Historical insights

After all, what is art?
Art is the creative process and it goes through all fields.
Einstein's theory of relativity – now that is a work of art!
Einstein was more of an artist in physics than on his violin.

Art is this:
art is the solution of a problem
that cannot be expressed explicitly
until it is solved.

–Piet Hein

Goals for the Lecture

Highlight some of the historical flashes of brilliance (Art!) that led to current research.

Provide sketches of results that illustrate some important techniques.

Provide motivation for Strong Properties.

1950s–the romance between matrices and graphs begins

The 1950s saw the emergence of two separate areas: Numerical Linear Algebra and Graph Theory.

Seymour Parter, one of founders of numerical linear algebra saw the power of bringing these two areas together.

Matrices arising in applications have structure that can be encapsulated by graphs leading to a better understanding of the algebraic, analytic and geometric properties of the matrices.

Art is this!

Parter was interested in generalizing an important result about Jacobi matrices.

A Jacobi matrix is an $n \times n$ matrix whose superdiagonal and subdiagonal entries are nonnegative, and all other off-diagonal entries are 0. Additionally, if each super- and subdiagonal entry is positive, then the matrix is an irreducible Jacobi matrix.

E.g.,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 2 & 0 & \pi \\ 0 & 0 & \sqrt{2} & \ln 2 \end{bmatrix} \text{ is a Jacobi matrix.}$$

An important algorithm, the Jacobi method, for finding eigenvalues of symmetric matrices uses the fact:

If A is an $n \times n$ symmetric matrix and has no multiple eigenvalues, then A is orthogonally similar (via a sequence of Given's rotation similarities) to a Jacobi matrix.

Parter wanted to find a way to deal with multiple eigenvalues. Key to this was generalizing:

Theorem (Cayley, Jacobi)

The eigenvalues of an irreducible, tridiagonal $n \times n$ matrix are real and simple.

Proof.

If $A = [a_{ij}]$ is a Jacobi matrix, then one can easily find $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ so that DAD^{-1} is a symmetric Jacobi matrix. So all eigenvalues of A are real, and the multiplicity of λ is $\text{nul}(A - \lambda I)$.

Note

$$(A - \lambda I)(n, 1) = \begin{bmatrix} a_{12} & 0 & \cdots & 0 \\ & a_{2,3} & \ddots & \vdots \\ & & \ddots & 0 \\ & & & a_{n-1,n} \end{bmatrix}$$

is lower triangular with nonzeros on the diagonal. So $\text{rk}(A - \lambda I) \geq n - 1$.



Graphs of matrices

Parter noted that the structure of the Jacobi matrix was path-like. This led him to essentially define:

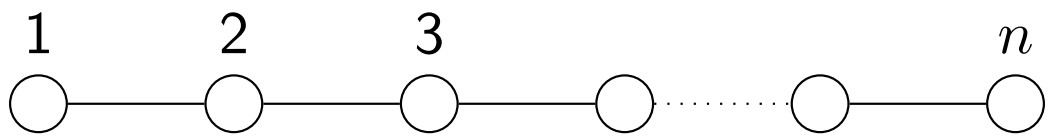
The graph $G(A)$ of an $n \times n$ symmetric matrix $A = [a_{ij}]$ consists of

Vertices $1, \ldots, n$

Edges: ij where $i \neq j$ and $a_{ij} \neq 0$.

Examples:

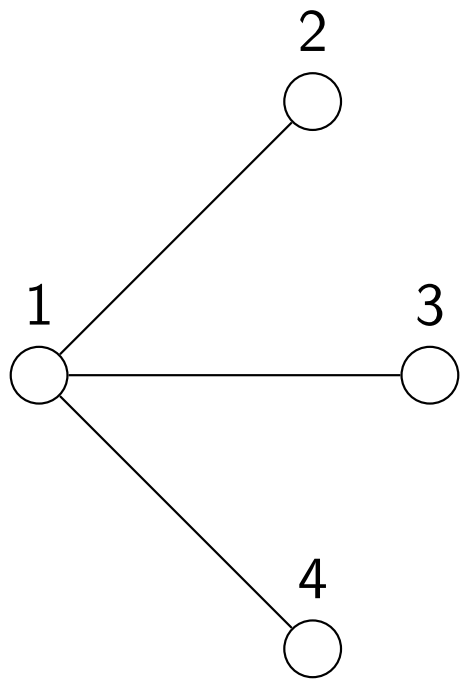
The graph of an $n \times n$ Jacobi matrix is the path P_n



Graph of

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

is the star $K_{1,3}$.



Given a graph G on n vertices,

$\mathcal{S}(G) := \{A : A = [a_{ij}]$ is $n \times n$ symmetric matrix with $G(A) = G\}$.

E.g., $\mathcal{S}(K_{1,3})$ is all matrices of the form

$$\begin{bmatrix} a & b & c & d \\ b & e & 0 & 0 \\ c & 0 & f & 0 \\ d & 0 & 0 & g \end{bmatrix}, \text{ where } b, c, d \neq 0.$$

Matrices whose graph is a tree

Recall, a tree is a connected graph with no cycles. Examples are paths, stars.

Parter focused on $\mathcal{S}(T)$ where T is a tree.

He proved two main results about symmetric A whose graph is a tree T:

Theorem (Parter, 1960)

$$\lambda \textit{ is a multiple eigenvalue of } A$$
$$\Downarrow$$
$$\exists\, i \textit{ such that each eigenvector } \mathbf{x} \textit{ of } A \textit{ corresponding to } \lambda \textit{ has } x_i = 0.$$

For spring mass systems corresponding to trees, a multiple angular frequency $\sqrt{|\lambda|}$ can only be achieved when there is some i such that the i-th mass is motionless for all vibrations that give that angular frequency.

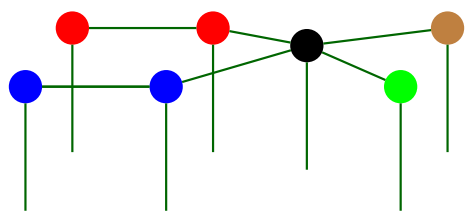
Theorem (Parter 1960; Wiener 1984)

λ *an eigenvalue of* A *with multiplicity* $m \geq 2$

$\Downarrow$

$\exists\ i$ *such that* λ *is an eigenvalue of* $A(i)$ *with multiplicity* $m+1$,
and
branches β_1, β_2, β_3 *at* i *such that each of*
$A[\beta_1]$, $A[\beta_2]$, $A[\beta_3]$ *has* λ *as an eigenvalue.*

For spring mass systems, this implies that a multiple frequency comes from at least three connected subregions that share the frequency. (Vertical lines are posts; other lines are springs. All movement is horizontal.)



Multiplicity 3 requires that the black vertex has no movement, and a common frequency at each of four subsystems with common color.

Proof of the first Parter theorem

Theorem (Parter, 1960)

$$\lambda \textit{ a multiple eigenvalue of } A$$
$$\Downarrow$$
$$\exists\, i \textit{ such that each eigenvector } \mathbf{x} \textit{ of } A$$
$$\textit{corresponding to } \lambda \textit{ has } x_i = 0.$$

Proof. Without loss of generality, $\lambda = 0$.

λ multiple implies there is a nonzero null vector $\mathbf{x}$ with at least one entry zero.

$\exists\, i$ and j such that $x_i = 0$, $x_j \neq 0$ and ij an edge of T. Wlog, $i = 1$, $j = 2$.

Then A and $\mathbf{x}$ have the form

$$\left[\begin{array}{c|c|c|c|c} a_{11} & a_1\mathbf{e}_{n_1}^\top & a_2\mathbf{e}_{n_2}^\top & \cdots & a_c\mathbf{e}_{n_c}^\top \\ \hline a_1\mathbf{e}_{n_1} & B_1 & O & \cdots & O \\ \hline a_2\mathbf{e}_{n_2} & O & B_2 & \ddots & \vdots \\ \hline \vdots & \vdots & \ddots & \ddots & O \\ \hline a_c\mathbf{e}_{n_c} & O & \cdots & O & B_c \end{array}\right] = \begin{bmatrix} a_{11} & \mathbf{w}^\top \\ \mathbf{w} & A(1) \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_c \end{bmatrix},$$

where the B_i correspond to the branches of T at 1.

Since $A\mathbf{x} = \mathbf{0}$, $\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$ is a vector in the null space of $A(1)$ that is not orthogonal to $\mathbf{w}$. Hence, the first column of A is not in the span of the other columns of A.

Therefore, every null vector of A has a 0 in its first entry. □

Fiedler's entrance

Theorem (Fiedler, 1969)

$$\exists \textit{ a matrix in } \mathcal{S}(G) \textit{ with a multiple eigenvalue}$$

$$\Updownarrow$$

$$G \textit{ is not a path.}$$

Actually, this proves something stronger.

Theorem (Fiedler, 1969)

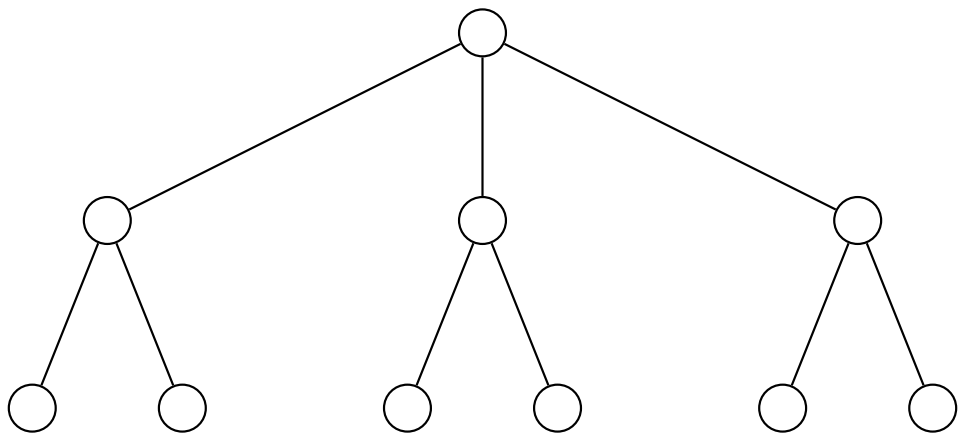
$$A \textit{ an } n \times n \textit{ symmetric matrix and } G(A) \textit{ not a path}$$

$$\Updownarrow$$

$$\exists \textit{ diagonal matrix } D \textit{ such that } D + A \textit{ has nullity at least } 2.$$

Fiedler & Eigenvectors

In 1975, Fiedler further studied $\mathcal{S}(T)$ when T is a tree. This time he focused on eigenvectors.



His results connected back to historical results (about vibrating surfaces) and initiated the study of nodal domains for graphs (which we discuss later).

To appreciate Fiedler's result let's look at an example.

$$A = \begin{bmatrix} -4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -10 & -3 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

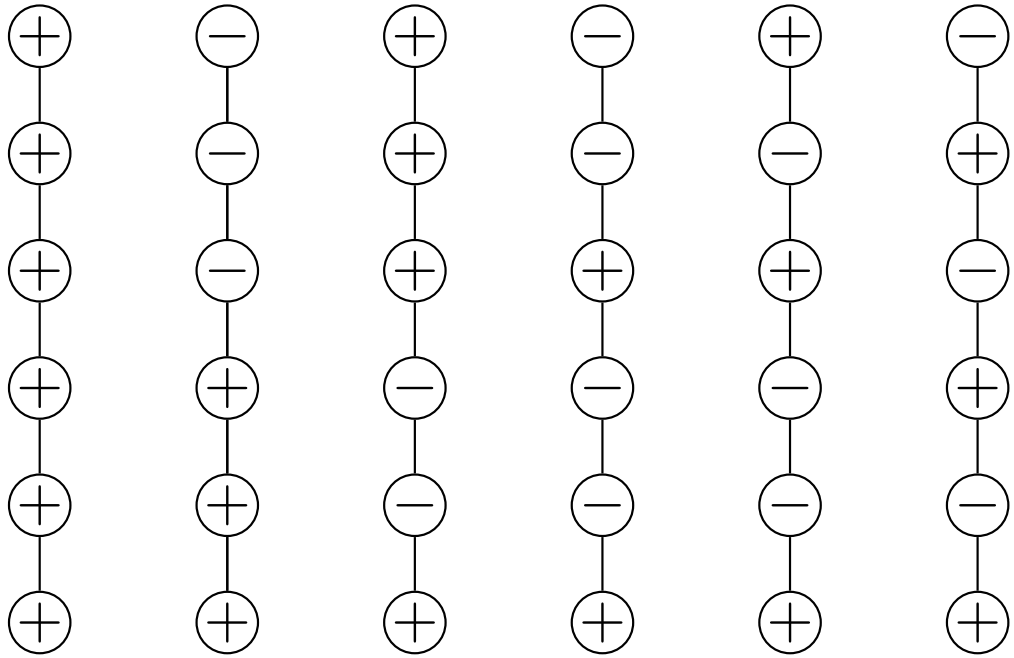
The columns of

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 8.0 & 0.25 & -2.0 & -8.0 & -8.0 & -16.0 \\ 64.0 & 0.12 & -8.0 & -16.0 & -8.0 & 4.0 \\ 260.0 & -0.031 & 1.0 & 16.0 & 32.0 & -0.50 \\ 130.0 & -0.031 & 2.0 & -64.0 & -130.0 & 0.50 \\ 8.0 & -0.0039 & 0.50 & -32.0 & 260.0 & -0.12 \end{bmatrix}$$

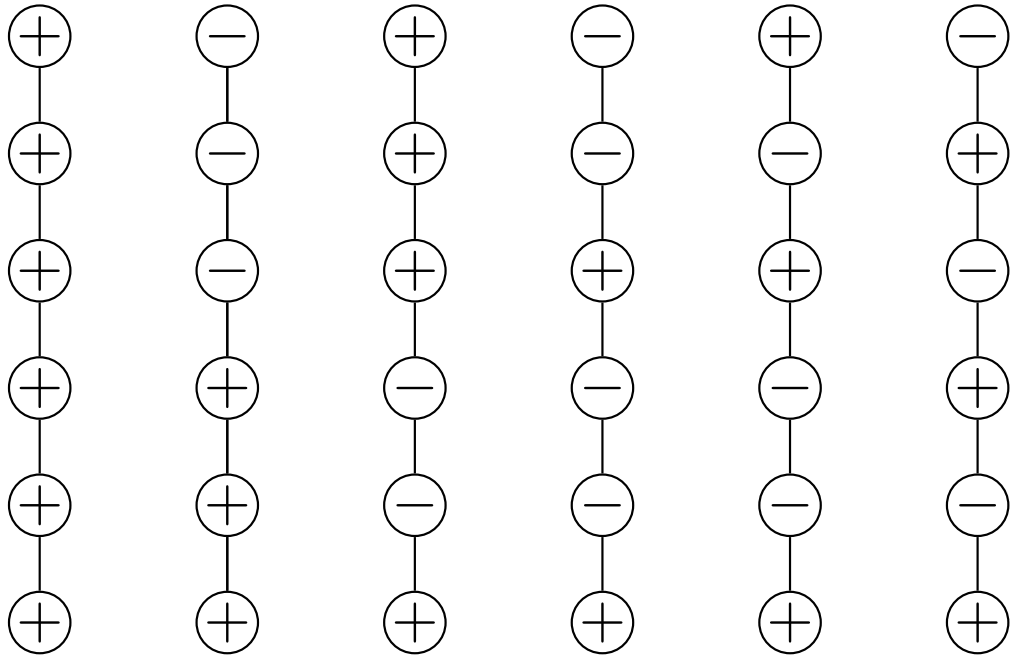
are a basis of eigenvectors of A; jth column corresponds to the j-th smallest eigenvalue.

Notice anything?

Note the graph of A is a path on 6 vertices.
Let's look at the sign patterns of these columns to label the vertices of the path. We obtain six labelings.



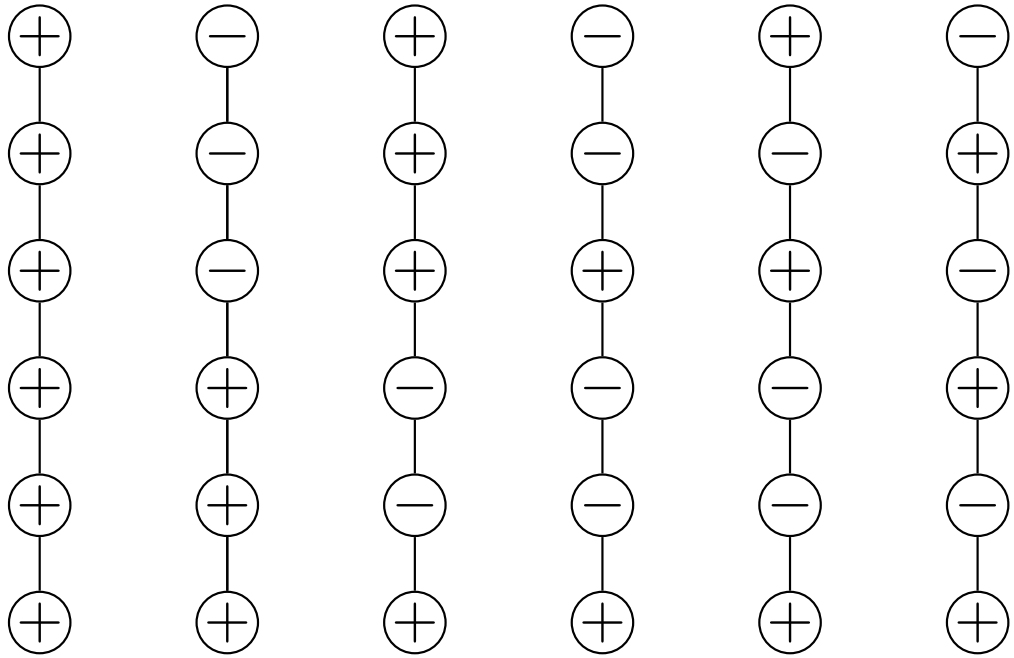
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Number of sign changes:

0 1 2 3 4 5

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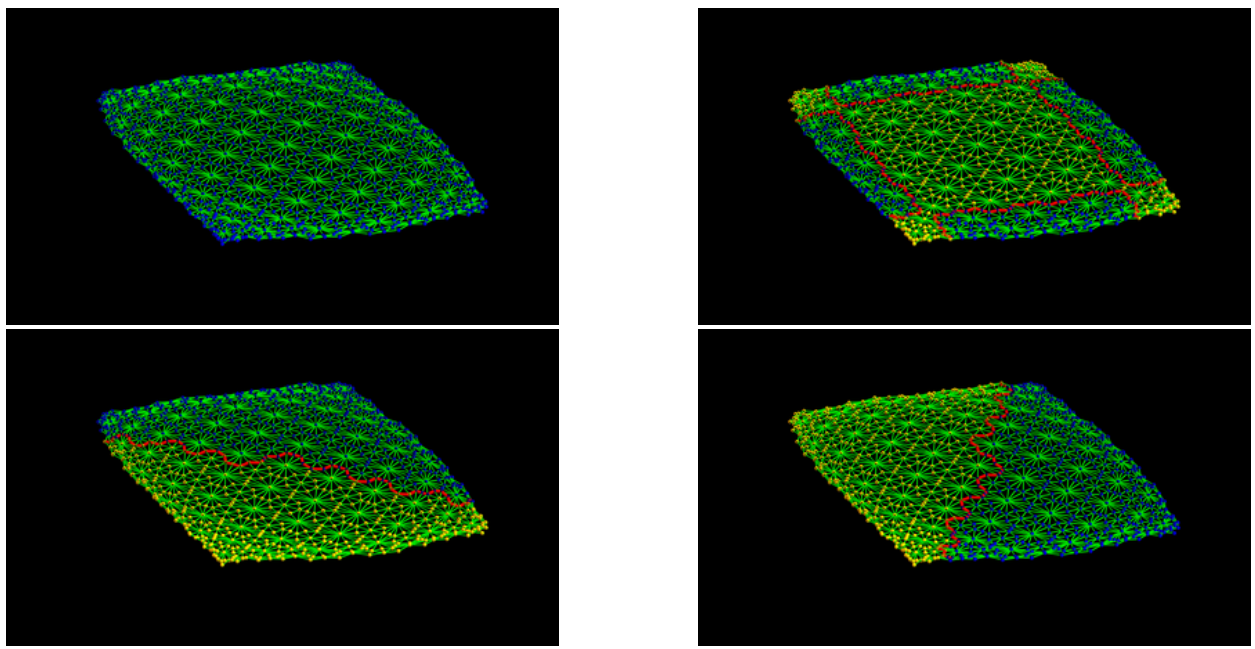
Number of e-values less than λ_k

0 1 2 3 4 5

It seems that the sign-patterns of an eigenvector of $A \in \mathcal{S}(T)$ (T a tree) tell one the location of the eigenvalue among all eigenvalues! This generalizes a famous result about the Laplace operator on a surface.

Courant's Nodal Domain Theorem, Courant and Hilbert, 1923]

Let f be an eigenfunction of the Laplace operator on the surface S whose corresponding eigenvalue is the k-th smallest. Then $S \setminus \{p \in S : f(p) = 0\}$ has at most k connected components.



Images due to Oliver Knill [KN]

Fiedler's nodal domain theorem for trees

Let $A \in \mathcal{S}(T)$, where T is a tree.
By diagonal scaling, we may assume that A is a Z-matrix; that is, all off-diagonal entries are non-positive.

Let $\mathbf{v}$ be an eigenvector of A corresponding to the eigenvalue λ with largest number of nonzeros.

$\mathbf{v}$ partitions the vertices of A into 3 sets:
$\mathcal{I}$: those i such that $v_i = 0$ and $v_j = 0$ for all j adjacent to i.
$\mathcal{P}$: those i such that $v_i = 0$ and $v_j \neq 0$ for some j adjacent to i.
$\mathcal{S}$: those i such that $v_i \neq 0$.

Let
$n_-(\lambda)$ be the number of eigenvalues of A less than λ;
$n_+(\lambda)$ be the number of eigenvalues of A greater than λ;
$n_0(\lambda)$ be the multiplicity of λ as an eigenvalue of A.

Theorem

If $\mathcal{I} = \emptyset$, then

- $n_+(\lambda) = |\mathcal{P}| +$ *the number of edges i–j in T with $v_i v_j > 0$.*
- $n_-(\lambda) = |\mathcal{P}| +$ *the number of edges i–j in T with $v_i v_j < 0$.*
- $n_0(\lambda) = (\text{number of connected components of } T[S]) - |P|$.
- *If λ is k-th smallest eigenvalue, then the number of connected components of $T[S]$ is at most k.*

Art is this!

$\exists$ much recent work on extending this to general graphs (see Urschel (2018) and references therein).

Minimum rank/Maximum Nullity

Theorem (Duarte, 1989)

Let T be a tree on n vertices, and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then $\exists$ a matrix in $\mathcal{S}(T)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.

The proof is by induction on n, but proves the stronger result:

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n \text{ and an index } i$$
$$\Downarrow$$
$$\exists\ A \in S(T) \text{ such that } A \text{ has the } \lambda \text{ as eigenvalues, and } A(i) \text{ has the } \mu \text{ as eigenvalues.}$$

Monfared-S (2015) show can extend to any graph. Proof is not constructive, but led to what is now called the Strong Spectral Property.

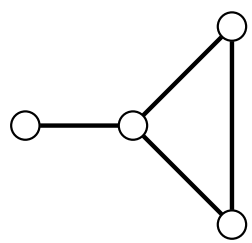
This is art!

Lower bounds on maximum nullity

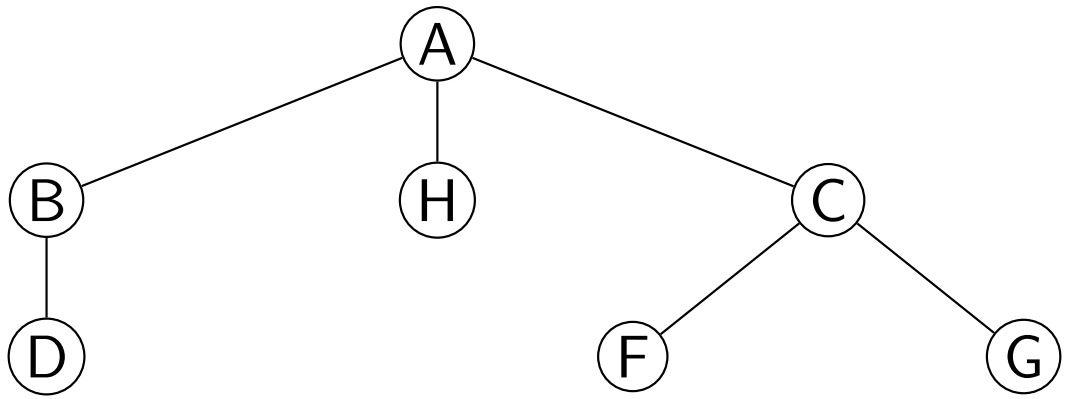
The path cover number, $P(G)$, of a graph G is the minimum number of induced paths of G whose vertices partition the vertex set of G.

E.g.,

- $P(G) = 1$ if and only if G is a path.
- $P(K_{1,n}) = n - 1$.
- $P(\text{Paw}) = 2$.



A peripheral path of a graph G is an induced path in G such that its terminal vertices have degree 1 in G and the path is incident to at most one vertex of degree 3 or more.



D–B–A–C–F is not peripheral.
D–B–A–H is peripheral.
D–B–A–C is not peripheral.

Theorem (Nylen 1996)

Let γ be peripheral path of G with vertex set α. Then

(a) $M(G) - 1 = M(G(\alpha))$

(b) *If G is a tree, $P(G) - 1 = P(G(\alpha))$.*

(c) *If G is a tree, then $M(G) = P(G)$.*

Proof.

(a): Let v the unique high degree vertex of γ.
Note $M(G(v))$ is the sum of the $M(C)$, where C is connected component of $G(v)$.
Two of these branches are the paths in $G[\alpha - \{v\}]$.
Thus,

$$M(G(v)) = M(G(\alpha)) + 2$$

Since, in general $M(G) \geq M(G(v)) - 1$, we conclude that

$$M(G) \geq M(G(\alpha)) + 1. \tag{1}$$

Take $A \in \mathcal{S}(G)$ we an assume that A has the form

$$\begin{bmatrix} a_{11} & \mathbf{x}_1{}^\top & \mathbf{x}_2{}^\top & \mathbf{x}_3{}^\top \\ \mathbf{x}_1 & B_1 & O & O \\ \mathbf{x}_2 & O & B_2 & O \\ \mathbf{x}_3 & O & O & B_3 \end{bmatrix},$$

where $v = 1$, B_1 and B_2 correspond to the legs of γ and are tridiagonal, and the only nonzero in $\mathbf{x}_i$ is the first position for $i = 1, 2$.

Let j and k be the pendent vertices of γ with $j < k$. Then the matrix $B(j, k)$ has the form:

$$\left[\begin{array}{ccc|cccc|c} a_{11} & a_{12} & 0 & a_{15} & 0 & 0 & 0 & \mathbf{y}^\top \\ \hline a_{12} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ 0 & a_{23} & a_{33} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ 0 & 0 & a_{34} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ \hline a_{15} & 0 & 0 & a_{55} & a_{56} & 0 & 0 & \mathbf{0}^\top \\ 0 & 0 & 0 & a_{65} & a_{66} & a_{67} & 0 & \mathbf{0}^\top \\ 0 & 0 & 0 & 0 & a_{67} & a_{77} & a_{78} & \mathbf{0}^\top \\ \mathbf{y} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_3 \end{array}\right]$$

$$\left[\begin{array}{c|ccc|cccc|c} a_{11} & a_{12} & 0 & a_{15} & 0 & 0 & 0 & \mathbf{y}^\top \\ \hline a_{12} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ 0 & a_{23} & a_{33} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ 0 & 0 & a_{34} & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\ \hline a_{15} & 0 & 0 & a_{55} & a_{56} & 0 & 0 & \mathbf{0}^\top \\ 0 & 0 & 0 & a_{65} & a_{66} & a_{67} & 0 & \mathbf{0}^\top \\ 0 & 0 & 0 & 0 & a_{67} & a_{77} & a_{78} & \mathbf{0}^\top \\ \hline \mathbf{y} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_3 \end{array}\right]$$

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a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}^\top \\
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\left[\begin{array}{c|c|c|c|c|c|c|c}
0 & 0 & 0 & a_{15} & 0 & 0 & 0 & \mathbf{y}^\top \\ \hline
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\left[\begin{array}{c|c|c|c|c|c|c|c}
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By row and column operations, we get that $A(j,k)$ is permutationly equivalent to a matrix of the form $D \oplus A(\alpha)$ where D is invertible diagonal matrix of order $|\alpha| - 1$. Hence, $\operatorname{nul} A(j,k) = \operatorname{nul} A(\alpha)$.

However, deleting a row and column of a matrix decreases the nullity by at most 1. So

$$\operatorname{nul} A - 1 \leq \operatorname{nul} A(j,k) = \operatorname{nul} A(\alpha).$$

Taking A with $\operatorname{nul} A = M(G)$ gives $M(G) \leq M(G(\alpha)) + 1$.

Combined with (1), we have $M(G) = M(G(\alpha)) + 1$.

Proof of (b).

Including γ into any path cover of $P(G(\alpha))$ gives a path cover of G. So

$$P(G) \leq P(G(\alpha)) + 1.$$



Surgery on any path cover of G with t paths can be done to give a path cover of G with at most t paths one of which is γ and the others form a path cover of $P(G(\alpha))$. So

$$P(G(\alpha)) \leq P(G) - 1.$$

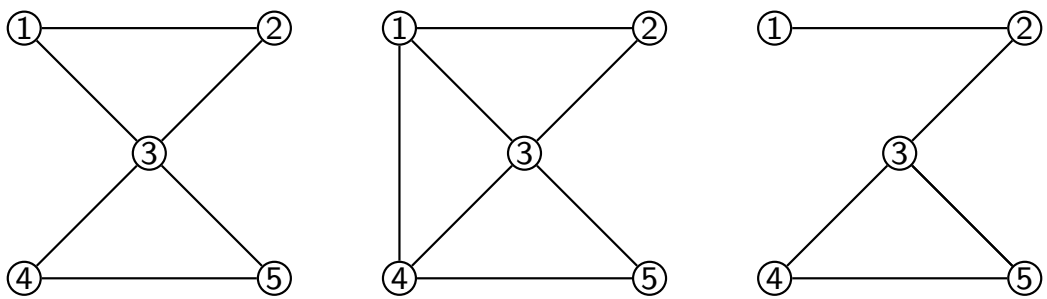
(c) follows from (b), the facts that every tree on $n \geq 2$ has a peripheral path, and $M(\text{Path}) = P(\text{Path}) = 1$.

Note that the Bow Tie graph has minimum rank at most 2 (hence maximum nullity at least 3). Max nullity $n-1$ requires, K_n. So $M(\text{Bow Tie}) = 3$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(\text{Bow Tie})$$

But inserting one edge (or deleting an edge) of Bow Tie creates a graph Bow Tie $+ 14$ (Bow Tie $- 13$) with an induced 4-path. and hence

$$M(\text{Bow tie} + 14) < M(\text{Bow Tie})$$
$$M(\text{Bow tie} - 13) < M(\text{Bow Tie})$$



Bow Tie Bow Tie $+ 14$ Bow Tie $- 13$

Ugh! Maximum nullity does not play well with respect to subgraph and supergraphs.

Problems worthy of attack prove their worth by hitting back.

Piet Hein

This can be remedied to a large extent by something called the Strong Arnol'd property. This is the mother of all strong properties.

The symmetric matrix A has the Strong Arnol'd property (SAP) provided

$$X \text{ symmetric, } I \circ X = O, \ A \circ X = O \text{ and } AX = O$$
$$\Downarrow$$
$$X = O.^{1}$$

Here $\circ$ denotes the entrywise (or Schur product).

[1] Sometimes in the literature the condition $I \circ X = O$ is not included.

This seems to be a rather strange definition.

"Art is this: art is the solution of a problem that cannot be expressed explicitly until it is solved."

The next result fits this definition of art.

Theorem (Colin de Verdiére (1990))

Let $A \in \mathcal{S}(G)$ and let H be a supergraph of G.

$$A \text{ has SAP}$$
$$\Downarrow$$
$$\exists\, B \in \mathcal{S}(H) \text{ with } nul(A) = nul(B).$$

SAP: Examples

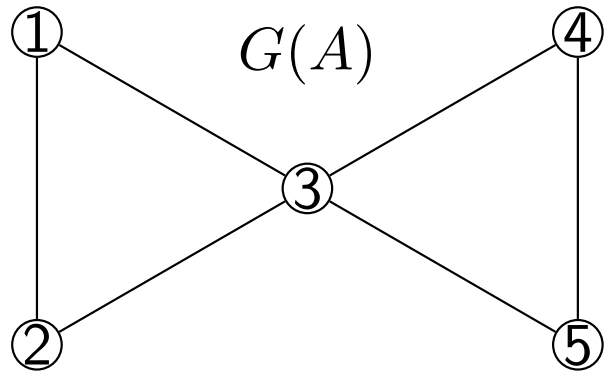
- Every matrix in $\mathcal{S}(K_n)$ has the SAP.
- Every invertible matrix has the SAP.
- Each symmetric matrix of nullity 1 has the SAP.

  Proof. Suppose A is a symmetric matrix of nullity 1, and X is a symmetric matrix with $AX = O$ and $I \circ X = O$.

  Then $X = O$ or $X = \mathbf{u}\mathbf{u}^\top$ for some $\mathbf{u}$. The latter does not occur, since $I \circ X = O$. □
- Every matrix in $\mathcal{S}(\text{path})$ has the SAP.

More examples

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix}$$



does not have the SAP. The matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is an **SAP-spoiler**; that is,

X is nonzero and symmetric,

$I \circ X = O$ and $A \circ X = O$, and

$AX = O$.

More examples

The matrix

$$\begin{bmatrix} -\sqrt{2} & 0 & 1 & -1 \\ 0 & -\sqrt{2} & 1 & 1 \\ 1 & 1 & -\sqrt{2} & 0 \\ -1 & 1 & 0 & -\sqrt{2} \end{bmatrix}$$

has nullity 2, the SAP, and its graph is a 4-cycle C_4.

Supergraph Theorem $\Rightarrow$ each graph obtained from C_4 by inserting edges allows a matrix of nullity 2.

SAP:Direct Sums

$A \oplus B$ has the SAP if and only if both A and B have the SAP and at least one is invertible.

Proof. If A doesn't have the SAP, then there is a spoiler $X \neq 0$ for A.

In this case $X \oplus O$ is a spoiler for $A \oplus B$.

Similarly, Y is a spoiler for $B \Rightarrow O \oplus Y$ is a spoiler for $A \oplus B$. $\square$

If both A and B are singular, then

$$\left[\begin{array}{c|c} O & \mathbf{u}\mathbf{v}^\top \\ \hline \mathbf{v}\mathbf{u}^\top & O \end{array}\right]$$

is a spoiler for $A \oplus B$ where $\mathbf{u}$, resp., $\mathbf{v}$, is a nonzero nullvector for A, resp., B.

Now suppose both A and B have the SAP and without loss, B is invertible. Let

$$W = \left[\begin{array}{c|c} X & U \\ \hline U^\top & Y \end{array}\right]$$

be a potential spoiler for $A \oplus B$. As A has no spoiler, $X = O$, Similarly, $Y = O$. But now B invertible and $BU^\top = O$ imply $U^\top = O$. So the $W = O$, and $A \oplus B$ has the SAP. $\square$

Consequence

We found $A \in \mathcal{S}(C_4)$ of nullity 2 with the SAP.

The direct sum theorem $\Rightarrow I_m \oplus A$ has the SAP.

The graph, G, of $I_m \oplus A$ is C_4 along with m isolated vertices.

The Supergraph theorem $\Rightarrow$ every supergraph of G allows a matrix with nullity 2.

Corollary

If H has a 4-cycle $\Rightarrow \exists\, B \in \mathcal{S}(H)$ with $\operatorname{nul} B = 2$.

We will see that the SAP is a simple manifestation of the Inverse Function Theorem in the appropriate context; that is, the SAP is not a weird condition!

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[KN] Knill, Oliver. "Chaldni figures of discrete square." Harvard Mathematics Department, people.math.harvard.edu/~knill/graphgeometry/chladni4/index.html. Accessed 6 Apr. 2026.

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