Generic Eigenvectors

Remember $\mathscr{G}$

A matrix A with the SSP allows us to find nearby matrices B with the same eigenvalues. In particular, B can achieve any super-pattern of A.

Can we say anything about the eigenvectors of B?

Motivation: Building new matrices using eigenvectors

Theorem (Fiedler (1974))

Let

- A *be a symmetric* $m \times m$ *matrix with*
  - *eigenvalues* $\alpha_1, \ldots, \alpha_m$, *and*
  - *a unit eigenvector* u *corresponding to the eigenvalue* α_1,
- B *be a symmetric* $n \times n$ *matrix with*
  - *eigenvalues* $\beta_1, \ldots, \beta_n$, *and*
  - *a unit eigenvector* v *corresponding to the eigenvalue* β_1.

Then for any $t \in \mathbb{R}$, *the matrix*

$$C = \begin{pmatrix} A & tuv^T \\ tvu^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_m$, $\beta_2, \ldots, \beta_n$, γ_1, γ_2, *where* γ_1, γ_2 *are eigenvalues of the matrix* $\hat{C} = \begin{pmatrix} \alpha_1 & t \\ t & \beta_1 \end{pmatrix}$

Proof

Let U and V be orthogonal matrices with

- $U^\top A U = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m) =: D_A$,
- $V^\top B V = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_n) =: D_B$

$$(U \oplus V)^\top \begin{pmatrix} A & tuv^T \\ tvu^T & B \end{pmatrix} (U \oplus V) = \begin{pmatrix} D_A & tU^\top uv^\top V \\ tV^\top vu^\top U & D_B \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc} \alpha_1 & & & t & & \\ & \ddots & & & & \\ & & \alpha_m & & & \\ \hline t & & & \beta_1 & & \\ & & & & \ddots & \\ & & & & & \beta_n \end{array}\right)$$

$$\sim \begin{pmatrix} \alpha_1 & t \\ t & \beta_1 \end{pmatrix} \oplus \operatorname{diag}(\alpha_2, \ldots, \alpha_m) \oplus \operatorname{diag}(\beta_2, \ldots, \beta_n)$$

Building new matrices using eigenvectors

Let U and V be orthogonal matrices with

- $U^\top A U = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m) =: D_A$,
- $V^\top B V = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_n) =: D_B$

For $S \in \mathbb{R}^{m \times n}$ we define:

$$M(S) := \begin{pmatrix} A & USV^\top \\ VS^\top U^\top & B \end{pmatrix} = (U \oplus V) \begin{pmatrix} D_A & S \\ S^\top & D_B \end{pmatrix} (U \oplus V)^\top$$

Observations

- Taking $S = tE_{11}$ recovers Fiedler's result.
- Choose $S = \sum_{j=1}^{k} t_j E_{jj}$.
  1. Let U_1 be the matrix consisting of the first k columns of U, and V_1 the matrix consisting of the first k columns of V. Note, the off-diagonal blocks of $M(S)$ depend only on U_1 and V_1.
  2. The eigenvalues of $M(S)$ can be explicitly determined.

We will return to this construction, after we find some results on the eigenvectors.

Nowhere-zero Eigenbasis

Theorem (Monfared, Shader (2016))

For any connected graph G on n vertices and any distinct real numbers $\lambda_1, \ldots, \lambda_n$, there exists $A \in S(G)$ with spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and an eigenbasis consisting of nowhere-zero vectors.

Generically realizable

- $\sigma = \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{R}^n$ be a multiset of real numbers,
- $\Lambda \in \mathbb{R}^{n \times n}$ be a diagonal matrix with diagonal elements λ_i (in some order).
- G connected graph

σ is generically realizable for G if for any finite set $\mathcal{Y} \subseteq \mathbb{R}^n \setminus \{\mathbf{0}\}$ there is an orthogonal matrix U so that

- $U\Lambda U^\top \in S(G)$,
- $U\mathbf{v}$ is nowhere-zero for all $\mathbf{v} \in \mathcal{Y}$.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$ has the SSP.

For every $\epsilon > 0$ there exists $t > 0$ so that

$$A = \begin{pmatrix} 1+\epsilon_1 & t & 0 & 0 \\ t & 2+\epsilon_2 & t & 0 \\ 0 & t & 3+\epsilon_3 & t \\ 0 & 0 & t & 4+\epsilon_4 \end{pmatrix} \in S(P_4)$$

has eigenvalues $\{1,2,3,4\}$ and $\|A - D\| < \epsilon$.

For every $\epsilon > 0$ there exists $t > 0$ so that

$$A = \begin{pmatrix} 1+\epsilon_1 & t^{\alpha_1} & 0 & 0 \\ t^{\alpha_1} & 2+\epsilon_2 & t^{\alpha_2} & 0 \\ 0 & t^{\alpha_2} & 3+\epsilon_3 & t^{\alpha_3} \\ 0 & 0 & t^{\alpha_3} & 4+\epsilon_4 \end{pmatrix} \in S(P_4)$$

has eigenvalues $\{1, 2, 3, 4\}$ and $\|A - D\| < \epsilon$. And we are free to choose $\alpha_i \in \mathcal{N}$.

A race to zero

$$A_n = \begin{pmatrix} 1+\epsilon_1 & t_n^{\alpha_1} & 0 & 0 \\ t_n^{\alpha_1} & 2+\epsilon_2 & t_n^{\alpha_2} & 0 \\ 0 & t_n^{\alpha_2} & 3+\epsilon_3 & t_n^{\alpha_3} \\ 0 & 0 & t_n^{\alpha_3} & 4+\epsilon_4 \end{pmatrix} \in S(P_4)$$

$U_n^\top A_n U_n = D.$

$n \to \infty$:

- $t_n \to 0$
- $A_n \to D$,
- $U_n \to I_n$,
- $\frac{1}{t^{s(i,j)}}(U_n)_{ij} \to c(i,j)$

The Supergraph Lemma (Revisited)

Let $A \in \mathcal{S}(G)$ be a matrix with the SSP, and H a subgraph of G on the same vertex set. Given $t \in \mathbb{R}$ and a function

$$f : E(H) \setminus E(G) \to \mathbb{N}$$

consider the family of matrices:

$$\mathcal{M}_f(t) = \left\{ M \in \mathcal{S}(H) : M_{i,j} = t^{f(i,j)} \text{ for } \{i,j\} \in E(H) \setminus E(G) \right\}.$$

For every $\epsilon > 0$, there exists $t_0 > 0$ and a matrix $M \in \mathcal{M}_f(t_0)$ with $\|A - M\| < \epsilon$, with the SSP, and the same spectrum as A.

Proof (Here we go again)

The SSP for A guarantees that

$$\mathfrak{f}\colon \mathcal{O}_1 \times \mathcal{O}_2 \to \mathrm{Sym}(n)$$
$$\mathfrak{f}(K,B) = e^{-K} A e^{K} + B$$

is locally surjective at $(0,0)$.
In particular, for any $N \in \mathcal{M}_f(t)$ with $\|N\|$ sufficiently small, there exist $K \in \mathrm{Skew}(n)$ and $B \in \langle \mathcal{S}(G) \rangle$ so that $\mathfrak{f}(K,B) = N$.
Observe: $N - K \in \mathcal{M}_f(t)$ and $N - K$ is similar to A.

Where are the eigenvectors?

Answer: Where we expect them to be.

- Let Λ be a diagonal matrix with diagonal entries:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_m,$$

- Let A_n be a sequence of symmetric matrices so that

$$U_n^\top A_n U_n = \Lambda$$

for an orthogonal matrix U_n with non-negative diagonal entries.

Then:

If $A_n \to \Lambda$ as $n \to \infty$, then $U_n \to I_n$ as $n \to \infty$.

Proof

Let $k \in \{1, 2, \ldots, m\}$ and p_k be a polynomial with $p_k(\lambda_\ell) = \delta_{k,\ell}$. Then $p_k(A_n)$ is the orthogonal projection onto the λ_k-eigenspace of A_n, and $p_k(\Lambda) = e_k e_k^\top$.

$$p_k(A_n) = p_k(U_n \Lambda U_n^\top) = U_n p_k(\Lambda) U_n^\top = (U_n e_k)(U_n e_k)^\top$$
$$p_k(A_n)^\top \to p_k(\Lambda).$$

In particular, $(U_n e_k)_k (U_n e_k)_\ell = p_k(A_n)_{k,\ell} \to p_k(\Lambda)_{k,\ell} = \delta_{k,\ell}$. Since $(U_n e_k)_k \geq 0$, this implies that $(U_n e_k)_k \to 1$, which implies in turn that $(U_n e_k)_\ell \to 0$ if $\ell \neq k$. Hence, $U_n e_k \to e_k$, proving $U_n \to I_n$.

Example

Let λ_i, $i = 1, 2, 3$, be three distinct numbers, and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Choose $\alpha, \beta, N \in \mathbb{N}$, with $N > 2\max\{\alpha, \beta\}$. Λ has the SSP, so the Supergraph Lemma gives us a convergent sequence of matrices

$$M_n = \begin{pmatrix} a_n(1,1) & t_n^{\alpha+N} & 0 \\ t_n^{\alpha+N} & a_n(2,2) & t_n^{\beta+N} \\ 0 & t_n^{\beta+N} & a_n(3,3) \end{pmatrix}$$

with the same eigenvalues as Λ, and a convergent sequence of orthogonal matrices $U_n = (u_n(i,j))_{i,j=1}^n$ so that $U_n^\top M_n U_n = \Lambda$, and $a_n(i,i) \to \lambda_i$, $U_n \to I_3$ as $n \to \infty$. We can already assert that the diagonal entries of U_n are positive, but the zero-non zero pattern of off-diagonal entries is a mystery.

Rate of decay of off-diagonal elements

$$\begin{pmatrix} a_n(1,1)-\lambda_1 & t_n^{\alpha+N} & 0 \\ t_n^{\alpha+N} & a_n(2,2)-\lambda_1 & t_n^{\beta+N} \\ 0 & t_n^{\beta+N} & a_n(3,3)-\lambda_1 \end{pmatrix} \begin{pmatrix} u_n(1,1) \\ u_n(2,1) \\ u_n(3,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

First we improve on $u_n(2,1) \to 0$, $u_n(3,1) \to 0$.

$$t_n^{\alpha} u_n(1,1) + (a_n(2,2)-\lambda_1)\frac{u_n(2,1)}{t_n^N} + t_n^{\beta} u_n(3,1) = 0 \Rightarrow \frac{u_n(2,1)}{t^N} \to 0$$

$$t_n^{\beta}\frac{u_n(2,1)}{t^N} + (a_n(3,3)-\lambda_1)\frac{u_n(3,1)}{t_n^{2N}} = 0 \Rightarrow \frac{u_n(3,1)}{t_n^{2N}} \to 0$$

Fine-tuning.

$$u_n(1,1) + (a_n(2,2)-\lambda_1)\frac{u_n(2,1)}{t_n^{\alpha+N}} + t_n^{\beta}\frac{u_n(3,1)}{t_n^{\alpha+N}} = 0 \Rightarrow \frac{u_n(2,1)}{t_n^{\alpha+N}} \to \frac{1}{\lambda_1-\lambda_2}$$

$$\frac{u_n(2,1)}{t_n^{\alpha+N}} + (a_n(3,3)-\lambda_1)\frac{u_n(3,1)}{t_n^{\alpha+\beta+2N}} = 0 \Rightarrow \frac{u_n(3,1)}{t_n^{\alpha+\beta+2N}} \to \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

Rate of decay of off-diagonal elements

Following the same idea we can determine the first nonzero term in the polynomial approximation of $u_n(i,j)$. In our example we would get:

$$u_n(1,1) \to 1 \qquad \frac{u_n(1,2)}{t_n^{\alpha+N}} \to \frac{1}{\lambda_2-\lambda_1} \qquad \frac{u_n(1,3)}{t_n^{\alpha+\beta+2N}} \to \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$\frac{u_n(2,1)}{t_n^{\alpha+N}} \to \frac{1}{\lambda_1-\lambda_2} \qquad u_n(2,2) \to 1 \qquad \frac{u_n(2,3)}{t_n^{\beta+N}} \to \frac{1}{\lambda_3-\lambda_2}$$

$$\frac{u_n(3,1)}{t_n^{\alpha+\beta+2N}} \to \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \qquad \frac{u_n(3,2)}{t_n^{\beta+N}} \to \frac{1}{\lambda_2-\lambda_3} \qquad u_n(3,3) \to 1$$

We are still free to choose α, β.

From above we can conclude that U_n have no entries equal to zero, and recover the result of Monfared and Shader for P_3.

Builidng on the ideas we have seen in the example, a more general result can be found.

Theorem (Levene, Oblak, Š. (2024))

Choose:

- *any* *connected graph* G *of order* m*,*
- *any* *distinct real numbers* $\lambda_1, \ldots, \lambda_m$*,*
- *any* *finite set* $\mathcal{Y} \subset \mathbb{R}^m \setminus \{\mathbf{0}\}$*.*

There exists a matrix $A \in \mathcal{S}(G)$ *with the SSP and an orthogonal matrix* U *so that* $U^\top A U = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ *and* $U\mathbf{v}$ *is nowhere-zero for all* $\mathbf{v} \in \mathcal{Y}$*.*

Returning to Example

Let $\mathcal{Y} \subset \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be a finite set, and U_n as in the example above.

We claim there exists an infinite subsequence of U'_n of U_n with $U'_n\mathbf{v}$ nowhere-zero for all $\mathbf{v} \in \mathcal{Y}$.
Assume the statement is false. Passing to a subsequence U'_n:

There exist $\mathbf{v} \in \mathcal{Y}$ and $k \in \{1, 2, 3\}$ so that $(U_n\mathbf{v})_k = 0$ for all n.

Let's check back with the information on U_n we found before.

Example continued

There exist $\mathbf{v} \in \mathcal{Y}$ and $k \in \{1, 2, 3\}$ so that $(U_n \mathbf{v})_k = 0$ for all n.

Choosing $\alpha \neq \beta$, in a fixed row of U_n every element has different rate of decay. This allows us to show $\mathbf{v} = 0$.
$k = 1$:

$$u_n(1,1) \to 1 \qquad \frac{u_n(1,2)}{t_n^{\alpha+N}} \to \frac{1}{\lambda_2 - \lambda_1} \qquad \frac{u_n(1,3)}{t_n^{\alpha+\beta+2N}} \to \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$u_n(1,1)\mathbf{v}_1 + u_n(1,2)\mathbf{v}_2 + u_n(1,3)\mathbf{v}_3 = 0$$

- $u_n(1,1) \to 1$, $u_n(1,2) \to 0$, $u_n(1,3) \to 0$ imply $\mathbf{v}_1 = 0$,
- from $\frac{u_n(1,2)}{t_n^{\alpha+N}}\mathbf{v}_2 + \frac{u_n(1,3)}{t_n^{\alpha+N}}\mathbf{v}_3 = 0$ we now get $\mathbf{v}_2 = 0$,
- and from here $\mathbf{v}_3 = 0$.

Similar arguments work for $k = 2, 3$.

Generically realizable $\mathscr{G}$

- Distinct eigenvalues are generically realizable for any connected graph G.
- Any multiset of n real numbers with at least two distinct eigenvalues is generically realizable for K_n.

- Establishing generic realizability can be challenging, but when we have it, it gives us a useful tool for matrix construction techniques.
- We will look at one application; construction of orthogonal symmetric matrices.

Orthogonal symmetric matrices

$\lambda_i \in (-1,1)$, $i = 1,\ldots,n$, $\lambda_i \neq \lambda_j$:

$$M_1 = \begin{pmatrix} \lambda_1 & \sqrt{1-\lambda_1^2} \\ \sqrt{1-\lambda_1^2} & -\lambda_1 \end{pmatrix}$$

$$M_2 = \oplus_{i=1}^{n} \begin{pmatrix} \lambda_i & \sqrt{1-\lambda_i^2} \\ \sqrt{1-\lambda_i^2} & -\lambda_i \end{pmatrix}$$

$$M_3 = P M_2 P^\top = \begin{pmatrix} \Lambda & S \\ S & -\Lambda \end{pmatrix}$$

$\Lambda := \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, $S := \mathrm{diag}(\sqrt{1-\lambda_1^2},\ldots,\sqrt{1-\lambda_n^2})$

Note, Λ has distinct diagonal elements. Can you guess what we do next?

Joins of graphs

Let G and H be connected graphs, $|G| = n$, $|H| \in \{n, n+1, n+2\}$.

- $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $S := \mathrm{diag}(\sqrt{1-\lambda_1^2}, \ldots, \sqrt{1-\lambda_n^2})$
- $|G| = n$, U orthogonal, $A = U\Lambda U^\top \in S(G)$.
- $|H| = n+2$, V orthogonal , $B = V(-\Lambda \oplus 1 \oplus -1)V^\top \in S(H)$

$$(U \oplus V) \begin{pmatrix} D & S & 0 & 0 \\ S & -D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (U^\top \oplus V^\top) = \begin{pmatrix} A & X \\ X^\top & B \end{pmatrix}$$

U and V can be chosen so that $X = U \begin{pmatrix} S & 0 & 0 \end{pmatrix} V^T$ has no zero elements. $\mathscr{G}$

$q(G \vee H) = 2$

- A necessary condition. If $q(G \vee H) = 2$, then G and H have compatible multiplicity matrices.
- If G and H have generically realisable compatible multiplicity matrices, then $q(G \vee H) = 2$.

Changing the rules of the game.

- If G and H have compatible 0–1 matrices, then $q(G \vee H) = 2$.
- If G and H are unions of paths, then $q(G \vee H) = 2 \Leftrightarrow G$ and H have compatible 0–1 matrices.

Joins of connected graphs

Theorem (Monfared, Shader (2016))

Let G and H be connected graphs satisfying $\big|\,|G| = |H|\,\big|$. Then $q(G \vee H) = 2$.

Theorem (Levene, Oblak, Š. (2024))

- *Let G and H be connected graphs satisfying $\big|\,|G| - |H|\,\big| \leq 2$. Then $q(G \vee H) = 2$.*
- $q(T_1 \vee T_2) = 2 \iff \big|\,|T_1| - |T_2|\,\big| \leq 2$ *for any trees T_1, T_2.*

Playing with the new rules

- Let $G = G_1 \cup G_2 \cup G_3$, with G_i connected and $|G_i| = 3$,
- Let $H = K_2 \cup 4K_1$

Then $q(G \vee H) = 2$.

Proof.

$$\tilde{V} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tilde{W} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

We can add 0, 1 or 2 to any of the sizes $(3, 3, 3)$ for G and any of the sizes $(2, 1, 1, 1, 1)$ for H and would still have $q(G \vee H) = 2$.

New game is harder to solve

Theorem (Levene, Oblak, Š., 2025)

Let $k \leq \ell$, $\psi = \lceil \ell/k \rceil$ and $(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^\ell$. The following statements satisfy the implications
(i) $\implies$ (ii) $\iff$ (iii) $\implies$ (iv):

(i) $|\mathbf{n}| - n_2^* - n_3^* \leq |\mathbf{m}| \leq |\mathbf{n}| + m_{\psi+1}^* + m_{\psi+2}^*$, $\mathbf{m} \geq (\psi^k)$ *and*

$$\sum_{j=1}^{\psi} n_j^* \geq k\psi + \max\{0, |\mathbf{n}| - |\mathbf{m}| - n_{\psi+1}^* - n_{\psi+2}^*\}.$$

(ii) $q(G \vee H) = 2$ *for all graphs G, H which are sized by $(\mathbf{m}, \mathbf{n})$.*

(iii) $q(G \vee H) = 2$ *for $G = \bigcup_{i=1}^{k} P_{m_i}$ and $H = \bigcup_{j=1}^{\ell} P_{n_j}$.*

(iv) $|\mathbf{n}| - n_2^* - n_3^* \leq |\mathbf{m}| \leq |\mathbf{n}| + m_{\psi+1}^* + m_{\psi+2}^*$, $\mathbf{m} \geq (\psi^k)$, *and if $k < \ell$, then we also have*

$$\iota(\mathbf{n}) \leq \max\left\{(\psi - 1)k, \frac{\psi}{\psi - 1}(\ell - k)\right\}.$$

Exploiting distinct eigenvalues in a submatrix

- $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in S(C_{2m})$
  - Eigenvalues of M: $\{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_m^{(2)}\}$,
  - $D_1 := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$
- $C \in S(P_{2m-2})$
  - C has distinct eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_{2m-2}\}$,
  - $D := \mathrm{diag}(\mu_1, \ldots, \mu_{2m-2})$,
  - $U^\top C U = D$.
- H connected, $|H| = 3m - 2$, V orthogonal $\mathscr{G}$, $N = V(D \oplus D_1)V^\top \in S(H)$.

Same toy, new game

$$(I_2 \oplus U^\top) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} (I_2 \oplus U) = \begin{pmatrix} A & BU \\ U^\top B^\top & D \end{pmatrix}$$

$$(I_2 \oplus V) \begin{pmatrix} A & BU & 0 \\ U^\top B^\top & D & 0 \\ 0 & 0 & D_1 \end{pmatrix} (I_2 \oplus V^\top) = \begin{pmatrix} A & X \\ X^\top & N \end{pmatrix}$$

Eigenvalues: $\{\lambda_1^{(3)}, \lambda_2^{(3)}, \ldots, \lambda_m^{(3)}\}$

Generic realisability (applied to H): $X = \begin{pmatrix} BU & 0 \end{pmatrix} V^\top$ has no zero elements.

Theorem

Let H be any connected graph with $3m-2$ vertices, $m \geq 2$. The multiplicity list $\{3^{(m)}\}$ is spectrally arbitrary for $K_2 \vee H$ and, in particular, $q(K_2 \vee H) \leq m$.

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