Patterns that require all simple singular values

Requires simple problem

Recall, that in graph and eigenvalue setting, Fielder characterized the paths as those graphs G for which every matrix in $\mathcal{S}(G)$ has all simple eigenvalues.

Here we study the analogous problem for patterns and singular values. Namely, we ask: which patterns P have the property that no matrix in $\mathcal{Z}(P)$ has a multiple singular value?

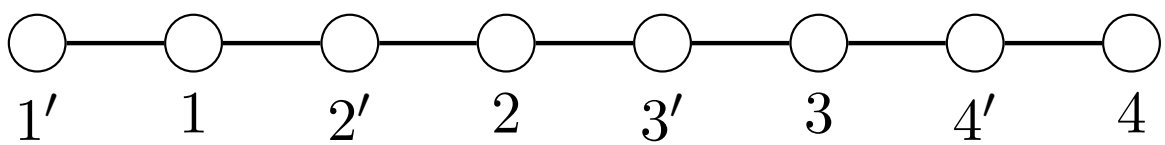
We first study this problem under the additional assumptions that $P = [p_{ij}]$ has term-rank m and $m = n$. In this setting, we may assume that P is normalized, that is $p_{ii} = *$ for $i = 1, \ldots, m$.

Let P be the $m \times m$ pattern with $*$ on the diagonal and the superdiagonal and zeros elsewhere.

The bigraph of P is a path on $2m$ vertices.

E.g., for $m = 4$, $P = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$.

The bigraph is



For $A \in \mathcal{Z}(P)$, $AA^\top \in \mathcal{S}(P_m)$, where P_m is the path



Thus, $AA^\top$ has no repeated eigenvalue. Hence, the pattern P requires all simple singular values.

Given Fiedler's theorem for graphs, one might expect these are the only P that require all simple singular values.

After all, for other patterns P, if $A \in \mathcal{Z}(P)$, the graph G of $AA^\top$ will not be a path, and $S(G)$ allows a matrix with a multiple eigenvalue.

Surprise!

Consider a matrix of the form

$$A = \begin{bmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & d & e & f \\ 0 & 0 & 0 & g \end{bmatrix}$$

where $a, \ldots, g \neq 0$. So A is invertible.

$$AA^\top = \begin{bmatrix} a^2+b^2 & bc & bd & 0 \\ bc & c^2 & cd & 0 \\ bd & cd & d^2+e^2 & fg \\ 0 & 0 & fg & f^2 \end{bmatrix}.$$

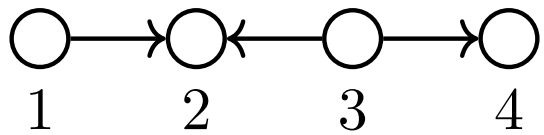
Let σ be a singular value of A and note that

$$(AA^\top - \sigma^2)[4,1] = \left[\begin{array}{cc|c} bc & bd & 0 \\ c^2-\sigma^2 & cd & 0 \\ \hline cd & d^2+e^2-\sigma^2 & fg \end{array}\right]$$

has determinant $fg \cdot bd \cdot \sigma^2 \neq 0$. Hence the multiplicity of σ as a singular value of A is 1.

The pattern of A is an example of one that requires simple singular values, that is different than the expected path pattern!

Let $P = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$. Note the unlooped digraph of P is a weak path (that is, a directed graph with no 2-cycles whose underlying graph is a path).



More generally, a weak path looks like

$$1 = i_0 \rightsquigarrow i_1 \nleftrightsquigarrow i_2 \rightsquigarrow i_3 \nleftrightsquigarrow i_3 \cdots \rightsquigarrow i_k = m$$

Weak paths imply requires all simple singular values

One can show that:

P a normalized pattern whose unlooped digraph is a weak path,
$A \in \mathcal{Z}(P)$ and
σ a singular value of A

$\Downarrow$

A invertible,
$(AA^\top - \sigma I)[m, 1]$ is block-lower triangular, and
each of its blocks is a 1×1 nonzero block, or
a block of the form $\begin{bmatrix} bc & bd \\ c^2 - \sigma_2 & cd \end{bmatrix}$ for some nonzero b and c.

$\Downarrow$

$AA^\top - \sigma I$ has nullity at most 1, and
σ is a singular value of A of multiplicity at most 1.

We wish to show that the normalized $m \times m$ patterns that require all singular values are simple are those whose digraph is a weak path.

Our plan of attack is:

(a) Find normalized patterns F such that F allows a matrix with the SSVP and a multiple singular value.

(b) Use the Direct Sum theorem and the Superpattern theorem to conclude that if the normalized pattern P requires all simple singular values, then P does not contain any such F as principal submatrix.

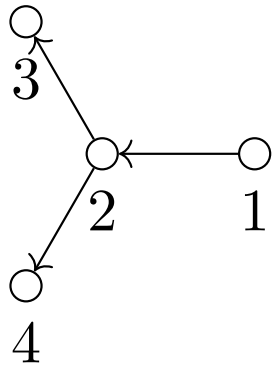
(c) Show that avoiding the F gives P a special form.

(d) Show that each P with the special form requires all simple singular values.

Requires simple $\Longrightarrow$ weak path

The pattern

$$F_1 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$



allows a matrix that has the SSVP and a multiple singular value. As does $F_1^\top$.

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Since $(AA^\top)(\{2\}) = 2I_3$, $\sqrt{2}$ is a multiple singular value of A.

To show that A has the SSVP, consider a matrix of the form

$$X = \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & 0 \\ d & e & 0 & f \\ g & h & i & 0 \end{bmatrix}$$

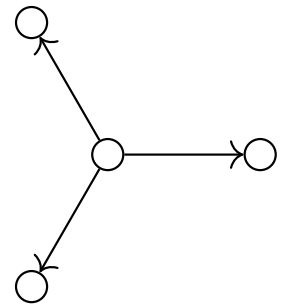
for which both $AX^\top$ and $X^\top A$ are symmetric. These yield the following system of equations

$$\begin{array}{ll} c = 0, & c = a + b, \\ a = \sqrt{2}d + c, & \sqrt{2}a = d + e, \\ b = \sqrt{2}g + c, & \sqrt{2}b = g + h \\ a = \sqrt{2}e, & e + f = 0 \\ b = \sqrt{2}h, & h + i = 0, \\ \sqrt{2}i = \sqrt{2}f. & \end{array}$$

whose only solution is the trivial one. Hence A has the SSVP.

The pattern

$$F_2 = \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$



allows a matrix with multiple singular values and the SSVP.

Note that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\top - I = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Hence F_2 (as does $F_2^\top$) allows a matrix with 1 as a singular value of multiplicity 2.

Now let $X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & b & c \\ d & e & 0 & f \\ g & h & i & 0 \end{bmatrix}$ be such that $XP^\top$ and $P^\top X$ are both symmetric.

The latter gives $a, d, g = 0$; $b = e$; $c = h$; and $f = i$.

The former gives $a + b + c = 0$; $d + e + f = 0$; and $g + h + i = 0$.

Together these imply $X = O$, and P has the SSVP.

We've also seen the each normalized pattern whose loopless digraph is a weak cycle allows a matrix with a multiple singular value and the SSVP.

Characterization of square patterns that require all simple singular values

Let P be an $m \times m$ normalized pattern that requires all singular values are simple.

Then the underlying graph of P is connected; otherwise, up to permutation, P is a direct sum, and it is easy to get a multiple singular value.

By the Direct Sum Theorem, and Superpattern Theorem, P does not contain any principal submatrix that is a superpattern of F_1, F_2, $F_2^\top$, or any adjacency matrix of a looped weak-cycle of length at least 2. Similarly, for the transposes of these.

Hence, the underlying graph of P is acyclic, and in the unlooped digraph of P, the out-degree and in-degree of each vertex is at most 2, and the sum of the in- and out-degree of each vertex is at most 2. It follows that the digraph of P is a weak path.

Theorem

Let P be a normalized $m \times m$ pattern. Then each matrix with pattern P has all simple singular values if and only if the unlooped digraph of P is a weak path.

Rectangular, full term-rank case

In the rectangular case, we need to with a graphical represention, as well as possiblity having more than one 'normalization.'
Here we look at the bigraph, $\mathcal{B}_P$ of the $m \times n$ pattern $P = [p_{i,j}]$. Vertices are $1, \ldots,$m and $1', \ldots, n'$, edges are i–j' if and only if $p_{i,j} = *$.

We'll assume P $m \leq n$ and P has term-rank m.

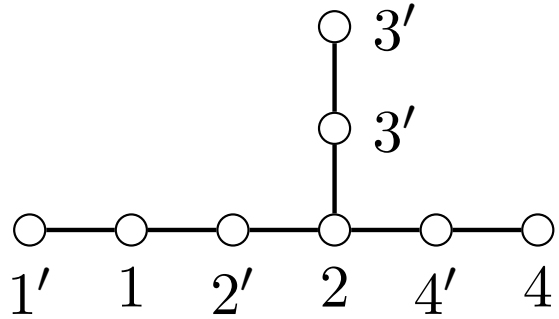
Note: independent nonzeros of P corresponding to matchings of $\mathcal{B}_P$. A submatrix $P[\alpha, \beta]$ (respectively, induced subgraph of $\mathcal{B}_P$ is **matched** is $P(\alpha, \beta)$ has a set of $m - |\alpha|$ independent nonzeros (respectively, the subgraph of $\mathcal{B}_P$ induced on the vertices not in $\alpha \cup \beta$ has a matching that covers each of its row-vertices).

Note if $P[\alpha, \beta]$ allows a matrix with the SSVP and a multiple singular value, and is matched, then by Supergraph theorem and Direct Sum theorem every superpattern of P allows a matrix with a multiple singular value.

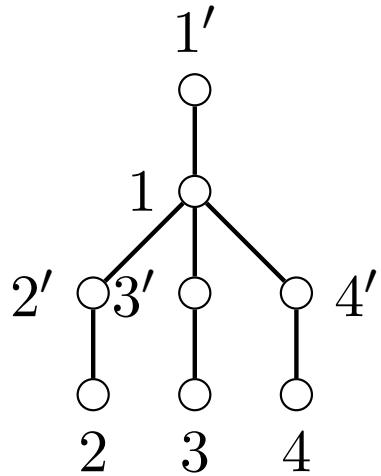
Forbidden subgraphs

We do have examples of patterns that allow the SSVP and a multiple singular value. Their bigraphs are:

$$F_1 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

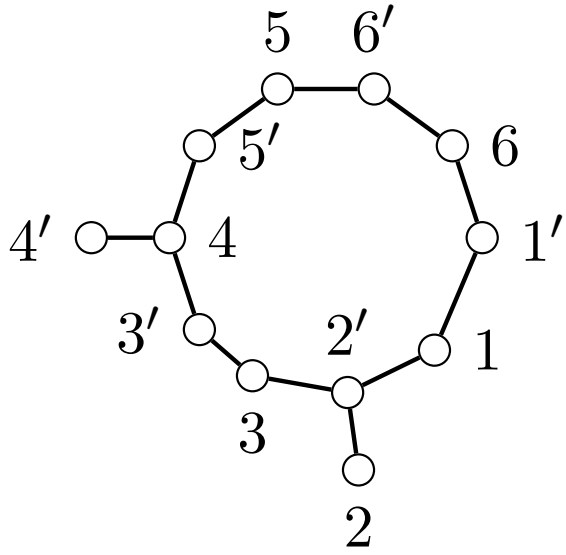


$$F_2 = \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$



Any weak cycles, e.g.,

$$C = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & 0 & * \end{bmatrix}$$



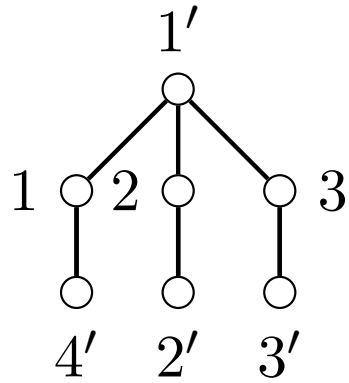
Their bigraphs are those unicyclic bigraphs with a unique perfect matching (exercise).

These are precisely the **bipartite, unicylilc caterpillars with a perfect matching**.

A new forbidden pattern

Consider the pattern

$$F_4 = \begin{bmatrix} * & 0 & 0 & * \\ * & * & 0 & 0 \\ * & 0 & * & 0 \end{bmatrix}$$



Note that the bi-adjacency matrix of F_4 satisfies $AA^\top = I_3 + J_4$, and hence P has singular values $\sqrt{2}$, 1 and 1.

It is also not difficult to show that P has the SSVP (Exercise).

Thus, by the Direct aum and Superpattern theorems, if $P[\alpha, \beta]$ is permutationally equivalent to F_4 and $P(\alpha, \beta)$ and matched, then P allows a matrix with a multiple eigenvalue.

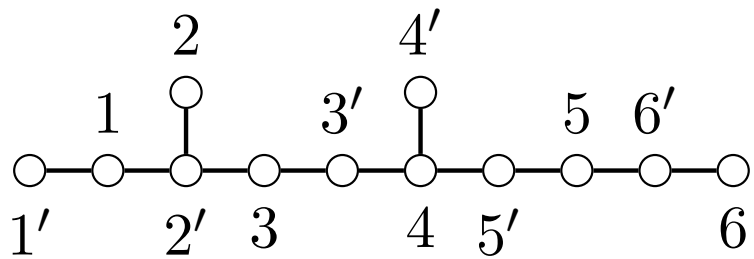
Bigraphs of weak paths

If P is an m $\times$ m pattern of a weak-path, then we know that P has term-rank m and allows a matrix with a multiple eigenvalue.

Bigraphs of weak paths are simply **caterpillars with a unique perfect matching**.

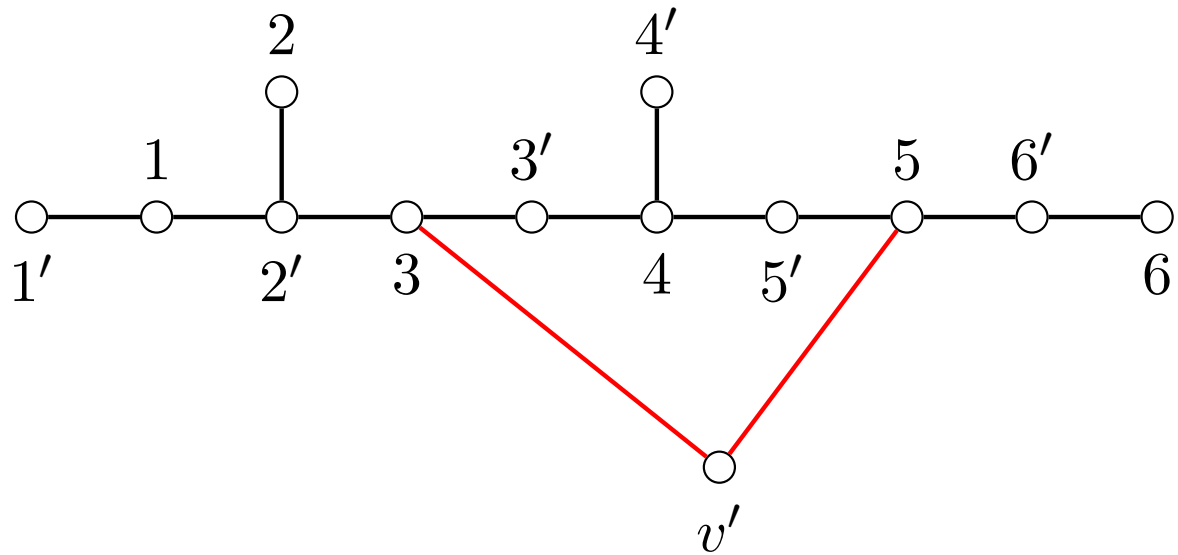
If P is an $m \times m$ pattern of a weak-path, then we know that P has term-rank m and allows a matrix with a multiple eigenvalue.
Bigraphs of weak cycles are simply **caterpillars with a unique perfect matching.**
For example,

$$W = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$



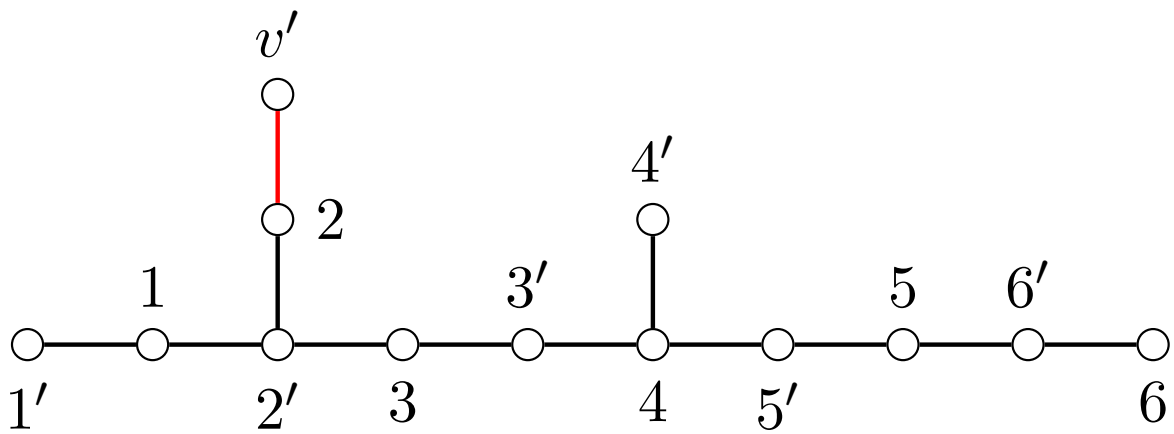
We wish to insert a new non-isolated column vertex v' to the bigraph of a weak path, and obtain a corresponding pattern that requires all simple singular values.

If v' has degree two or more, then we have a matched unicyclic caterpillar.

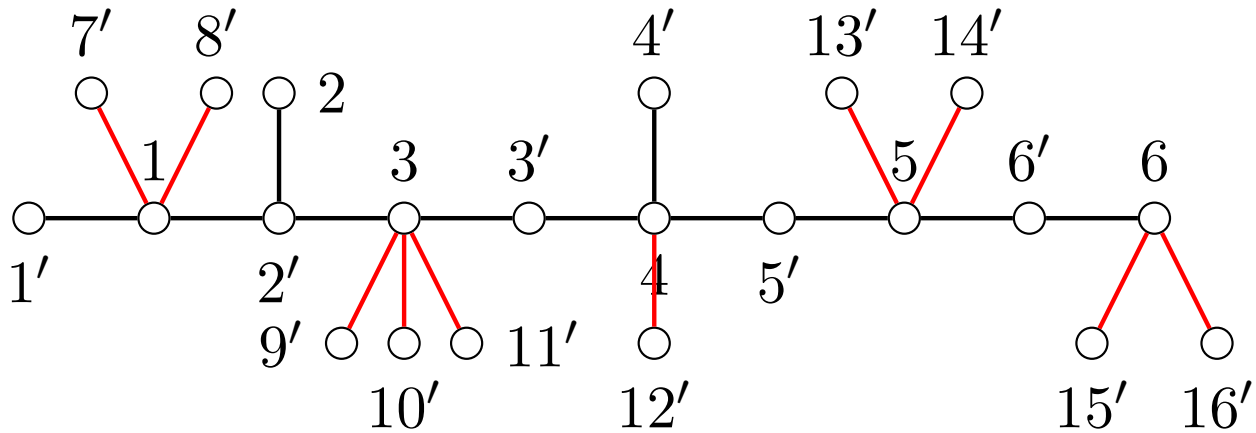


We wish to insert a new non-isolated column vertex v' to the bigraph of a weak path, and obtain a corresponding pattern that requires all simple singular values.

If v' is incident to a leg of the caterpillar, then we have a matched F_4:



Thus, the only way to add new non-isolated column vertices and maintain the requires all simple property is if these are pendent and adjacent to the row vertices on the spine of the caterpillar.



Matrix-wise this results in a matrix of the form

$$W = [A\ B] = \left[\begin{array}{cccccc|cccccccccc} * & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{array}\right].$$

$WW^T = AA^\top +$ a diagonal matrix.

So $WW^\top(m,1) - \sigma^2 I$ and $AA^\top(m,1) - \sigma_2 I$ will both be block lower triangular with the exact same diagonal blocks!

So $WW^\top(m,1) - \sigma^2 I$ has rank $m-1$.

Thus, each singular value of W is simple!

To reduce to the above, one needs to show that if the $m \times n$ P requires all simple singular values, then there is a choice of a matching that when P is normalized by that matching the leading $m \times m$ has connected bigraph.

Thus, one can completely characterize those $m \times n$ patterns P with $m \leq n$ with **term-rank equal to** m that require all simple eigenvalues.

Open problem : what if term-rank is less than m?

Only need to consider term-rank $m - 1$.

Open question becomes: which $m \times n$ patterns of term-rank $m - 1$ require all simple singular values?

Difficulty, use of strong properties is out!

Future work needed

1. Establish and explore consequences of the Decontraciton Theorem (probably Butterfly contractions are the way to go).

2. Relate this work to the Asymmetric Strong Arnol'd Property work

3. Study associated minor monotone parameters, eg. maximum multiplicity of $A \in \mathcal{Z}(P)$ with the SSVP, the largest hap of A in $\mathcal{Z}(P)$ (Recall $m - \mathrm{hap}(P)$ is the minimum number of distinct singular values for a matrix with pattern P?

4. Unified treatment of the inverse singular value problem for the matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

arising in control theory,