The Strong Singular Value Property

The SVD constitutes one of science's superheroes in the fight against monstrous data, and it arises in seemingly every scientific discipline.

Carla Martin and Mason Porter

At the most fundamental level, the singular value decomposition is the following result:

> *Let L be a linear transformation from a finite dimensional inner product space U to a inner product space V. Then there exists an orthogonal basis of U whose image under L is a set of mutually orthogonal vectors in V.*

More precisely:

Theorem (SVD–outer product)

Let A be an $m \times n$ real matrix of rank r. Then there exist orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ in $\mathbb{R}^m$, orthonormal vectors $\mathbf{v}_1, \ldots \mathbf{v}_r$ in $\mathbb{R}^n$ and positive scalars $\sigma_1 \geq \cdots \geq \sigma_r > 0$ such that $A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i^\top \mathbf{v}_i^\top$.

Theorem (SVD-matrix factorization)

For $m \times n$ real matrix A with $m \leq n$, there exist orthogonal matrices U and V and real numbers $\sigma_1 \geq \cdots \geq \sigma_m \geq 0$ such that $A = USV^\top$ where

$$S = \begin{bmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_m) & O_{m,n-m} \end{bmatrix}.$$

$A = USV^\top$ is the singular value decomposition of A.

$\sigma_1, \ldots, \sigma_m$ are the singular values of A.

Throughout we use $\|\mathbf{v}\|$ to denote $\sqrt{\mathbf{v}^\top \mathbf{v}}$.

Known properties of the SVD:

(a) The singular values of A are the square roots of the eigenvalues of $AA^\top$.

(b) The $\mathbf{u}_j$ whose whose σ_j equal σ form an orthonormal basis of the eigenspace of $AA^\top$ corresponding to eigenvalue σ^2.

(c) The $\mathbf{v}_j$ whose σ_j equal σ form an orthonormal basis of the eigenspace of $A^\top A$ corresponding to eigenvalue σ^2.

(d) For nonzero $\mathbf{v}$, gain$(\mathbf{v}) := \|\mathbf{Av}\|/\|\mathbf{v}\|$. The minimum gain for A is σ_m; the maximum for A is σ_1.

(e) The spectral norm

$$\|A\|_2 := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

of A is σ_1.

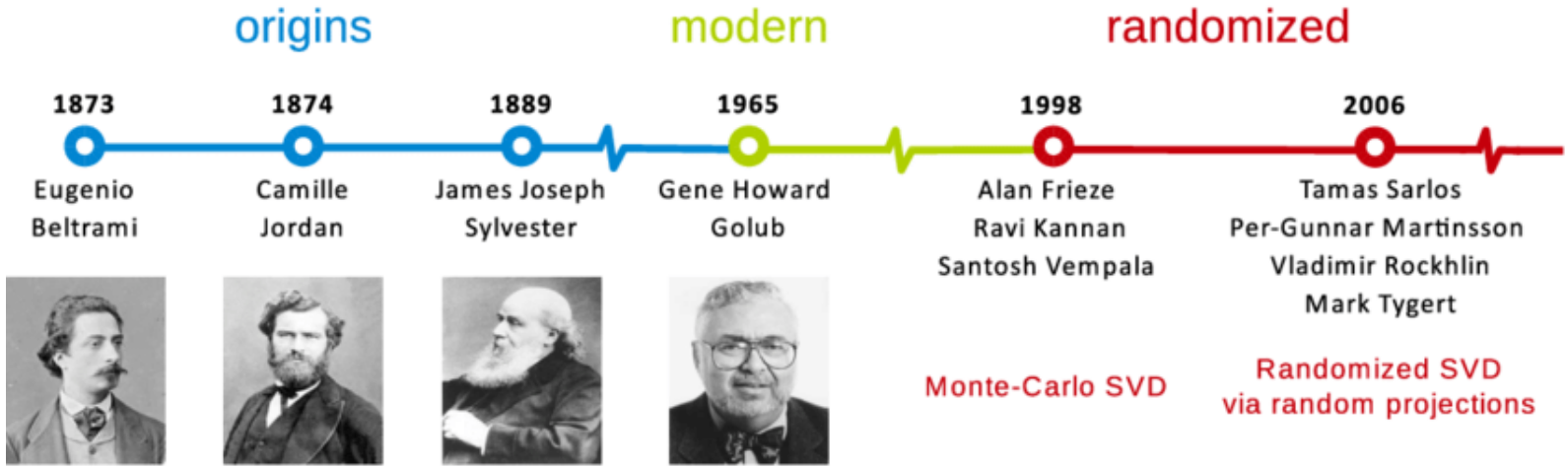


Figure: Timeline Development of SVD

See [Martin-Porter 2012] for a beautiful survey.[1]

Theorem (Eckart-Young (1936))

Let A be an $m \times n$ real matrix of rank r with singular value decomposition $A = \sum_{j=1}^{r} \mathbf{u}_j \mathbf{v}_j^\top$. If $k \leq r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^\top$, then

$$\min_{\text{rank}(B)=k} = \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

[1] Figure from Steven L. Brunton, J. Nathan Kutz (2019).

Example where inverse singular value problems arise.

Let's look at an aircraft's control system.

Control Surface	Motion Controlled	Primary Function
Elevator	**Pitch** (Nose Up/Down)	Climbing and descending.
Aileron	**Roll** (Banking Left/Right)	Banking to initiate turns.
Rudder	**Yaw** (Nose Left/Right)	Steering and coordination.

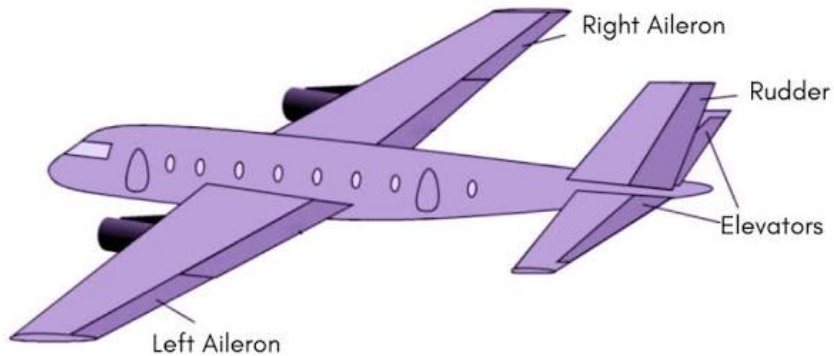
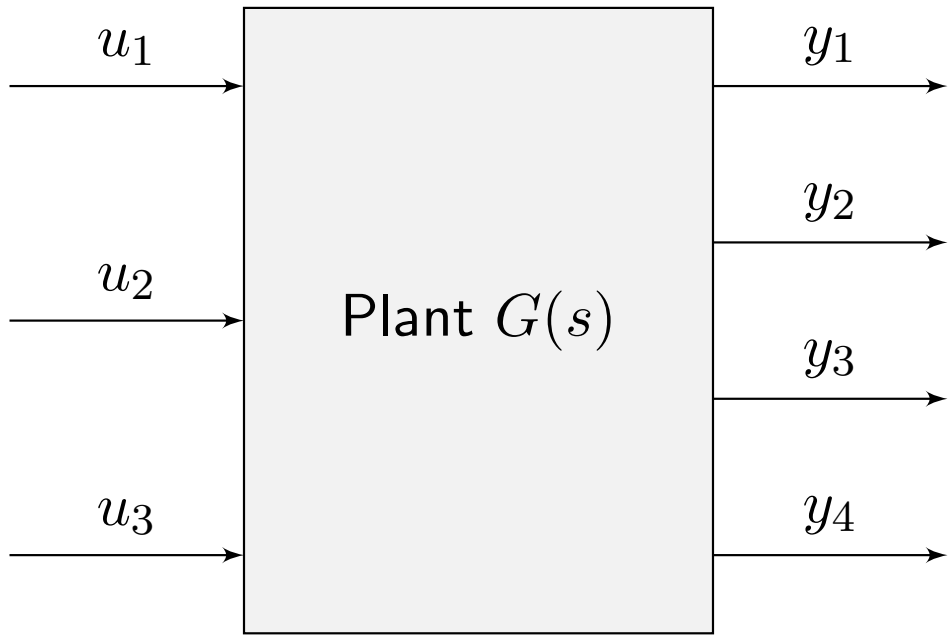


Image from: `http://globalaviation.cognitivenavigation.com/control-surfaces/`

So we have three actuators $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ that move the elevator, aileron, rudder; and four basic gauges $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ that give info about the pitch angle, airspeed, altitude and vertical velocity. This is a Multi-input Multi-output system (MIMO). Engineers represent this with the diagram below.



The dynamics are modeled as:
The internal dynamics of the plant are described by the state-space equations:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{2}$$

where:

- $\mathbf{x} \in \mathbb{R}^3$: Internal state vector (the physical values of air speed etc).
- $A \in \mathbb{R}^{3\times3}$: State matrix (governs autonomous stability).
- $B \in \mathbb{R}^{3\times3}$: Input matrix (how actuators push the states).
- $C \in \mathbb{R}^{4\times3}$: Output matrix (how sensors observe the states).
- $D \in \mathbb{R}^{4\times3}$: Direct feedthrough matrix (instantaneous input-to-output link).

Related singular values and importance:

(a) Singular values of B represent control authority. The right singular vectors corresponding to large singular values of B describe actuator movements that have the most influence on the system.

(b) Singular values of C represent sensor sensitivity. Related to which changes in state will be most easily observed.

(c) Singular values of A represent the internal coupling and conditioning of the stand-alone plant. Is the plane sensitive to small changes?

(d) Singular values of D represent the system's behavior to instantaneous shocks.

Quantities of interest to engineers.

(a) Maximum Singular Value: The maximum possible gain across all input directions.

(b) Minimum Singular Value: The gain in the "hardest to control" direction. If $\underline{\sigma} \approx 0$, the system is near-singular and difficult to control.

(c) Transmission Zeros: Frequencies where the system matrix loses rank:

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} < \text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

These give rise to inverse problems: How can one design the control system so the control authority lies in a certain range, and the sensors are sensitive to the most important types of changes?

In our specific 4×3 aircraft flight control example, the matrices could be:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & -0.2 & 0.1 \\ 0.1 & 0 & -0.05 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 0.1 & 0 \\ -0.1 & 0 & 0.05 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0 & 0.2 \\ 0 & 0.8 & 0 \end{bmatrix}$$

and $D = O$ (indicating no instantaneous input-to-output).

Could also include a feed back equation:

$$\mathbf{u} = -K\mathbf{x} + \mathbf{r}$$

to help control the system.

The singular values of K indicate the strength of transforming input signals into output signals; larger singular values correspond to strong response.

There exists structure!

Certain entries are zero because there is no first order relation between an actuator and a gage, an actuator and an internal variable (e.g., the rudder has virtually no influence on vertical speed).

Certain entries are positive or negative, indicating the type of change (an increase in the slope of the elevator increases vertical speed).

So we are naturally led to inverse singular value problems for families of structured matrices.

The Inverse Singular Value Problem

Let $\mathcal{F}$ be a family of $m \times n$ matrices. The Inverse Singular Value Problem for $\mathcal{F}$ asks:

> *Which sets of $\min\{m, n\}$ of nonnegative real numbers are the singular values of some matrix in $\mathcal{F}$?*

We will restrict our discussion to the family of real matrices with a given zero/nonzero pattern.
Recall the zero/nonzero pattern of A is obtained from A by replacing each nonzero entry with a $*$. Given an $m \times n$ $(0, *)$-matrix P, $\mathcal{Z}(P)$ is the set of all $m \times n$ real matrices with pattern P.

Examples:
$\mathcal{Z}(*J_{34})$ is the set of all 3×4 nowhere zero matrices.

$$\mathcal{Z}\left(\begin{bmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\right) \text{ is all } \begin{bmatrix} a & b & 0 \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix} \text{ where } a, b, c, d, e \neq 0.$$

Simple examples

(a) $P = O_{m,n}$ is the only pattern that requires singular value set $\{0, \ldots, 0\}$.

(b) The $m \times n$ pattern P allows a matrix with each singular value equal to a given positive constant if and only if $\mathcal{Z}(P)$ contains a matrix whose rows are orthonormal. In this case, every multiset of m positive reals is the set of singular values of a matrix with pattern P.

(c) The multiset S of m nonnegative real numbers is the set of singular values of $[*I_m \; O]$ if and only if S has no zero.

(d) For $i = 1, 2$, let P_i be an $m_i \times n_i$ pattern The multiset S of $m_1 + m_2$ real numbers is the set of singular values of matrix with pattern $P_1 \oplus P_2$ if and only if S is $T_1 \cup T_2$ and where T_i is the set of singular values for some matrix in $\mathcal{Z}(P_i)$ for $i = 1, 2$.

(e) If P is $m \times n$, and the multiset S of m nonnegative reals is the set of singular values of a matrix in $\mathcal{Z}(P)$, then the number of nonzeros in S is at most the term-rank of $\mathcal{Z}(P)$, and at least the triangle number of $\mathcal{Z}(P)$.

Questions that come to mind:

- Does every pattern support every list of distinct positive reals as a singular value set?
- What is the maximum multiplicity of a singular value over all matrices with pattern P
- Which patterns P requires all singular values to be simple?
- Is there a strong property for singular values? If so, what are the analogous superpattern, proximity, bifurcation theorems, and their consequences?

The Strong Singular Value Property: SSVP

Remainder of this talk is based on PhD Dissertation of Caleb Cheung (see Cheung-S 2025, and Cheung-S 2026).

Other group of researchers studied independently at same time (see (Yue et al 2026]).

What we call the Strong Singular Value Property is SSVP is called the Orthogonal Equivalence Transversality Property (OETP) in their work.

The Strong Singular Value Property: SSVP

Let A be an $m \times n$ real matrix.
If U is an $m \times m$ orthogonal matrix, and V is an $n \times n$ orthogonal matrix, then UAV and A have the same singular values.

This leads to looking at the function
$f\colon \operatorname{Skew}(m) \times \operatorname{Skew}(n) \to \mathbb{R}^{m \times n}$ by $f(K, L) = e^K A e^L$.

Then $f(O, O)$ and $Df_{(O,O)} = KA + AL$.

Thus image$(f) = \{KA + AL\colon K \in \operatorname{Skew}(m), L \in \operatorname{Skew}(n)\}$

We set

$$\mathfrak{f}\colon\ \mathrm{Skew}(m) \times \mathrm{Skew}(n) \times \mathcal{O} \to \mathbb{R}^{m\times n}$$

by

$$(K, L, B) = e^K A e^L + B$$

and $\mathcal{O}$ is is a sufficiently small neighborhood of O in $\langle \mathcal{Z}(P) \rangle$.

Note: $D\mathfrak{f}_{(O,O,O)}(K, L, B) = e^K A e^L + B$.

It follows that $\mathfrak{f}$ is locally surjective if and only if

$$\{KA + AL : K \in \mathrm{Skew}(m), L \in \mathrm{Skew}(n)\} + \langle \mathcal{Z}(P) \rangle = \mathbb{R}^{m\times n}. \tag{3}$$

We take Equation (3) as the definition of the strong singular value property; that is, A has the SSVP if and only if Equation (3) holds.

Complementary form of SSVP

We can view $\mathbb{R}^{m\times n}$ as an inner product space with

$$\langle M, N\rangle = \text{tr } M^\top N.$$

With respect to this

(a) $\langle \mathcal{Z}(P)\rangle^\perp = \{X \in \mathbb{R}^{m\times n} : A \circ X = 0\}$

(b)

$$\{KA + AL\colon K \in \text{Skew}(n), L \in \text{Skew}(n)\}^\perp$$

is

$$\{Y \in \mathbb{R}^{m\times n} \text{ both } A^\top Y \text{ and } YA^\top X \text{ are symmetric}\}.$$

Why?
$\{KA + AL\colon K \in \text{Skew}(n), L \in \text{Skew}(n)\}^\perp$
equals the set of X such that

$\text{tr}((KA + AL)^\top X) = 0$ for all $K \in \text{Skew}(m)$, $L \in \text{Skew}(n)\}$.

This in turn is the set of all X such that

$$\mathrm{tr}(KA)^\top X) = 0 \text{ and } \mathrm{tr}((AL)^\top X) = 0 \text{ for all } K \text{ and all } L.$$

So the desired perp is the intersection of

$$\{X : \mathrm{tr}(A^\top X)K = O \text{ for all } K \in \mathrm{Skew}(m)\}$$

and

$$\{X : \mathrm{tr}(XA^\top)L = O \text{ for all } L \in \mathrm{Skew}(n)\}.$$

The first of these is all the matrices X such that $A^\top X$ is symmetric.

The second is all the matrices such that $XA^\top$ is symmetric. □

The SSVP spoiler space for A is the set of all $m \times n$ matrices X such that $A \circ X = O$ and both $A^\top X$ and $X A^\top$ are symmetric.

A has the SSVP if and only if O is the only matrix in the spoiler space of A.

Examples:

(a) Each nowhere zero matrix has the SSVP.

(b) If A is $m \times n$, $m \leq n$ and A has a row of zeros, then A does not have the SSVP.

As $m \leq n$ and A has a row of zeros, there exist j and nonzero $\mathbf{v}$ such that the jth row of A is zero, and $A\mathbf{v} = \mathbf{0}$. Now $\mathbf{v}\mathbf{e}_j^\top$ is nonzero SSVP spoiler for A.

(c) If A is $m \times n$ with the SSVP and linearly independent rows, and B is $m \times p$, then $\begin{bmatrix} A & B \end{bmatrix}$ has the SSVP.

If $\begin{bmatrix} X & Y \end{bmatrix}$ is in the SSVP spoiler space, then X is in the SSVP spoiler space for A, and $Y^\top A = BX$.

Hence $X = O$, and the linear independence of rows of A imply $Y = O$.

So $\begin{bmatrix} A & B \end{bmatrix}$ has the SSVP.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ has the SSVP, but $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$ does not. So assumption that rows of A are linearly independent is needed.

(d) Each full upper Hessenberg matrix A has the SSVP.

E.g., $P = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$. A matrix in spoiler space has form

$X = \begin{bmatrix} 0 & 0 & r & s \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$(2,3)$-entry of $A^\top X$ is ra_{12}; the $(3,2)$-entry of $A^\top X$ is 0. So $A^\top X$ symmetric $\implies r = 0$.

$(3,4)$-entry of $A^\top X$ is $ta_{2,3}$ $(4,3)$-entry of $A^\top X$ is 0. So $A^\top X$ symmetric $\implies t = 0$.

$(2,4)$-entry of $A^\top X$ is $sa_{12} + ta_{22} = sa_{12}$; the $(4,2)$-entry of $A^\top X$ is 0. So $A^\top X$ symmetric $\implies s = 0$.

Thus, $X = O$ and A has the SSVP.

(e) If A is $m \times n$ with $m \leq n$ and A has the SSVP, then A has term-rank m.

Assume A has term-rank less than m. Then by König-Egervary, we may assume that A has the form $\begin{bmatrix} B & O_{r,s} \\ C & D \end{bmatrix}$, where $r + s = n + 1$.

Thus, there exists nonzero $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}^\top B = \mathbf{0}^\top$, and $D\mathbf{v} = \mathbf{0}$. Now $X = \begin{bmatrix} O & \mathbf{v}\mathbf{u}^\top \\ O & O \end{bmatrix}$ is a nonzero SSVP spoiler matrix for A.

When defining parameters related to the SSVP, it is desirable that each pattern has a matrix with the SSVP. In light of the above, we often restrict to $m \times n$ patterns that have term-rank m.

(f) $D = \text{diag}(d_1, \ldots, d_m)$ has the SSVP if and only if each d_i is nonzero and $|d_i| \neq |d_j$ for $i \neq j$.

$\Longrightarrow$. Since the SSVP, by (b), it has no row of zeros. So each d_i is nonzero. If $d_i = \pm d_j$ and $i \neq j$, then $d_i E_{ij} + d_j E_{ij}$ is a spoiler for A. So $d_i \neq \pm d_j$.

$\Longleftarrow$: D diagonal, and both DX and XD symmetric imply that $D^2 X = X D^2$. The conditions on entries of D imply X is diagonal. $D \circ X = O$ implies $X = O$.

(g) A $m_1 \times n_2$ and B $m_2 \times n_2$ with $m_1 + n_2 \leq n_1 + n_2$. Then $A \oplus B$ has the SSVP if and only if A and B don't have a common singular value, $m_1 \leq n_1$ and $m_2 \leq n_2$ and both A and B each has the SSVP and at least one is square and invertible.

(h) Under same conditions as in (g). There exist nowhere zero matrices C and C' such that $\begin{bmatrix} A & C \\ C' & B \end{bmatrix}$ has the SSVP if and only if for all (i, j) either row i or column j of $A \oplus B$ has a nonzero entries.

Spoiler space description of direct sums

Let be an M $m \times m$ symmetric matrix. and N be an $n \times n$ symmetric matrix. The $m \times n$ matrix Y intertwines M and N provided $MY = YN$.

Theorem (Sylvester)

Y intertwines M and N if and only if Y is a linear combination of outer-products $\mathbf{u}\mathbf{v}^\top$ where $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of M and N, respectively, corresponding to a common eigenvalue.

Corollary

Let A, respectively B, be an $m \times n$, respectively $r \times s$, matrix with linearly independent rows. Then the SSVP spoiler space of $A \oplus B$ is the set of all matrices of the form $\begin{bmatrix} X_1 & Y \\ Z & X_2 \end{bmatrix}$ where X_1 is in the spoiler space of A, X_2 is in the spoiler space of B, Y intertwines $AA^\top$ and $Z = (A^\top A)^{-1}Y$. In particular, if A and B both have the SSVP, then the dimension of the spoiler space of $A \oplus B$ equals $\sum m_\sigma(A)m_\sigma(B)$ where the sum is over the common singular values σ of A and B.

Fundamental theorem for SSVP

The conditions $A \circ X = O$ and both $A^\top X$ and $X A^\top$ symmetric correspond to the solutions to a system of linear homogeneous equations.

Hence we have:

Theorem (SSVP is an open property)

If A has the SSVP, then there exists an $\epsilon > 0$ such that each matrix B with $\|B - A\| < \epsilon$ has the SSVP.

Fundamental theorems for SSVP

Theorem (Superpattern theorem for SSVP)

Let A be an $m \times n$ matrix with pattern P and the SSVP. Then for each superpattern $\hat{P}$ of P there is a matrix with pattern $\hat{P}$, the SSVP, and the same singular values as A.

Proof The function $\mathfrak{f} = e^K A e^L + B$ is locally surjective near $(0, 0, 0)$.

Let $\text{adj}_{\hat{P}}$ be the adjacency matrix of $\hat{P}$. By the Local Surjectivity theorem, for ϵ sufficiently small,

$$A + \epsilon \text{adj}_{\hat{P}} = e^K A e^L + B$$

for some skew-symmetric K and L and some $B \in \langle \mathcal{Z}(P) \rangle$ with $A + \epsilon \text{adj} \hat{P} = e^K A e^L + B$.

The matrix $e^k A e^L = A + \epsilon \text{adj}_{\hat{P}} - B$ will have the same singular values as A and pattern $\hat{P}$. □

Realizability of sets of distinct positive reals

Corollary

If P is an $m \times n$ pattern of term-rank n, and S is a set of m distinct positive reals, then there is a matrix with pattern P with singular values S.

Proof.

Let D be the diagonal matrix whose entries are the elements of S
D has the SSVP. Now apply the Superpattern theorem. □

Open question.

If P has term-rank r, and S is a set of r distinct positive reals, then is there a matrix whose pattern is P having singular values S along with 0 of multiplicity $m - r$?

Theorem (Augmentation theorem)

Let A be an $m \times n$ matrix with pattern P, the SSVP, and linearly independent rows.

Let σ be a singular value of A of multiplicity $k \geq 1$.

Let $\alpha \subseteq \{1, \ldots, n\}$, $\beta \subseteq \{1, \ldots, m\}$ with

$$|\mathit{supp}(\mathbf{x}) \cap \alpha| + |\mathit{supp}(\mathbf{y}) \cap \beta| \geq 2$$

for all nonzero $\mathbf{x}$ in the eigenspace of $AA^\top$ corresponding to σ^2 and all nonzero $\mathbf{y}$ in the eigenspace of $A^\top A$ corresponding to σ^2.

Then the pattern obtained from $[*] \oplus P$ by replacing the 0 im position $(1, i)$ $i \in \alpha$ by $*$ and the 0 in position $(j, 1)$ $i \in \beta$ by $*$ contains a matrix with the SSVP whose singular values are those of A along with σ.

Example:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

The eigenspaces of $AA^\top$ and $A^\top A$ corresponding to $\sigma = 1$ are both $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$.

So each superpattern of any of

$$\begin{bmatrix} * & * & * & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

allows a matrix with singular values 1,1,1,$\sqrt{2}$.

Matrix Liberation

Let A be a matrix with pattern P and $\widehat{P}$ be a superpattern of P. Recall A has the SSVP with respect to $\widehat{P}$ provided $X = O$ is the only $m \times n$ matrix such that $\widehat{P} \circ X = O$ and both $A^\top X$ and $XA^\top$ are symmetric.

Theorem (Matrix Liberation Theorem)

Let A be an $m \times n$ matrix with pattern P. Let D be a matrix of the form $KA + AL$ such that $K \in \mathrm{Skew}(m)$ and $L \in \mathrm{Skew}(n)$. Let P_D be the pattern of P in the direction of D.

If A has the SSVP w.r.t. P_D, then there exists a matrix with pattern P_D, the SSVP and singular values the same as A.

Liberation Sets

An SSVP liberation set for the $m \times n$ matrix A is a subset $\mathcal{L}$ of the zero entries of A with the property:
for each $(i, j) \in \mathcal{L}$. A has the SSVP with respect to the pattern obtained from that of A by replacing the 0's by $*$'s in positions $\mathcal{L} \setminus \{(i, j)\}$.

Theorem (Liberation Sets theorem)

If $\mathcal{L}$ is an SSVP liberation set for A, then there is a matrix B with the SSVP and same singular values as A whose pattern is that obtained from A by changing the positions in $\mathcal{L}$ to stars.

Example

Let A be an $r \times r$ matrix whose bigraph is the path

$$1\text{—}1'\text{—}2 \cdots r' \text{—}r,$$

and B be an $s \times s$ matrix whose bigraph is the path

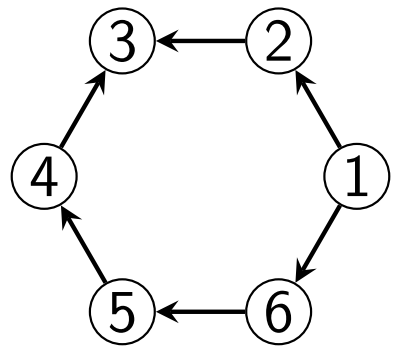
$$1\text{—}1\text{—}2 \cdots s'\text{—}s.$$

Assume that the only common singular value of A and B is their spectral radius. Let P be the pattern of $A \oplus B$.

By Sylvester's theorem, $A \oplus B$ has the SSVP with respect to $P + *E_{ij}$ for each $(i,j) \in (\{1,\ldots,r\} \times \{r+1,\ldots s\}) \cup (\{r+1,\ldots,s\} \cup \{1,\ldots,r\})$.

Hence, each pattern obtained from P by replacing two zeros in the above described positions by $*$s allows a matrix with the same singular values as $A \oplus B$ and the SSVP.

In particular, each normalized pattern whose unlooped digraph is a **weak cycle** (that is, having no 2-cycles and its underlying graph is a cycle) allows a matrix with the SSVP and a multiple eigenvalue.



Proximity theorem

Theorem (Proximity theorem)

Let A be an $m \times n$ matrix with pattern P and the SSVP. Then there exists an $\epsilon > 0$ such that if C is a matrix with $\|C - A\| < \epsilon$, then there is a matrix with pattern P and the same singular values as B.

Proof.

Idea: Look at the map $\mathfrak{g}_{(K,L,B)} = e^K(A+B)e^L$.

$D(g)_{(O,O,O)} = D\mathfrak{f}_{(O,O,O)}$. A has SSVP $\implies$ $D\mathfrak{g}$ is onto.

By Local Surjectivity Theorem, $\mathfrak{g}$ is locally surjective.

So there exist K, L, B with $e^K(A+B)e^L = C$. Now $A + B = e^{-K}Ce^{-L}$ has pattern P and same singular values as C.



Bifurcation for singular values

Given an $m \times n$ matrix A with distinct singular values

$$\sigma_1 > \cdots > \sigma_q,$$

the ordered multiplicity list of A is $\mathbf{m}_A^\sigma = (m_1, \ldots, m_q)$ where m_i is the multiplicity of σ_i as a singular value of A.

A **refinement** of $\mathbf{m}_A^\sigma$ is any tuple of positive reals obtained by replacing some of the m_i with a partition of m_i.

Examples:
The refinements of $(1, 1, 4, 1)$ are:

$(1, 1, 4, 1)$,
$(1, 1, 1, 3, 1)$, $(1, 1, 3, 1, 1)$, $(1, 1, 2, 2, 1)$,
$(1, 1, 1, 2, 1, 1)$, $(1, 1, 1, 1, 2, 1)$, $(1, 1, 2, 1, 1, 1)$,
$(1, 1, 1, 1, 1, 1, 1)$.

$(1, 2, 1, 2, 1)$ is not a refinement of $(1, 1, 4, 1)$.

Theorem

Let A be an $m \times n$ matrix with the SSVP and pattern P. Then for each refinement $\mathbf{m}'$ of $\mathbf{m}_A^\sigma$ there is a matrix in $\mathcal{Z}(P)$ with ordered multiplicity list $\mathbf{m}'$.

Proof.

Let $A = USV^\top$ be the singular value decomposition of A. The diagonal entries of S are the singular values of A. They can be perturbed by an arbitrarily small amount to a "diagonal" matrix S with ordered singular value list $\mathbf{m}'$.

As U and V are orthogonal $A' = US'V^\top$ can be taken arbitrarily close to A. Now apply the Proximity theorem. $\square$

Problems to Ponder

(a) What is the maximum multiplicity of a singular value of $A \in \mathcal{Z}(P)$?

(b) What is the maximum multiplicity of a singular value over matrices $A \in \mathcal{Z}(P)$ that have the SSVP?

(c) What's the maximum multiplicity of the largest singular value of $A \in \mathcal{Z}(P)$? (Note if $A \neq O$ is $m \times n$, then max multiplicity $\sigma_1(A) = m$ if and only if the rows of A are orthonormal.)

(d) What's the maximum multiplicity of the largest singular value of $A \in \mathcal{Z}(P)$ having the SSVP?

(e) Same questions for smallest singular value σ_m?

(f) What is the minimum number of distinct singular values of a matrix in $\mathcal{Z}(P)$?, of a matrix in $\mathcal{Z}(P)$ with the SSVP?

Consequences

Corollary

Let P be an $m \times n$ pattern such that $P[\{1, \ldots, r\}] = *J_r$, and $P(\{1, \ldots, r\})$ has term-rank $m - r$. For each tuple $\mathbf{m} = (m_1, \ldots, m_q)$ of positive integers with

(a) $m_1 + \cdots + m_q = m$,

(b) $\sum_{i:m_i>1} m_i \leq r$.

there is a matrix with pattern P and ordered multiplicity list $\mathbf{m}$.

Proof.

Suffices to prove for $n = m$.

Let $B = I - \frac{2}{r}J_r$, and let D be a diagonal matrix whose diagonal entries are distinct positive reals and different than 1.

Then $B \oplus D$ has the SSVP and singular values 1 of multiplicity r and the diagonal entries of D each of multiplicity 1. Now apply the Supergraph and Proximity theorems. □

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