Think Positive

Goals

- The NIEP
- Different concepts of extremality in the NIEP
- Basic properties of Hessenberg matrices
- Strong spectral property for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$
- Application the strong property in the context of Hessenberg matrices, and characteristic polynomials of nonnegative Hessenberg matrices.

The NIEP

The nonnegative inverse eigenvalue problem (NIEP):
When is a multiset of n complex numbers

$$\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

the spectrum of an $n \times n$ (entrywise) nonnegative matrix?

Necessary conditions

The spectrum of a nonnegative matrix

$$\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

satisfies the following conditions:

1. σ is closed under complex conjugation.
2. $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$ for $k = 1, 2, \ldots$
3. The Perron eigenvalue $\lambda_1 = \max\{|\lambda_i|; \lambda_i \in \sigma\} \in \sigma$ lies in σ.
4. JLL-inequalities (Johnson (1981),Loewy and London (1978)):

$$s_k^m \leq n^{m-1} s_{km}$$

for all $k, m = 1, 2, \ldots$

Above conditions are also sufficient for 2×2 *and* 3×3 *matrices.*

Short and very Incomplete History

- Suleimanova, 1949: posed,
- Loewy and London, 1978: solution for $n = 3$,
- Laffey and Meehan, 1998: solution for $n = 4$,
- J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estevez, C. Marijuan, M. Pisonero, 2007: Another solution for $n = 4$ (characterization of the characteristic polynomials of 4×4 nonnegative matrices).
- $n \geq 5$, unsolved.

Nonzero Spectrum

Boyle, Handelman (1991)

A multiset of complex numbers $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is **the nonzero spectrum** of some nonnegative matrix if:

1. σ has a Perron eigenvalue $\lambda_1 > |\lambda_i|$, $i = 2, 3, \ldots, n$.
2. σ is closed under complex conjugation.
3. For all positive integers n and k,

$$s_n(\sigma) \geq 0,$$

and $s_n(\sigma) > 0$ implies $s_{nk}(\sigma) > 0$.

Laffey (2011): A constructive proof of the above result.

Choosing the Perron Eigenvalue: We can always go up

(Guo 1997): Let $\{\lambda_2, \ldots, \lambda_n\}$ be a multiset of complex numbers that is closed under the complex conjugation.

- There exists $\lambda_1 \geq |\lambda_i|$, $i = 2, \ldots, n$, so that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative matrix,
- $\{\lambda_1 + t, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative matrix for all $t \geq 0$.

The spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of a nonnegative matrix A is Perron Extreme, if $\{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$ is not the spectrum of a nonnegative matrix for all $\epsilon > 0$. In this case we also say that the matrix A is Perron extreme.

The spectrum is Perron extreme if and only if it is not the spectrum of any matrix with positive entries.

Local Similarity: Looks familiar?

(Laffey 1998, Laffey, Loewy, Š. 2009) If A is a nonnegative matrix that it is **not locally similar** to a positive matrix, then there exists a nonzero nonnegative matrix Y such that:

1. $AY - YA = 0$,
2. $A \circ Y = 0$.

- $(I_n - \varepsilon L)A(I_n - \varepsilon L)^{-1} = A + \varepsilon[A, L] + \ldots$
- If we can find L with $[A, L]$ positive outside the support of A, then A is locally similar to a positive matrix.
- Apply a theorem of the alternative.

Safely inside

Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be the spectrum of a nonnegative matrix A and $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$. The following are equivalent:

- σ is not Perron extreme.
- σ is the spectrum of a positive matrix.
- For every $q(x) \in \mathbb{R}_{n-1}[x]$ there exists $\varepsilon > 0$ so that $p(x) + \varepsilon q(x)$ is the spectrum of a positive matrix.

Coefficient Theorem for Weighted Digraphs

Let $A \in M_n(\mathbb{R})$ have the characteristic polynomial

$$p_A(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \ldots + k_n,$$

and the corresponding weighted digraph Γ. Let $\mathcal{C}_i(\Gamma)$ be the set of composite cycles with exactly i vertices, and for $\gamma \in \mathcal{C}_i(\Gamma)$ let γ_t denote the number of cycles in γ, and $\mathrm{wt}(\gamma)$ the product of the weights of all arcs belonging to γ. Then, for each $1 \leq i \leq n$,

$$k_i = \sum_{\gamma \in C_i(\Gamma)} (-1)^{\gamma_t} \mathrm{wt}(\gamma).$$

Coefficients vs Pattern, real matrices

Let $A \in M_n(\mathbb{R})$ with directed graph Γ and characteristic polynomial

$$p(x) = x^n + k_1 x^{n-1} + \ldots + k_n.$$

1. The shortest cycle in Γ is of order $m \Rightarrow k_1 = \ldots = k_{m-1} = 0$,
2. GCD of all orders of cycles in Γ is $d \geq 2 \Rightarrow \ k_i = 0$ for all i that are not divisible by d.

Coefficients vs Pattern, nonnegative matrices

Let $A \geq 0$ with the directed graph Γ and the characteristic polynomial

$$p(x) = x^n + k_1 x^{n-1} + \ldots + k_n.$$

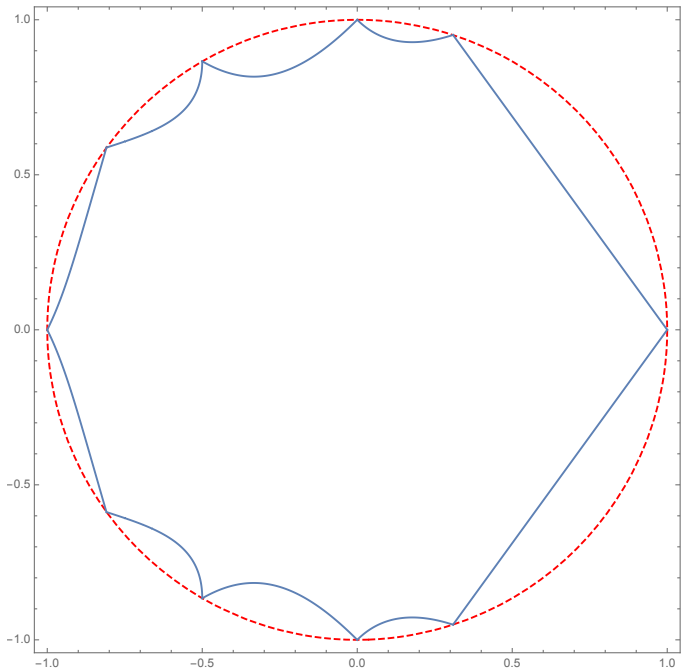
1. The shortest cycle in Γ is of order $m \geq 2 \iff$ $k_1 = \ldots = k_{m-1} = 0$, $k_m < 0$.
2. GCD of all orders of cycles in Γ is $d \geq 2 \iff$ $k_i = 0$ for all i that are not divisible by d.
3. Loewy and London 1978, J. Torre-Mayo et al. 2007: Γ has no 2-cycles, all loops have the same weight $\iff$ $k_2 = \frac{n-1}{2n} k_1^2$

In all cases above, A is Perron Extreme. End of story?

Further restrictions on the pattern

Let $A \geq 0$:

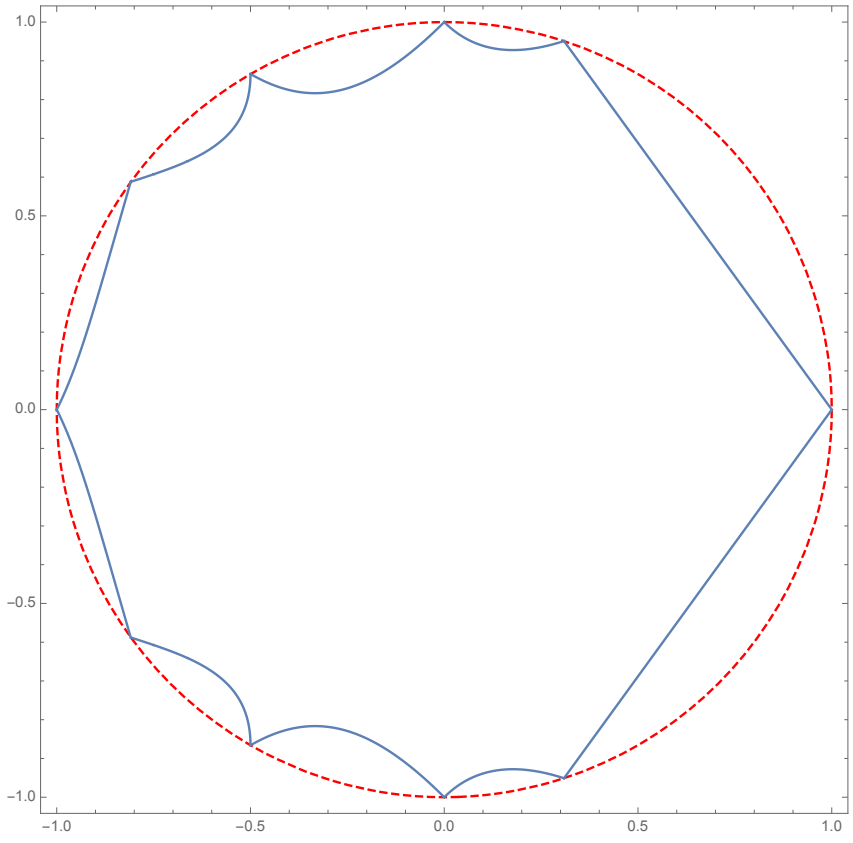
- The multiplicity of the Perron eigenvalue is $\geq 2 \Rightarrow$ A is reducible.
- A has an eigenvalue on the border of the Karpelevič region $\Rightarrow$ strong restrictions on the corresponding digraph.



Further restrictions on the pattern

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Any stochastic matrix with the characteristic polynomial

$$f_\alpha(t) = t^5 - t(1-\alpha) - \alpha$$

is permutationally similar to a matrix of the form

$$A = \begin{pmatrix} 0 & \alpha_1 & 1-\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1-\alpha_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 \alpha_2 = \alpha$.

Walking on the border

Let

$$p(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \ldots + k_n.$$

be the characteristic polynomial of a nonnegative matrix.

If $k_1 = k_2 = \ldots = k_{t-1} = 0$, we can say that $p(x)$ is not extreme, if for every $q(x) \in \mathbb{R}_{n-t}[x]$ there exists $\varepsilon > 0$ so that $p(x) + \varepsilon q(x)$ is the characteristic polynomial of a nonnegative matrix.

Let $\mathcal{P}$ be a nonzero subspace of $\mathbb{R}_{n-1}[x]$.

- We say that $p(x)$ has slack on $\mathcal{P}$ if for every $q(x) \in \mathcal{P}$ there exists $\varepsilon > 0$ so that $p(x) + \varepsilon q(x)$ is the characteristic polynomial of a nonnegative matrix.
- We say that $p(x)$ is tight for $\mathcal{P}$ if does not have slack on $\mathcal{P}$.

$p(x)$ is tight for $\mathbb{R}_{n-1}[x] \iff p(x)$ is Perron extreme.

More generally...

Let

$$p(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \ldots + k_n.$$

be the characteristic polynomial of a matrix with sign pattern (pattern) Q.

Let $\mathcal{P}$ be a nonzero subspace of $\mathbb{R}_{n-1}[x]$.

- We say that $p(x)$ has slack for $(\mathcal{P}, Q)$ if for every $q(x) \in \mathcal{P}$ there exists $\varepsilon > 0$ so that $p(x) + \varepsilon q(x)$ is the characteristic polynomial of a matrix with sign pattern Q.
- We say that $p(x)$ is tight for $(\mathcal{P}, Q)$ if does not have slack for $(\mathcal{P}, Q)$.

In this lecture we will stay positive, but we can explore more general sign patterns in questions.

Long way around

J. Torre-Mayo et al. 2007: Let

$$p(x) = x^n + k_t x^{n-t} + \ldots + k_n.$$

- $2t > n$: $p(x)$ is the characteristic polynomial of a nonnegative matrix if and only if $k_i \leq 0$, $i = t, \ldots, n$.
- $2t = n$: $p(x)$ is the characteristic polynomial of a nonnegative matrix if and only if $k_i \leq 0$, $i = t, \ldots, n-1$, and $k_{2t} \leq \frac{k_t^2}{4}$.
- $2t - 1 = n$: Solved.

Examples:

- $p(x) = x^n$ is tight for $\mathcal{P}$, for all subspaces of $\mathbb{R}_{n-1}[x]$.
- $p(x) = x^7 - x^2 - x - 1$ is tight for $\mathbb{R}_k[x]$, $k = 3, 4, 5, 6$, but has slack on $\mathbb{R}_2[x]$.
- $p(x) = x^4 - 2x^2 - x + 1$ has slack on $\mathcal{P} = \operatorname{span}\{x\}$.

Focus on selected coefficients

Let $A \in \mathcal{M}_n(\mathbb{R})$, $\mathcal{S} \subseteq \operatorname{supp}(A)$, $\mathcal{C} \subset [n]$, and

$$p_A(\lambda) = \lambda^n + f_1\lambda^{n-1} + \ldots + f_n.$$

We say that the pair $(\mathcal{S}, A)$ targets $\mathcal{C}$ if the coefficient of x^{n-i} in the characteristic polynomial of B equals f_i for all $i \in [n] \setminus \mathcal{C}$ and all matrices B with $\operatorname{supp}(A - B) \subset \mathcal{S}$.

If A and B only differ in the entries corresponding to $\mathcal{S}$, then the coefficients corresponding to $[n] \setminus \mathcal{C}$ agree.

Let $J[\mathcal{S}, \mathcal{C}]$ be the submatrix of J on the rows corresponding to $\mathcal{S}$ and columns corresponding to $\mathcal{C}$.
A has full-rank Jacobian for $(\mathcal{S}, \mathcal{C})$ if $\operatorname{rank}(J[\mathcal{S}, \mathcal{C}]|_{X=A}) = |\mathcal{C}|$.

If $A \geq 0$ has full rank Jacobian for $(\mathcal{S}, \mathcal{C})$, then $p_A(x)$ has slack on $\operatorname{span}\{x^{n-i}; i \in \mathcal{C}\}$.

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_{4,1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{5,3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{7,5} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a_{9,6} & 0 & 0 & 0 \end{pmatrix}$$

$$p_A(x) = x^9 - x^6(a_{5,3} + a_{7,5}) - x^5(a_{4,1} + a_{9,6}) + x^2(a_{4,1}a_{7,5} + a_{5,3}a_{9,6}) + xa_{4,1}a_{9,6}.$$

$$J' = \begin{pmatrix} 0 & -1 & a_{9,6} & a_{7,5} \\ -1 & 0 & 0 & a_{9,6} \\ -1 & 0 & 0 & a_{4,1} \\ 0 & -1 & a_{4,1} & a_{5,3} \end{pmatrix}, \det J' = (a_{4,1} - a_{9,6})^2.$$

If $a_{4,1} \neq a_{9,6}$, then A has slack on $\operatorname{span}\{x^6, x^5, x^2, x\}$.

Hessenberg Matrices

Let $\mathcal{H}_n$ denote the set of all $n \times n$ matrices of the form:

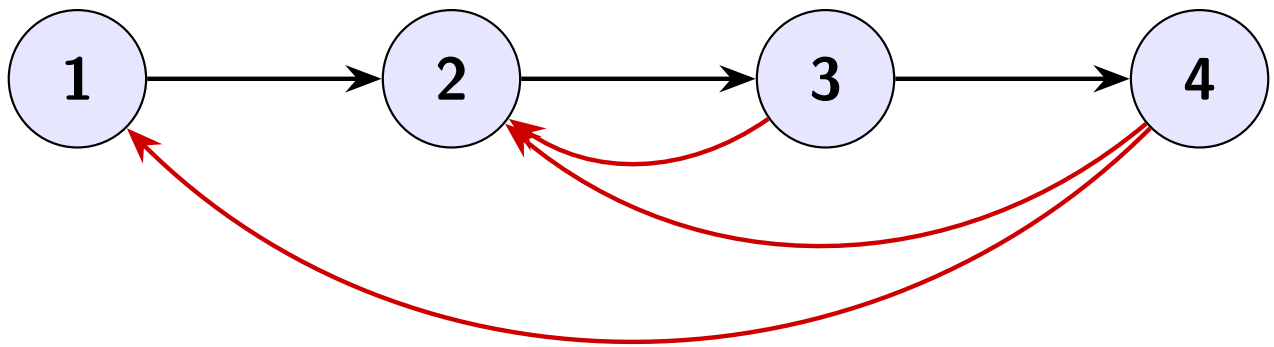
$$A = \begin{pmatrix} a_{1,1} & 1 & & \\ a_{2,1} & a_{2,2} & 1 & \\ \vdots & \vdots & \ddots & 1 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

and $\mathcal{H}_n^+$ the set of all nonnegative matrices in $\mathcal{H}_n$.

There is one-to-one correspondence between $a_{i,j}$, $i \geq j$, and simple cycles in Γ.

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a_{3,2} & 0 & 1 \\ a_{4,1} & a_{4,2} & 0 & 0 \end{pmatrix}$$



Going in Circles, but Staying on Track.

For $k \in \{-(n-1), \ldots, n-1\}$ we define

$$\mathscr{D}_k = \{(i,j) \in [n] \times [n]; i - j = k\},$$

so $\mathscr{D}_0$ corresponds to the main-diagonal, $\mathscr{D}_1$ to sub-diagonal, $\mathscr{D}_{-1}$ super-diagonal, etc.

Let $A = (a_{i,j}) \in \mathcal{H}_n$ with the corresponding directed graph Γ.

$(i,j) \in \mathscr{D}_k \Rightarrow a_{i,j}$ corresponds to an edge in a cycle of order $k+1$.

$$A = \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \\ * & * & * & * & * \end{pmatrix}, \mathscr{D}_2 \text{ - three cycles}, \mathscr{D}_3 \text{ - four cycles}$$

Characteristic Polynomials of matrices in $\mathcal{H}_n^+$: we can always go down

J. Torre-Mayo et al. 2007: If $p(x)$ is the characteristic polynomial of a matrix in $\mathcal{H}_n^+$, then $p(x)-q(x)$ is the characterisitc polynomial of a matrix in $\mathcal{H}_n^+$ for all $q(x) \in \mathbb{R}_{n-1}^+[x]$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \det(xI_4 - A) = x^4 - 2x^3 - x^2 + 2x,$$

$$x^4 - (2+\alpha_1)x^3 - (1+\alpha_2)x^2 + (2-\alpha_3)x - \alpha_4$$

is the characteristic polynomial of a matrix in $\mathcal{H}_4^+$ for all $\alpha_i \geq 0$.

c-extreme

Let $p(x)$ be the characteristic polynomial of a matrix in $\mathcal{H}_n^+$. We say that $p(x)$ is c-extreme, if $p(x) - \varepsilon q(x)$ is not the characteristic polynomial of a matrix in $\mathcal{H}_n^+$ for all $\varepsilon > 0$ and all nonzero $q(x) \in \mathbb{R}_{n-1}^+[x]$. If a matrix $A \in \mathcal{H}_n^+$ has c-extreme characteristic polynomial, then A is c-extreme.

$p(x)$ is c-exteme $\iff$ $p(x)$ is tight for $\operatorname{span}\{x^i\}$ for $i = 0, \ldots, n-1$.

Example

The polynomial

$$p(x) = x^6 - 11x^5 + 35x^4 - 26x^3 + 11x^2 - 35x + 25$$

is c-extreme.

Why? $p(x) = (x-5)^2(x-1)^2\left(x^2+x+1\right)$ has repeated Perron Eigenvalue.

If the Perron eigenvalue is a repeated root of the characteristic polynomial $p(x)$ of a matrix in $\mathcal{H}_n^+$, then $p(x)$ is c-extreme.

Repeated Perron eigenvalue may not be easy to recognize from the coefficients.

Similarity between matrices in $\mathcal{H}_n$

Let $\mathcal{T}_n$ denote the set of all $n \times n$ lower-triangular matrices with all the entries on the diagonal equal to one.

For co-spectral matrices $A, B \in \mathcal{H}_n$ there exists unique $T \in \mathcal{T}_n$ satisfying $TAT^{-1} = B$.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \end{pmatrix}$$

Similarities that do not affect short cycles

Let $A, B \in \mathcal{H}_n$ be cospectral matrices, $T \in \mathcal{T}_n$ satisfying $TAT^{-1} = B$, and k a nonnegative integer.

1. If $\mathscr{D}_i \cap \operatorname{supp}(T) = \emptyset$ for $i = 1, 2, \ldots, k+1$, then $\mathscr{D}_j \cap \operatorname{supp}(A - B) = \emptyset$ for $j = 0, \ldots, k$.
2. If $\mathscr{D}_j \cap (\operatorname{supp}(A) \cup \operatorname{supp}(B)) = \emptyset$ for $j = 0, 1, 2, \ldots, k$, then $\mathscr{D}_i \cap \operatorname{supp}(T) = \emptyset$ for $i = 1, \ldots, k+1$.

$$B = \begin{pmatrix} a_{1,1} & 1 & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 1 & 0 & 0 \\ * & a_{3,2} & a_{3,3} & 1 & 0 \\ * & * & a_{4,2} & a_{4,3} & 1 \\ * & * & * & a_{5,4} & a_{5,5} \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \end{pmatrix}$$

Similarities that preserve gcd of cycle lengths

Let $A, B \in \mathcal{H}_n$ be cospectral matrices, $T \in \mathcal{T}_n$ satisfying $TAT^{-1} = B$, and $d \geq 2$. Define

$$\mathscr{G}_d = \cup_{k=1}^{\lfloor \frac{n}{d} \rfloor} \mathscr{D}_{kd-1} \text{ and } \mathscr{F}_d = \cup_{k=1}^{\lfloor \frac{n-1}{d} \rfloor} \mathscr{D}_{kd}.$$

Assume $\mathrm{supp}(A) \subseteq \mathscr{G}_d \cup \mathscr{D}_{-1}$. Then, $\mathrm{supp}(B) \subseteq \mathscr{G}_d \cup \mathscr{D}_{-1}$, if and only if $\mathrm{supp}(T) \subset \mathscr{F}_d \cup \mathscr{D}_0$.

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ * & 0 & * & 0 & 1 \\ 0 & * & 0 & * & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ * & 0 & * & 0 & 1 \end{pmatrix}$$

Underlying Framework

Let $\mathcal{L}$, $\widehat{\mathcal{Q}}$ be vector subspaces of $\mathcal{M}_n(\mathbb{R})$, $C \in \mathcal{M}_n(\mathbb{R})$, and $\widehat{\mathcal{A}} := C + \widehat{\mathcal{Q}}$ an affine space. Assume:

$$(I_n - L)^{-1} M (I_n - L) \in \widehat{\mathcal{A}} \tag{1}$$

for all $M \in \widehat{\mathcal{A}}$ and all $L \in \mathcal{L}$ with $(I_n - L)$ invertible.

For a specific choice of $\widehat{\mathcal{A}}$ and $\mathcal{L}$, we say that the framework is well defined if (1) holds.

Example: The set $\mathcal{H}_n$ is not a vector space, but it is an affine space. If we take $C = J_n$, $\widehat{\mathcal{Q}}$ the space of lower-triangular matrices, and $\mathcal{L}$ the space of all strictly lower-triangular matrices, then the above framework is well defined.

Local similarity map

Assume the framework is well defined for $\mathcal{L}$ and $\widehat{\mathcal{A}} = C + \widehat{Q}$.
Let $\mathcal{B}_{\mathcal{L}}(0, \varepsilon) := \{L \in \mathcal{L}; \|L\| < \varepsilon\}$, and define $g : \mathcal{B}_{\mathcal{L}}(0, \varepsilon) \to \widehat{\mathcal{Q}}$ by

$$g(L) = (I_n - L)(A + C)(I_n - L)^{-1} - C.$$

Note, $Dg(0) = [A + C, L]$ with the image

$$\mathcal{U} = \{[A + C, L]; L \in \mathcal{L}\}.$$

The orthogonal complement $\mathcal{U}^{\perp}$ of $\mathcal{U}$ **with respect to** $\widehat{\mathcal{Q}}$ is equal to

$$\mathcal{U}^{\perp} = \{X \in \widehat{\mathcal{Q}}; \operatorname{tr}\Big(L[X^{\top}, A + C]\Big) = 0 \text{ for all } L \in \mathcal{L}\}.$$

Examples

- If $\mathcal{L} = \{L \in \mathcal{M}_n(\mathbb{R}); \operatorname{supp}(L) \subseteq \mathcal{S}\}$ for some $\mathcal{S} \subseteq [n] \times [n]$, then

$$\mathcal{U}^{\perp} = \{X \in \widehat{\mathcal{Q}}; [X, (A+C)^{\top}] \circ L = 0 \text{ for all } L \in \mathcal{L}\}$$

- If $\mathcal{L} = \{L \in \mathcal{M}_n(\mathbb{R}); L\mathbf{v} = 0\}$ for some $\mathbf{v} \in \mathbb{R}^n$, then

$$\mathcal{U}^{\perp} = \{X \in \widehat{\mathcal{Q}}; [X^{\top}, A+C] = \mathbf{v}\mathbf{x}^{\top} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

  A notable special case is $\mathbf{v} = \mathbf{1}_n$, for the corresponding similarities to preserve matrices with constant row sums (such as stochastic matrices).

- If $\mathcal{L} = \{L \in \mathcal{M}_n(\mathbb{R}); L\mathbf{v} = 0, \mathbf{w}^{\top}L = 0\}$ for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then

$$\mathcal{U}^{\perp} = \{X \in \widehat{\mathcal{Q}}; [X^{\top}, A+C] = \mathbf{v}\mathbf{x}^{\top} + \mathbf{y}\mathbf{w}^{\top}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\}.$$

  In the context of stochastic matrices, we may let $\mathbf{v} = \mathbf{1}_n$ and $\mathbf{w}$ to be the left eigenvector of $A+C$, to restrict to similarities that preserve the stationary distribution of stochastic matrices.

Perturbation map

Let $f : \mathcal{Q} \to \mathcal{Q}$ be defined by

$$f(B) = A + B.$$

Note, $Df(0)$ is the identity map, with the image $\mathcal{Q}$. Let $\mathcal{Q}^{\perp}$ denote the orthogonal complement of $\mathcal{Q}$ **with respect to** $\widehat{\mathcal{Q}}$.

Example

Let $\mathcal{Q} = \{B \in \widehat{Q}; \mathrm{supp}(B) \subseteq \mathcal{S}\}$, and consider $Q^{\perp}$ for different choices of $\widehat{Q}$.

1. If $\widehat{\mathcal{Q}} = \{B \in M_n(\mathbb{R}); \mathrm{supp}(B) \subseteq \widehat{\mathcal{S}}\}$ for some $\mathcal{S} \subseteq \widehat{\mathcal{S}} \subseteq [n] \times [n]$ then

$$\mathcal{Q}^{\perp} = \{X \in \widehat{Q}; \mathcal{S} \cap \mathrm{supp}(X) = \emptyset\}.$$

2. If $\widehat{\mathcal{Q}} = \{B \in M_n(\mathbb{R}); B\mathbf{v} = 0\}$ for some $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$ with $\mathbf{v}^{\perp}\mathbf{v} = 1$, then

$$Q^{\perp} = \{X_1(I_n - \mathbf{v}\mathbf{v}^{\perp}); \mathcal{S} \cap \mathrm{supp}(X_1) = \emptyset\}.$$

3. If $\widehat{\mathcal{Q}} = \{B \in M_n(\mathbb{R}); B\mathbf{v} = 0, \mathbf{w}^{\perp}B = 0\}$ for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{\mathbf{n}}$ with $\mathbf{w}^{\perp}\mathbf{w} = \mathbf{v}^{\perp}\mathbf{v} = 1$, then

$$Q^{\perp} = \{(I_n - \mathbf{w}\mathbf{w}^{\perp})X_1(I_n - \mathbf{v}\mathbf{v}^{\perp}); \mathcal{S} \cap \mathrm{supp}(X_1) = \emptyset\}.$$

The strong spectral property (the nSSP) for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$

Let $\mathcal{L}$, $\widehat{\mathcal{Q}}$ be vector subspaces of $\mathcal{M}_n(\mathbb{R})$, and $C \in \mathcal{M}_n(\mathbb{R})$. We define the affine spaces $\mathcal{A} := C + \mathcal{Q}$, and $\widehat{\mathcal{A}} := C + \widehat{\mathcal{Q}}$, and assume that the underlying framework is well defined for $\mathcal{L}$ and $\widehat{\mathcal{A}}$.

A matrix $M = A + C \in \mathcal{A}$ has the (non-symmetric) strong spectral property (the nSSP) for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$ if $\mathcal{U}^{\perp} \cap \mathcal{Q}^{\perp} = \{0\}$, where the orthogonal complements are taken with respect to $\widehat{Q}$.

Example

Let

$$\widehat{\mathcal{Q}} = \{B \in M_n(\mathbb{R}); \operatorname{supp}(B) \subseteq \mathcal{S}\},$$
$$\mathcal{L} = \{L \in \mathcal{M}_n(\mathbb{R}); \operatorname{supp}(L) \subseteq \mathcal{S}_1\},$$
$$\mathcal{Q} = \{B \in \widehat{Q}; \operatorname{supp}(B) \subseteq \mathcal{S}_2\}$$

for some $\mathcal{S} \subseteq [n] \times [n]$, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, and $C \in M_n(\mathbb{R})$, so that the underlying framework is well defined for $\mathcal{L}$ and $\widehat{\mathcal{A}} := C + \widehat{\mathcal{Q}}$.

Then $M = A + C \in \mathcal{A}$ has the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$ if $X = 0$ is the only matrix that satisfies:

$$\mathcal{S} \setminus \mathcal{S}_2 \subseteq \operatorname{supp}(X) \subseteq \mathcal{S}$$
$$\mathcal{S}_1 \cap \operatorname{supp}([X, M^\top]) = \emptyset$$

Tools of the Trade

Let $M = A + C \in \mathcal{A}$ have the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$.

Proximity Lemma. There exists $\varepsilon > 0$ such that for any $N \in \widehat{\mathcal{A}}$ with $\|N - M\| < \varepsilon$, a matrix $M' \in \mathcal{A}$ exists such that M' and N are similar, and M' has the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$.

Superspace Lemma. Let $M = A + C \in \mathcal{A}$ have the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ such that for any $N \in \widehat{\mathcal{A}}$ with $\|N - M\| < \varepsilon$, there exists a matrix $B' \in \mathcal{Q}$ such that $\|B'\| < \delta$, $M' := N - B'$ and M are similar, and M' has the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$.

Back to $\mathcal{H}_n$

Choose $M \in \mathcal{H}_n$, an integer $t \geq 0$, and set

- $\mathcal{L} = \{L \in \mathcal{M}_n(\mathbb{R}); \mathrm{supp}(L) \subseteq \cup_{j=t+1}^{n-1} \mathscr{D}_j\}$
- $\widehat{\mathcal{Q}} = \{Q \in \mathcal{M}_n(\mathbb{R}); \mathrm{supp}(Q) \subseteq \cup_{j=t}^{n-1} \mathscr{D}_j\}$
- $\mathcal{Q} = \{B \in \widehat{\mathcal{Q}} : \mathrm{supp}(B) \subseteq \mathrm{supp}(A)\}$.
- $C \in \mathcal{H}_n$ defined by $\cup_{j=t}^{n-1} \mathscr{D}_j \cap \mathrm{supp}(C) = \emptyset$ and $M - C \in \widehat{\mathcal{Q}}$.
  Restriction of C to $\widehat{\mathcal{Q}}$ is equal to zero, and C agrees with M outside the pattern of $\widehat{\mathcal{Q}}$.
- $\widehat{\mathcal{A}} = C + \widehat{\mathcal{Q}}$, and $\mathcal{A} = C + \mathcal{Q}$.

The underlying framework is well defined, and M has the nSSP for $(\mathcal{L}, \mathcal{A}, \widehat{\mathcal{A}})$, if zero is the only matrix $X \in \widehat{\mathcal{Q}}$ that satisfies $M \circ X = 0$ and $\cup_{j=t}^{n-1} \mathscr{D}_j \cap \mathrm{supp}([X, M^\top]) = \emptyset$.

Positive in the corner

$M \in \mathcal{H}_n^+$.

If zero is the only matrix X that satisfies:

- $X \geq 0$, $\operatorname{supp}(X) \subseteq \cup_{j=t}^{n-1} \mathscr{D}_j$,
- $M \circ X = 0$,
- $\cup_{j=t+1}^{n-1} \mathscr{D}_j \cap \operatorname{supp}([X, M^\top]) = \emptyset$

then

- M is similar to $N \in \mathcal{H}_n^+$ with $\cup_{j=t}^{n-1} \mathscr{D}_j \subseteq \operatorname{supp}(N)$.
- The characteristic polynominal of M has slack on $\mathbb{R}_{n-t-1}[x]$.

If the above holds for M, we will say that M has the strong spectral property (SSP) for $\cup_{j=t}^{n-1} \mathscr{D}_j$.

Example

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{2,1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & m_{4,2} & m_{4,3} & 0 & 1 \\ 0 & 0 & m_{5,3} & 0 & 0 \end{pmatrix}, m_{i,j} > 0.$$

- has the SSP for $\cup_{j=1}^{4} \mathscr{D}_j$ if and only if $m_{2,1} \neq m_{4,3}$.
- M does not have the SPP for $\cup_{j=t}^{4} \mathscr{D}_j$ for $t = 2, 3, 4$.

Assuming $m_{2,1} \neq m_{4,3}$:

- M is similar to a matrix that is positive outside the diagonal.
- The characteristic polynomial of M has slack on $\mathbb{R}_3[x]$.

Corner positive

Let $p(x)$ be the characteristic polynomial of $A \in \mathcal{H}_n^+$.
The following are equivalent:

1. $p_A(x)$ has slack on $\mathbb{R}_s[x]$.
2. A is similar to a nonnegative matrix that with positive entries on $\mathscr{D}_i$, $i = n-s-1, \ldots, n-1$.
3. A is similar to a nonnegative matrix that with at least one positive entry on each $\mathscr{D}_i$, $i = n-s-1, \ldots, n-1$.

$$A = \begin{pmatrix} ? & 1 & 0 & 0 & 0 \\ ? & ? & 1 & 0 & 0 \\ + & ? & ? & 1 & 0 \\ + & + & ? & ? & 1 \\ + & + & + & ? & ? \end{pmatrix}$$

Not All Entries Get to Play

Let $M \in \mathcal{H}_n$, consider the nSSP for $\cup_{j=t}^{n-1} \mathscr{D}_j$, and set $r = n - t$.

- L_r: the leading $r \times r$ submatrix of M,
- T_r: the trailing $r \times r$ submatrix of M, and
- W_r: the lower-triangular part of $M[\{t, \ldots, n\}, [r]]$, the lower triangular part of the $r \times r$ submatrix of M in the lower-left corner.

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & * & 1 & 0 \\ * & * & * & * & * & * & 1 \\ * & * & * & * & * & * & * \end{pmatrix}$$

Not All Entries Get to Play

$M \in \mathcal{H}_n$ has the nSSP for $\cup_{j=t}^{n-1} \mathscr{D}_j$, if $X' = 0$ is the only lower-triangular $r \times r$ matrix that satisfies:

- $X' \circ W_r = 0$
- $X' L_r^\top - T_r^\top X'$ is upper-triangular, i.e. the strictly lower-triangular part of $X' L_r^\top - T_r^\top X'$ is equal to zero.

- If a matrix M has the nSSP for $\cup_{j=t}^{n-1} \mathscr{D}_j$, then all matrices (of any size) that have the same leading and trailing $(r-1) \times (r-1)$ submatrices and the same pattern of W_r also have the same property.
- For $2r > n$ those submatrices are not independent.

Nonzero entries in W_r

$M \in \mathcal{H}_n$, $n = t + r$.

- If M has the nSSP for $\cup_{j=t}^{n-1} \mathscr{D}_j$, then W_r must have at least r nonzero terms. The space of lower-triangular $r \times r$ matrices has dimension $\frac{r(r+1)}{2}$. The condition that $X' L_r^\top - T_r^\top X'$ is upper-triangular gives $\frac{r(r-1)}{2}$ linear equations.
- If W_r has at least one nonzero term on each $\mathscr{D}_s$, $s = 0 \ldots, r-1$, then M has the nSSP for $\cup_{j=t}^{n-1} \mathscr{D}_j$.

GCD of cycles is $d \geq 2$

$$\mathscr{G}_d = \cup_{k=1}^{\lfloor \frac{n}{d} \rfloor} \mathscr{D}_{kd-1} \text{ and } \mathscr{F}_d = \cup_{k=1}^{\lfloor \frac{n-1}{d} \rfloor} \mathscr{D}_{kd},$$

$$M \in \mathcal{H}_n^+ \text{ with } \operatorname{supp}(M) \subseteq \mathscr{G}_d \cup \mathscr{D}_{-1}.$$

If zero is the only matrix X that satisfies:

- $X \geq 0$, $\operatorname{supp}(X) \subseteq \mathscr{G}_d$,
- $M \circ X = 0$,
- $\mathscr{F}_d \cap \operatorname{supp}([X, M^\top]) = \emptyset$

then

- M is similar to $N \in \mathcal{H}_n^+$ with $\mathscr{G}_d \subseteq \operatorname{supp}(N)$.
- the characteristic polynomial of M has slack on $\operatorname{span}\{x^{n-kd}, k = 1, \ldots, \lfloor \frac{n}{d} \rfloor\}$.

If the above holds for M, we will say that M has the SSP for $\mathscr{G}_d$.

Example

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{2,1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & m_{4,3} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, m_{i,j} > 0$$

- M has the SSP for $\mathscr{G}_2$ if and only if $m_{2,1} \neq m_{4,3}$.
- If $m_{2,1} \neq m_{4,3}$, then:
  - M is similar to a matrix that is positive on $\mathscr{D}_1 \cup \mathscr{D}_3$.
  - the characteristic polynomial of M has slack on $\operatorname{span}\{x, x^3\}$.

Example

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ m_{2,1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & m_{5,2} & 0 & 0 & 0 \end{pmatrix}, m_{i,j} > 0$$

M has the SSP for $\mathscr{G}_2$.

If $M \in \mathcal{H}_n$ with $\mathrm{supp}(M) \subset \mathscr{G}_d$ has at least one nonzero entry on every diagonal corresponding to $\mathscr{G}_d$, then it has the SSP for $\mathscr{G}_d$.

Final Thoughts

- $(0, +, -)$ sign patterns can be studied with the same methods.
- The strong spectral properties allow us to identify matrices that have slack on $\mathbb{R}[x]_k$ or $\operatorname{span}\{x^n - kd, k = 1, 2, \ldots, \lfloor \frac{n}{d} \rfloor\}$.
- They do not identify matrices that have slack on $\operatorname{span}\{x^{i_0}\}$.
- New methods are needed to study c-extreme matrices.

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