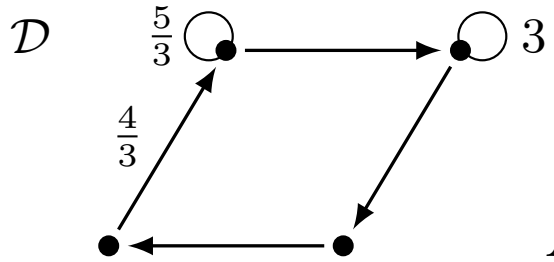
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$q(A) = 2 \quad \Rightarrow \quad q_2(\mathcal{A}) = 1$

A allows a rank 3 Jacobian $\quad \Rightarrow \quad$ for each superpattern $\hat{\mathcal{A}}$,

$$q_{seq}(\hat{\mathcal{A}}) = \langle 0, 1, 1, 1 \rangle \text{ or } q_{seq}(\hat{\mathcal{A}}) = \langle 1, 1, 1, 1 \rangle$$

A derivative-Jacobian method: repeated roots



$$A = \begin{bmatrix} \frac{5}{3} & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_2 & 0 & 0 & 0 \end{bmatrix}$$

A has eigenvalues $1, 1, \frac{4}{3} \pm \frac{2\sqrt{7}}{3}$: $\quad c_A(1) = 0$ and $c'_A(1) = 0$.

$$c_X(z) = z^4 - (x_1 + 3)z^3 + (3x_1)z^2 - x_2$$

$$c_X(1) = -2 + 2x_1 - x_2$$

$$c'_X(1) = -5 + 3x_1$$

$$J = \left.\frac{\partial(c_X(1), c'_X(1))}{\partial(x_1, x_2)}\right|_{(x_1,x_2)=(\frac{5}{3},\frac{4}{3})} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$$

J has rank 2 $\Rightarrow$ every digraph having $\mathcal{D}$ as a subdigraph allows an eigenvalue 1 with multiplicity 2.

a derivative-Jacobian method: order 2 version

Theorem

Let $\mathcal{D}$ be digraph on n vertices. Suppose $\mathcal{D}$ has a subdigraph G with at least two arcs and $\lambda \in \mathbb{R}$ is an eigenvalue of A having multiplicity two. Suppose X is a matrix obtained by replacing two nonzero arc weights a and b of A with real variables x_1 and x_2 and the characteristic polynomial of X is $c(z)$. If the 2×2 Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial c(\lambda)}{\partial x_1} & \frac{\partial c(\lambda)}{\partial x_2} \\ \frac{\partial c'(\lambda)}{\partial x_1} & \frac{\partial c'(\lambda)}{\partial x_2} \end{bmatrix}$$

has full rank when evaluated at $(x_1, x_2) = (a, b)$, then there exists $B \in \mathcal{A}(\mathcal{D})$ having λ as an eigenvalue with multiplicity two.

A cycle application

Theorem

Suppose $\mathcal{D}$ is a digraph on n vertices which has a subdigraph having exactly two cycles.
If the two cycles have distinct lengths and intersect in at least one vertex, then every superpattern of $\mathcal{D}$ allows a nonzero real eigenvalue of multiplicity two.

Proof.

$$\begin{aligned} c(z) &= z^n + c_1 z^{n-\ell_1} + c_2 z^{n-\ell_2} \\ c(1) &= 1 + c_1 + c_2 \\ c'(1) &= n + (n-\ell_1)c_1 + (n-\ell_2)c_2 = c(1) - \ell_1 c_1 - \ell_2 c_2 \\ J &= \begin{bmatrix} 1 & 1 \\ n-\ell_1 & n-\ell_2 \end{bmatrix} \end{aligned}$$

There is matrix realization with $c_1 = \frac{\ell_2}{\ell_2 - \ell_1}, c_2 = \frac{-\ell_1}{\ell_2 - \ell_1}$



A cycle application

Theorem

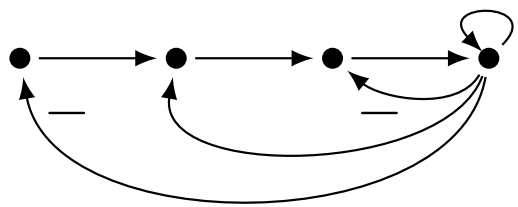
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Intercyclic digraph with distinct cycle orders

Theorem

Let $m \geq 2$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq n$. Suppose $\mathcal{A}$ is an $n \times n$ pattern with $D(\mathcal{A})$ intercyclic and composed of exactly m cycles $C_1, C_2, \ldots, C_m$ and such that for $1 \leq r \leq m$, cycle C_r has sign $(-1)^{r+1}$, length ℓ_r and an arc that does not belong to any other cycle. Then every superpattern of $\mathcal{A}$ has a realization having 1 as an eigenvalue with multiplicity equal to m.

$$\begin{bmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ - & + & - & + \end{bmatrix}$$

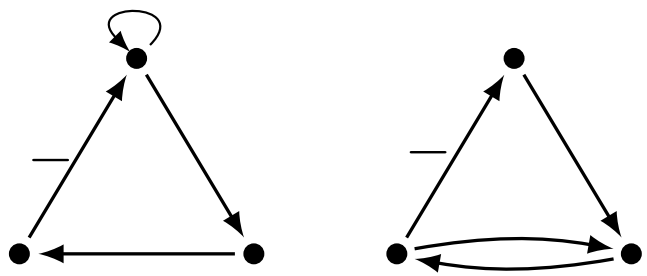


Subdigraphs with a high eigenvalue multiplicity

Lemma

Let $\mathcal{A}$ be an $n \times n$ sign pattern, $\eta \subset [n]$ and $\mathcal{B}$ a subpattern of $\mathcal{A}_\eta$. If there exists $B \in \mathcal{B}$ having $\lambda \in \mathbb{R}$ as an eigenvalue with multiplicity equal to $m \geq 1$ and having full-rank derivative-Jacobian, then there exists $A \in \mathcal{A}$ having λ as an eigenvalue with multiplicity equal to m and whose remaining eigenvalues are sufficiently close to those of $B \oplus O_{n-|\eta|}$.

Examples:



$D(\mathcal{H}_1)$ $D(\mathcal{H}_2)$

Let $\mathcal{A}$ be an $n \times n$ sign pattern and suppose $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_1)$ or $D(\mathcal{H}_2)$. If $n = 4$ or $c(\mathcal{A}) = 3$, then $q_3(\mathcal{A}) = 1$.

Some existence notes: cyclic sequences

Theorem

For a positive integer m, let $s = \langle 0, \ldots, 0, 1 \rangle$ have length $m \geq 2$. It is possible to construct examples with allow sequences $\langle s, s, \ldots, s \rangle$, $\langle 0, s, s, \ldots, s \rangle$, and $\langle 1, s, s, \ldots, s \rangle$.

Open question: what other cyclic sequences are possible?

Some existence notes: sequences with trailing zeros

Theorem

Given $n \geq 3$, sequences $\langle s_1, s_2, 0, \ldots, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$ are not realizable by any $n \times n$ irreducible sign pattern.

Theorem

Let $n \geq 4$ and $s = \langle s_1, s_2, 1, 0, \ldots, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$. Then $\langle 1, 1, 1, 0, 0, \ldots, 0\rangle$ is the only sequence s not realizable by an $n \times n$ irreducible sign pattern. Furthermore, if $\mathcal{A}$ be an $n \times n$ irreducible sign pattern, then

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 1, 0, 0, \ldots, 0\rangle \iff D(\mathcal{A})$ *is a star digraph with no loops and both positive and negative 2-cycles.*

(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 1, 0, 0, \ldots, 0\rangle \iff D(\mathcal{A})$ *is a star digraph with at least one negative 2-cycle and exactly one loop that is located on the central vertex.*

(iii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 0, 0, \ldots, 0\rangle \iff D(\mathcal{A})$ *is a star digraph with either*

- (a) *no loops and all 2-cycles of the same sign, or*
- (b) *no negative 2-cycles and exactly one loop that is located on the central vertex.*

Sequences with $n-4$ trailing zeros

Theorem

Let $n \geq 4$ and $\mathcal{A}$ be an $n \times n$ star sign pattern with $c(\mathcal{A}) = 3$. Then $q_{\text{seq}}(\mathcal{A}) \in \{\langle 1,1,1,1,0,\ldots,0\rangle, \langle 0,1,1,1,0,\ldots,0\rangle, \langle 0,0,1,1,0,\ldots,0\rangle, \langle 0,0,0,1,0,\ldots,0\rangle\}$.

Theorem

Let $n \geq 5$ and $s = \langle s_1, s_2, s_3, 1, 0, 0, \ldots, 0\rangle$ have length n with $s_1, s_2, s_3 \in \{0,1\}$. Then there exists an $n \times n$ irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$ if and only if $s \neq \langle s_1, 1, 0, 1, 0, \ldots, 0\rangle$.

Characterizing possible small order allow sequences

Theorem

Let $\mathcal{A}$ be a 2×2 irreducible sign pattern. Then $q_{\text{seq}}(\mathcal{A}) \in \{\langle 0,1\rangle, \langle 1,1\rangle\}$

Theorem

Let $\mathcal{A}$ be a 3×3 irreducible sign pattern. Then

$$q_{\text{seq}}(\mathcal{A}) \in \{\langle 1,1,1\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle, \langle 0,0,1\rangle\}$$

Theorem

Let $s = \langle s_1, s_2, s_3, s_4\rangle$ with $s_i \in \{0,1\}$. There exists a 4×4 irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$ if and only if $s \notin \{\langle 0,0,0,0\rangle, \langle 1,0,0,0\rangle, \langle 0,1,0,0\rangle, \langle 1,1,0,0\rangle, \langle 1,1,1,0\rangle, \langle 1,1,0,1\rangle\}$.

Zero-nonzero patterns

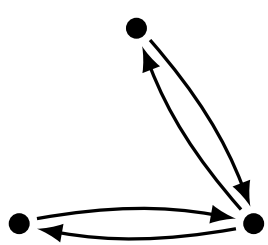
$$\mathcal{A} = \begin{bmatrix} + & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}$$

$q_{seq}(\mathcal{A}) = \langle 0, 0, 1, 0 \rangle \qquad q_{seq}(\mathcal{B}) = \langle 0, 1, 1, 0 \rangle$

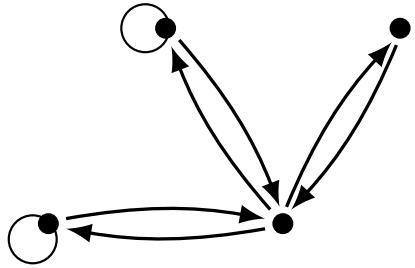
There is no zero-nonzero pattern with $q_{seq} = \langle 0, 0, 1, 0 \rangle$.

Open question: Is there an allow sequence that is realizeable by a zero-nonzero pattern but not a sign pattern?

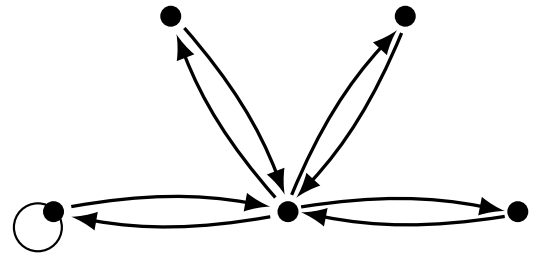
Characterizing the allow sequence of star digraphs



$\mathcal{S}_{3,0}$



$\mathcal{S}_{4,2}$



$\mathcal{S}_{5,1}$

Theorem

Let $n \geq 3$, $0 \leq \ell \leq n$ and $\mathcal{S}_{n,\ell}$ be star digraph on n vertices with ℓ loops. Then

$$q_{\text{seq}}(\mathcal{S}_{3,\ell}) = \begin{cases} \langle 1,0,1 \rangle & \text{if } \ell = 0 \\ \langle 0,1,1 \rangle & \text{if } \ell = 1 \text{ and a centre loop} \\ \langle 1,1,1 \rangle & \text{otherwise,} \end{cases}$$

and for $n \geq 4$,

$$q_{\text{seq}}(\mathcal{S}_{n,\ell}) = \begin{cases} \langle 1,0,1,0,0,\ldots 0 \rangle & \text{if } \ell = 0 \\ \langle 0,1,1,0,0,\ldots 0 \rangle & \text{if } \ell = 1 \text{ and a centre loop} \\ \langle 0,1,1,1,0,0,\ldots 0 \rangle & \text{if } \ell = 1 \text{ and no centre loop} \\ \langle \underbrace{1,\ldots,1}_{t+3}, \underbrace{0,0,\ldots,0}_{n-t-3} \rangle & \text{if } 2 \leq \ell \leq n-3\text{; and } t \text{ non-centre loops} \\ \langle 1,\ldots,1 \rangle & \text{if } \ell \geq n-2. \end{cases}$$

Unattainable allow sequences

Attainable allow sequences have been characterized for digraphs up to order 4.

Theorem

For $n \geq 4$, the following sequences can not be realized by a strongly connected digraph:

$$\langle 1, 0, 0, 0, \ldots, 0 \rangle$$

$$\langle 1, 1, 0, 0, \ldots, 0 \rangle$$

$$\langle 0, 1, 0, 0, \ldots, 0 \rangle$$

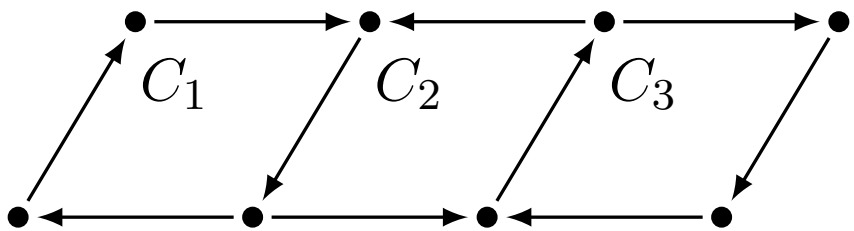
$$\langle 0, 0, 1, 0, \ldots, 0 \rangle$$

$$\langle 1, 1, 1, 0, \ldots, 0 \rangle$$

Another cycle application

Theorem

Suppose $\mathcal{D}$ is digraph on n vertices which contains a subdigraph with exactly three cycles C_1, C_2 and C_3. Suppose (1) these cycles have the same length ℓ, (2) C_1 and C_3 are disjoint, and (3) C_2 intersects both C_1 and C_3. Then every superpattern of $\mathcal{D}$ allows a nonzero real eigenvalue with multiplicity two.



Another cycle application: argument

Theorem

Suppose $\mathcal{D}$ is digraph on n vertices which contains a subdigraph with exactly three cycles C_1, C_2 and C_3. Suppose (1) these cycles have the same length ℓ, (2) C_1 and C_3 are disjoint, and (3) C_2 intersects both C_1 and C_3. Then every superpattern of $\mathcal{D}$ allows a nonzero real eigenvalue with multiplicity two.

Proof.

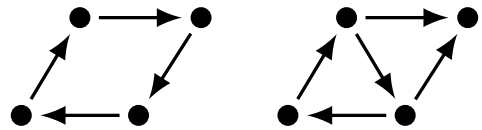
$c_X(z) = z^n - (x_1 + x_2 + x_3)z^{n-\ell} + x_1x_3z^{n-2\ell}$

Taking $x_1 = x_3 = -1$ and $x_2 = 4$ gives a realization A with $c_A(z) = z^{n-2\ell}(z^{2\ell} - 2z^\ell + 1)$ and hence 1 is a repeat eigenvalue

$$\begin{aligned} c_X(1) &= 1 - (x_1 + x_2 + x_3) + x_1x_3 \\ c'_X(1) &= n - (n-\ell)(x_1 + x_2 + x_3) + (n - 2\ell)x_1x_3. \end{aligned}$$

$$\begin{aligned} J &= \left.\frac{\partial(c_X(1), c'_X(1))}{\partial(x_1, x_2)}\right|_{(x_1,x_2,x_3)=(-1,4,-1)} \\ &= \begin{bmatrix} -2 & -1 \\ -2(n-\ell) - \ell & -(n-\ell) \end{bmatrix}, \qquad \det(J) = \ell \neq 0 \end{aligned}$$

q_{seq} with exactly one one



Theorem

Let $n \geq 2$ and $\mathcal{D}$ be a digraph on n vertices.

(i) *$\mathcal{D}$ requires n distinct eigenvalues $\iff q_{seq} = \langle 0, \ldots, 0, 1 \rangle$ $\iff$ either $\mathcal{D}$ is an n-cycle or $\mathcal{D}$ is unicyclic with the unique cycle being an $(n-1)$-cycle.*

(ii) *$\mathcal{D}$ requires exactly one eigenvalue $\iff q_{seq} = \langle 1, 0, \ldots, 0, 0 \rangle$ $\iff$ $\mathcal{D}$ has no cycles.*

(iii) *Suppose $1 < k < n$. $\mathcal{D}$ requires exactly k distinct eigenvalues $\iff$*

$$q_{seq} = \langle 0, \ldots, 0, k, 0, \ldots, 0 \rangle$$

$\iff$ $\mathcal{D}$ is unicyclic with the unique cycle being a $(k-1)$-cycle.

A cycle with loops...

Theorem

If $\mathcal{D}$ is a 2-cycle, then

$$q_{\mathrm{seq}}(\mathcal{D}) = \begin{cases} \langle 0,1 \rangle & \text{if } \mathcal{D} \text{ has no loops} \\ \langle 1,1 \rangle & \text{otherwise.} \end{cases}$$

If $\mathcal{D}$ is an 3-cycle, then

$$q_{\mathrm{seq}}(\mathcal{D}) = \begin{cases} \langle 0,0,1 \rangle & \text{if } \mathcal{D} \text{ has no loops} \\ \langle 0,1,1 \rangle & \text{otherwise.} \end{cases}$$

If $\mathcal{D}$ is a 4-cycle, then

$$q_{\mathrm{seq}}(\mathcal{D}) = \begin{cases} \langle 0,0,0,1 \rangle & \text{if } \mathcal{D} \text{ has no loops} \\ \langle 0,0,1,1 \rangle & \text{if } \mathcal{D} \text{ has 1 loop} \\ \langle 0,1,1,1 \rangle & \text{otherwise.} \end{cases}$$

Cycles on 4, 5 and 6 vertices

$n = 4$

Loops	Sequence
0	$\langle 0, 0, 0, 1 \rangle$
1	$\langle 0, 0, 1, 1 \rangle$
2	$\langle 0, 1, 1, 1 \rangle$
3	$\langle 0, 1, 1, 1 \rangle$
4	$\langle 0, 1, 1, 1 \rangle$

$n = 5$

Loops	Sequence
0	$\langle 0, 0, 0, 0, 1 \rangle$
1	$\langle 0, 0, 0, 1, 1 \rangle$
2	$\langle 0, 0, 0, 1, 1 \rangle$
3	$\langle 0, 0, 1, 1, 1 \rangle$
4	$\langle 0, 0, 1, 1, 1 \rangle$
5	$\langle 0, 0, 1, 1, 1 \rangle$

$n = 6$

Loops	Sequence
0	$\langle 0, 0, 0, 0, 0, 1 \rangle$
1	$\langle 0, 0, 0, 0, 1, 1 \rangle$
2	$\langle 0, 0, 0, 1, 1, 1 \rangle$
3	$\langle 0, 0, 0, 1, 1, 1 \rangle$
4	$\langle 0, 0, 1, 1, 1, 1 \rangle$
5	$\langle 0, 0, 1, 1, 1, 1 \rangle$
6	$\langle 0, 0, 1, 1, 1, 1 \rangle$

Cycle results

Theorem

If $\mathcal{D}$ is an n-cycle with any number of loops, and $n \geq 3$, then $q_1(\mathcal{D}) = 0$ and if $n \geq 5$ then $q_2(\mathcal{D}) = 0$.

Lemma

If $n \geq 2$ and $D(\mathcal{A})$ is an n-cycle with ℓ loops, then $q(\mathcal{A}) \geq n - \ell$.

Theorem

Let $n \geq 2$. If $\mathcal{D}$ is an n-cycle with no loops then $q_{\text{seq}}(\mathcal{D}) = \langle 0, \ldots, 0, 1 \rangle$ and $q_n(\mathcal{D}') = 1$ if $\mathcal{D}$ is a spanning subdigraph of $\mathcal{D}'$.

If $\mathcal{D}$ is a n-cycle with exactly one loop, then $q_{\text{seq}}(\mathcal{D}) = \langle 0, \ldots, 0, 1, 1 \rangle$ and $q_{n-1}(\mathcal{D}') = 1$ if $\mathcal{D}$ is a spanning subdigraph of $\mathcal{D}'$.

If $n \geq 4$ is even and $\mathcal{D}$ is an n-cycle with exactly two loops, then $q_{\text{seq}}(\mathcal{D}) = \langle 0, \ldots, 0, 1, 1, 1 \rangle$ and $q_{n-2}(\mathcal{D}') = 1$ if $\mathcal{D}$ is a spanning subdigraph of $\mathcal{D}'$.

Cycle with 2 loops: the strong multiplicity property

$$A = \begin{bmatrix} w & 1 & 0 & \dots & 0 \\ 0 & -w & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 \\ b & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Suppose $X \circ A = 0$ and $AX^T = X^T A$ and $\mathrm{tr}(X^T A^k) = 0$ for $0 \leq k \leq n-1$. For $3 \leq i < n$, $(AX^T)_{i,i+1} = X_{i+1,i+1}$ and $(X^T A)_{i,i+1} = X_{i,i}$.
Thus $X_{3,3} = X_{4,4} = \cdots = X_{n,n}$ with $X_{1,1} = X_{2,2} = 0$ and $\mathrm{tr}(X^T) = 0 \Longrightarrow X_{3,3} = X_{4,4} = \cdots = X_{n,n} = 0$.

A is nonderogatory so $AX^T = X^T A$ implies $X^T = \sum_{i=0}^{n-1} c_i A^i$.

Considering the (n,n) entry of X^T implies $c_0 = 0$. Considering the $(1,n)$ entry of X^T implies $c_{n-1} = 0$.

Thus $X^T = \sum_{i=1}^{n-2} c_i A^i$

n-cycle with weight b having 2 loops weighted w and $-w$

$$\mathrm{tr}(A^i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2w^i & \text{if } i \text{ is even and } i \leq n-2 \\ 2w^i + bn & \text{if } i \text{ is even and } i = n-1 \end{cases}$$

Hence it follows that $\mathrm{tr}(X^T A^2) = w^2 \mathrm{tr}(X^T) + c_{n-2} bn$ hence

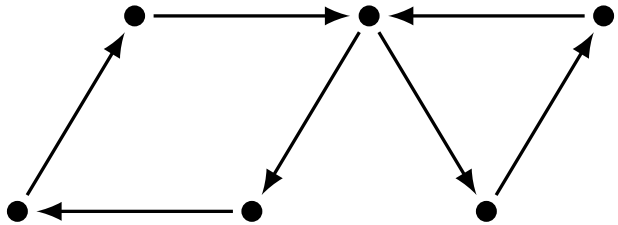
$$\mathrm{tr}(X^T) = \mathrm{tr}(X^T A^2) = 0 \Longrightarrow c_{n-2} = 0.$$

Iterating for $1 \leq t \leq n-1$,

$$\mathrm{tr}(X^T A^t) = w^2 \mathrm{tr}(X^T A^{t-2}) + c_{n-t} bn \Longrightarrow c_{n-t} = 0$$

Therefore $X = 0$ and so A has the nSMP.

nSMP vs the derivative Jacobian method



$$A = \begin{bmatrix} 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & -3 & 0 \end{bmatrix}$$

A has a repeat eigenvalue of 1. Derivative Jacobian method implies every superpattern allows a repeat eigenvalue of 1.

but A does not have nSMP since $X \circ A = O$ and $X^T A = O = A X^T$ but $X \neq O$.

Further work

- Find the minimun number of arcs needed in $\mathcal{D}$ such that $q_{seq}(\mathcal{D}) = \langle 1, \ldots, 1 \rangle$.
- Characterize the digraphs with $q_{seq}(\mathcal{D}) = \langle 1, \ldots, 1 \rangle$.
- Find the proportion of acheivable allow sequences for a digraph on $n \geq 5$ vertices.
- Characterize the allow sequences for all n-cycles with k loops, $2 \leq k \leq n-1$.
- Characterize the allow sequences for doubly directed path digraphs.
- Characterize the acheivable allow sequences for $n \geq 5$.

Results in this lecture are taken from the following:

- Breen, J., C. Brouwer, M. Catral, M. Cavers, P. van den Driessche, and K.N. Vander Meulen. "Minimum number of distinct eigenvalues allowed by a sign pattern." Linear Algebra and its Applications 654 (2022): 311-338.
- Breen, Jane, Carraugh Brouwer, Minerva Catral, Michael Cavers, Pauline van den Driessche, and K.N. Vander Meulen. "The allow sequence of distinct eigenvalues for a sign pattern." The Electronic Journal of Linear Algebra 40 (2024): 48-80.
- Cavers, Michael, Casey Charette, Ezekiel Colvin, and K.N. Vander Meulen. "Combinatorial considerations for the number of distinct eigenvalues of a matrix." Electronic Journal of Linear Algebra 41 (2025): 669-693.
- Cavers, M., and K.N. Vander Meulen. "Nonzero matrix patterns that require distinct eigenvalues." Linear Algebra and its Applications (2026).
- Eschenbach, Carolyn A. "Eigenvalue classification in qualitative matrix analysis." PhD diss., Clemson University, 1987.
- Eschenbach, C.A., and C.R. Johnson. "Sign patterns that require repeated eigenvalues." Linear algebra and its applications 190 (1993): 169-179.
- Eschenbach, Carolyn A., Frank J. Hall, and Zhongshan Li. "Eigenvalue frequency and consistent sign pattern matrices." Czechoslovak Mathematical Journal 44, no. 3 (1994): 461-479.
- Li, Zhongshan, and Laura Harris. "Sign patterns that require all distinct eigenvalues." JP J. Algebra Number Theory Appl 2, no. 2 (2002): 161-179.