Inverse Problems for Symplectic Eigenvalues[1]

Physics is Geometry;
Symplectic Geometry
— V. Arnol'd

[1] Talk based on work by Gutpa, Hogben, S, Wong (2026)

Newtonian mechanics

The natural setting here is Euclidean space; that is, $\mathbb{R}^n$ endowed with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top I \mathbf{v}$ (i.e., the dot product).

The matrices that preserve the dot product are the $n \times n$ orthogonal matrices Q; that is, those with $Q^\top Q = I$.

This leads to results like the following.

Spectral Theorem

For $n \times n$ symmetric A there exists an orthogonal matrix Q and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$Q^\top A Q = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n).$$

This then allows one to study the dynamics of systems such as

$$\ddot{\mathbf{x}} = A\mathbf{x}.$$

The primitives in Newtonian mechanics are position and velocity.

Hamiltonian mechanics

Hamiltonian mechanics offers a different perspective.

The primitives here are position and momentum.

Many view this perspective as more versatile and elegant, offering a smoother transition to quantum mechanics.

The natural setting for Hamiltonian mechanics is Symplectic space: $\mathbb{R}^{2n}$ endowed with the bilinear form $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top \Omega \mathbf{v}$, where

$$\Omega = \left[\begin{array}{c|c} O & -I_n \\ \hline I_n & O \end{array}\right].$$

The different perspectives between Euclidean and Symplectic geometry are evident even in two dimensions.

On the one hand, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

Euclidean geometry is about lengths and angles. Orthogonal matrices preserve length and angle.

On the other hand,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top \Omega \mathbf{v} = \det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$$

which is the signed area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. So those linear operators that preseve $\langle \cdot, \rangle$ preserve (signed) area.

Symplectic geometry is about areas (volumes). Symplectic matrices conserve "area", which corresponds to energy in the physics setting.

The matrices that preserve this bilinear form, i.e., S with $S^\top \Omega S = \Omega$ are called the symplectic matrices.

Fact 1. S symplectic $\implies \det\ S = 1.$

By definition, $S^{-1} = \Omega^\top S^\top \Omega$. So the eigenvalues of S are nonzero and closed under inversion. The result now follows from the fact that $\det\ S$is the product of the eigenvalues of S.

Fact 2.

$$S = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \text{ is symplectic}$$

$$\Updownarrow$$

$A^\top C$ is symmetric, $B^\top D$ is symmetric, and $A^\top D - B^\top C = I$.

Fact 3. The symplectic matrices, $\mathrm{Sp}(\mathbb{R}, 2n)$, under multiplication form a group and are generated by

$$\Omega, \quad \begin{bmatrix} A & O \\ O & A^{-1} \end{bmatrix}, \quad \begin{bmatrix} I & B \\ O & I \end{bmatrix},$$

where A is invertible and B is symmetric.

The analog of the spectral theorem is the following.

Williamson's Theorem [1936]
Let M be a $(2p) \times (2p)$ symmetric positive definite matrix. Then there exists a symplectic matrix S and positive real numbers $d_1, \ldots, d_p$ such that

$$S^\top M S = \left[\begin{array}{c|c} \mathrm{diag}(d_1, \ldots, d_p) & O \\ \hline O & \mathrm{diag}(d_1, \ldots, d_p) \end{array}\right]$$

The d_i are uniquely determined from M and are called the symplectic eigenvalues of M.

Sketch of proof.

There exists a (symmetric) positive definite square root, $M^{1/2}$, of M.

The matrix $M^{-1/2}\Omega M^{-1/2}$ is skew-symmetric.

There exists an orthogonal Q and a positive diagonal D such that

$$Q(M^{-1/2}\Omega M^{-1/2})Q^\top = \left[\begin{array}{c|c} O & D^{-1} \\ \hline -D^{-1} & O \end{array}\right].$$

Set $S = \left[\begin{array}{c|c} D^{1/2} & O \\ \hline O & D^{1/2} \end{array}\right] QM^{-1/2}$.

Verify that $S\Omega S^\top = \Omega$, and $SMS^\top = D \oplus D$.

Fact 4.
The symplectic eigenvalues of M are the singular values of the skew-symmetric matrix $M^{1/2}\Omega M^{1/2}$.

Fact 5.
The symplectic eigenvalues of M are the moduli of the eigenvalues of ΩM. Each of the eigenvalues of ΩM is nonzero and purely imaginary.

This is because $M^{1/2}\Omega M^{1/2}$ is similar to $M\Omega$ and also to $Q^\top \begin{bmatrix} O & -D \\ D & O \end{bmatrix} Q$.

Additionally the symplectic eigenvalues of M are the moduli of the roots of such that $\det(M + x\Omega)$.

Examples.

1. Take $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\Omega M = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, which has characteristic polynomial $x^2 + 1$. Hence, the eigenvalues of ΩM are $\pm i$, and 1 is the only symplectic eigenvalue of M.

2. Take $M = D \oplus E$, where $D = \text{diag}(d_1, \ldots, d_p)$, $E = \text{diag}(e_1, \ldots, e_p)$ and each e_i and d_i is positive.
Then $\Omega M = \begin{bmatrix} O & E \\ -D & O \end{bmatrix}$ which has eigenvalues $\pm\sqrt{d_j e_j}\sqrt{-1}$.
Hence the symplectic singular values of $D \oplus E$ are $\sqrt{d_j e_j}$ $j = 1, \ldots, p$.
In particular, 1 is the only symplectic eigenvalue of $D \oplus D^{-1}$.

The symplectic spectrum of a $2p \times 2p$ positive definite matrix A is the multiset consisting of the diagonal entries of the matrix D in the Williamson decomposition of A.

The symplectic spectra of a $2p \times 2p$ matrix is a multi-set of size $\mathbf{p}$.

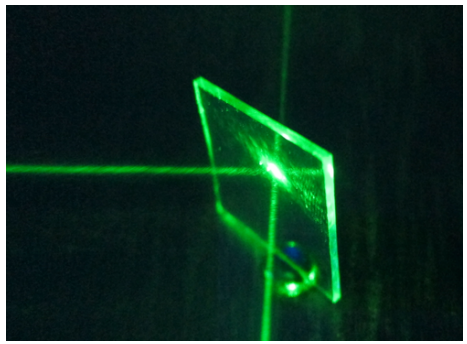
Fact 6. Symplectic congruences preserve symplectic eigenvalues That is, if M is positive definite, and S is symplectic, then $S^\top M S$ and S have the same symplectic eigenvalues.

Symplectic matrices are to positive definite matrices and symplectic eigenvalues, as orthogonal matrices are to symmetric matrices and eigenvalues.

Gaussian States in continuous-variable quantum information

Gaussian states are the linear operators/matrices used to transform signals in optical components like beam splitters and mirrors.

A Gaussian state is just a $2p \times 2p$ matrix M that encodes the variance and correlations of a system's position and momentum.



The diagonal entries of M describe the "spread" or uncertainity of position and momentum, and the off-diagonal elements show the correlation between these quantities for different states.

A symplectic eigenvalue can be thought of the noise of a single quantum mode (on the 2-dimensional M-invariant subspace corresponding to its eigenvectors)

Image from Zaereth, CC0, via Wikimedia Commons

Taking the convention that noise in a vacuum is 1, the Heisenberg Uncertainty Principle requires each symplectic eigenvalue of a Gaussian state M to be at least 1.

Thus, M is a Gaussian state if and only if M is positive definite and each of its symplectic eigenvalues is at least 1.

A **pure state** corresponds to a symplectic eigenvalue 1. Symplectic eigenvalues of the Gaussian state M greater than 1 correspond to mixed states where there is noise.

If each symplectic eigenvalue is 1, then M is pure Gaussian state.

Examples of Williamson form of Gaussian states

Vacuum	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Squeezed vacuum 1 mode	$\begin{bmatrix} e^{-2r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Squeezed vacuum	$\begin{bmatrix} \cosh(2r) & \sinh(2r) & 0 & 0 \\ \sinh(2r) & \cosh(2r) & 0 & 0 \\ 0 & 0 & \cosh(2r) & -\sinh(2r) \\ 0 & 0 & -\sinh(2r) & \cosh(2r) \end{bmatrix}$
Thermal State	$(\bar{n}+1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

($\bar{n}$ corresponds to temperature)

Inverse problems for symplectic eigenvalues

Very natural to ask: what can one say about the symplectic eigenvalues of a matrix? of a matrix with a given structure?

Simple results in the classic symmetric setting have more interesting results in the symplectic setting.

Example: If A is a symmetric matrix with exactly one distinct eigenvalue, then A is diagonal; that is, the graph of A is a disjoint union of isolated vertices).

Why? The spectral theorem implies that the symmetric A has all equal eigenvalues $\iff$ $A = Q^\top kIQ = kI$ for some scalar k.

More interesting examples in symplectic setting

Williamson's theorem says A positive definite has all equal symplectic eigenvalues $\iff$ $A = S^\top kIS = kS^\top S$ for some positive k and some symplectic matrix S.

The set of $S^\top S$ as S runs over the symplectic matrices, has a richer structure than the set of $Q^\top Q$ as Q runs over the orthogonal matrices!.

For each $p \times p$ diagonal matrix D whose diagonal entries are positive, $D \oplus D^{-1}$ has symplectic spectrum 1 of multiplicity p.

More generally, if A is a symmetric matrix, then $S := A \oplus A^{-1}$ is symplectic, and $S^\top S = A^2 \oplus A^{-2}$ has 1 as its only symplectic eigenvalue.

A matrix is a SympPD matrix provided it is positive definite and has just one symplectic eigenvalue. In other words, M is SympPD if and only if it is a positive scalar multiple of a pure Gaussian state.

Theorem Let M be a $2p \times 2p$ symmetric positive definite matrix.

Then the following are equivalent:

(a) Each symplectic eigenvalue of M is 1.
(i.e., M is a pure Guassian state).

(b) M is symplectic (that is, $M\Omega M^\top = \Omega$)

(c) $M = S^\top S$ for some symplectic matrix S.

(d) $(\Omega M)^2 = -I$.

(e) $M^{-1} = \left[\begin{array}{c|c} M_{22} & -M_{12}^\top \\ \hline -M_{12} & M_{11} \end{array}\right]$ where each M_{ij} is $p \times p$.

Labeled graphs

Note that not all pairs of coordinates in symplectic space are the same.
$|\langle \mathbf{e}_i, \mathbf{e}_j \rangle| = \delta_{p,|j-i|}$.
So the pairs (i, j) with $|j - i| = p$ are special.

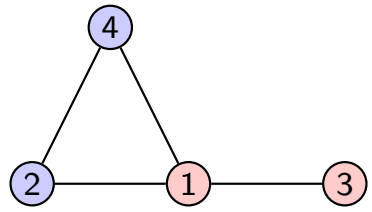
That is, Ω couples the rows of the $2p \times 2p$ matrix: rows i and row $i + p$ are coupled (for $i = 1, \ldots, p$).

In the symplectic setting when we wish related the structure of $(2p) \times (2p)$ matrix A to its symplectic eigenvalues, then we must use labeled graphs.

The labeled graph of the $(2p) \times (2p)$ matrix $\mathcal{S}_A^L$ has vertices labeled by $1, \ldots, 2p$ and an edge from i to j provided $a_{ij} \neq 0$. In a labeled graph, we couple vertices i and $i + p$ for $i = 1, \ldots, p$.

Labeling matters!

For the paw with labeling:



The matrices with this labeled graph have the form

$$M = \begin{bmatrix} a & f & e & g \\ f & c & 0 & h \\ e & 0 & b & 0 \\ g & h & 0 & d \end{bmatrix} \quad a, \ldots, h \neq 0$$

One can verify that

$$(\Omega M)^2 = \begin{bmatrix} -ab + e^2 & -bf & 0 & -bg \\ -df + eg + gh & -cd + h^2 & bg & 0 \\ 0 & ef + cg - fh & -ab + e^2 & -df + eg + gh \\ -ef - cg + fh & 0 & -bf & -cd + h^2 \end{bmatrix} \neq -I.$$

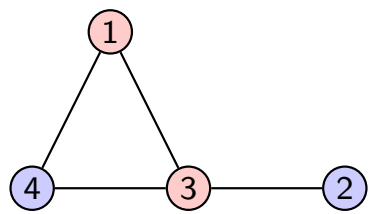
So this is not a labeled graph of a pure Gaussian state.

However,

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} = SS^\top, \text{ where } S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is a pure}$$

Gaussian state.

Its labeled graph is a Paw but with the labeling:



Labeled graphs of pure Gaussian states

1. N positive definite $\implies$ $N \oplus N^{-1}$ is a pure Gaussian state. Its labeled graph is $G(A) \sqcup G(A^{-1})$.
   E.g., one realizes $K_p \sqcup K_p$ with $A = I_p - \frac{2}{p} J_p$.

2. For each symmetric $p \times p$ matrix $B = [b_{ij}]$

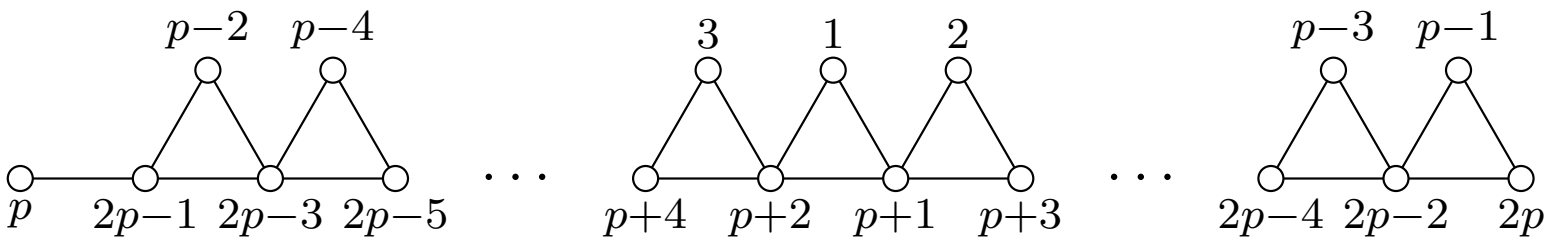
$$\begin{bmatrix} I & B \\ B & I + B^2 \end{bmatrix} = \begin{bmatrix} I & B \\ O & I \end{bmatrix} \begin{bmatrix} I & B \\ O & I \end{bmatrix}^{\top}$$

   is a pure Gaussian state. If $B \geq 0$, its graph has edges i—$(p + j)$ if there is a walk from i to j in $G(B)$ of length at most 1 from i to j and edges $(p + i)$–$(p + j)$ if there is a walk from i to j in $\mathcal{G}(B)$ of length 1 or 2.

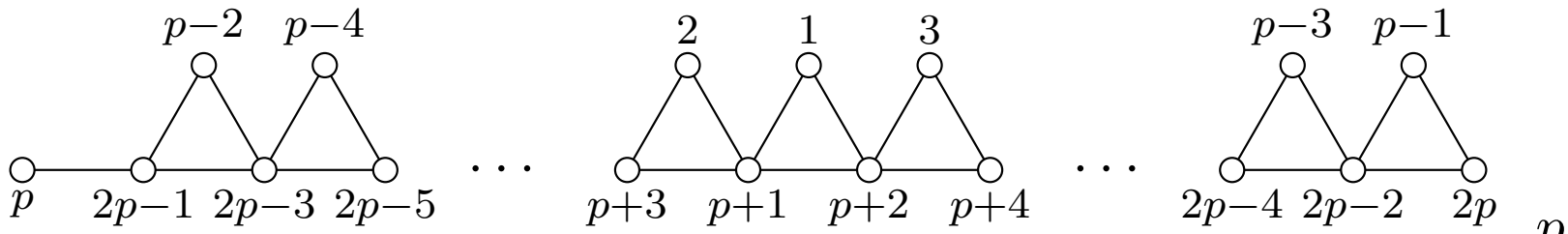
One interesting example uses

$$B = E_{11} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

The labeled graph TP_n is the following:



p odd



even p

TP_n has $2n$ vertices, is connected, and has $3n-2$ edges. This is rather sparse.

Question:
What is the fewest nonzero entries in a $(2p) \times (2p)$ SympPD matrix whose graph is connected?

Above gives $8p-4$ nonzero entries. Conjectured that this is the correct answer.

Evidence:
Can show $8-4$ is a lower bound if M comes from the above construction with $B \geq O$.

Can show $8p-4$ is the correct bound for $p \leq 3$.

Later we will prove this conjecture, by proving a stronger result about positive definite matrices and their inverses.

Another easy question (in the classic setting):
For which graphs G on n vertices is every collection of n real numbers the spectrum of a matrix with graph G?

Answer: Precisely the empty graphs on n vertices. (These correspond to diagonal matrices).

Analogous question in symplectic setting: For which labeled graphs G on $2p$ vertices is every collection of p real numbers the symplectic spectrum of a matrix with labeled graph G?

Answer: Much more interesting!

Such labeled graphs are called symplectic spectrally arbitrary (SSA)-graphs.

Example

Let B be a nonnegative symmetric $n \times n$, and let D be an $n \times n$ diagonal matrix with all diagonal entries positive.

Then

$$S = \left[\begin{array}{c|c} I & B \\ \hline O & I \end{array}\right]$$

is symplectic,
and

$$S^\top (D \oplus D) S$$

is a symmetric positive definite matrix whose symplectic eigenvalues are the diagonal entries of D.

Additionally, the labeled graph of $S^\top (D \oplus D) S$ and that of $S^\top S$ are the same!

Thus, the labeled graph of $S^\top S$ is SSA.

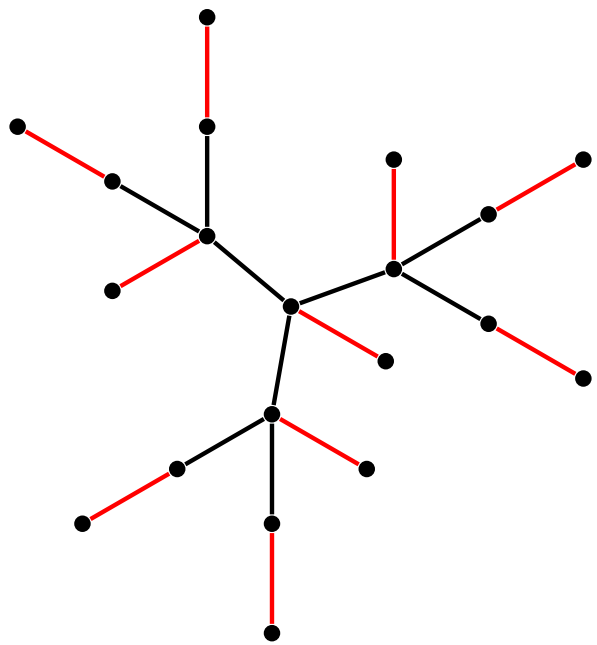
E.g., the labeled graph TP_{2n} is SSA.

To date, if G allows a pure Gaussian state, then G is spectrally arbitrary!

But not every multiplicity least is symplectically spectrally arbitrary. Consider the labeled graph shown below; this is obtained from the Barioli-Fallat tree by appending leafs to each vertex.

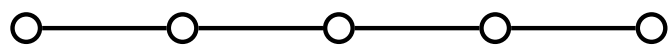
One can show (exercise) that $(1, 2, 4, 2, 1)$ is a symplectic ordered multiplicity list of this labeled graph, but with the constraint on the symplectic eigenvalues $\vartheta_1 < \ldots < \vartheta_5$ satisfy

$$\sqrt{\vartheta_1} + \sqrt{\vartheta_5} = \sqrt{\vartheta_2} + \sqrt{\vartheta_4}.$$



Reasonable question (classic setting):
For which graphs G does every symmetric matrix with graph G have no multiple eigenvalue?

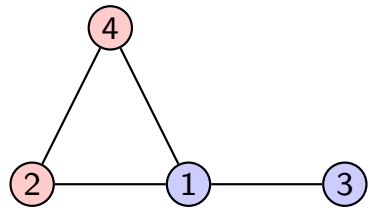
Answer: Fiedler's theorem: Precisely the paths P_n

P_5 

$$\begin{bmatrix} a & g & 0 & 0 & 0 \\ g & b & h & 0 & 0 \\ 0 & h & c & i & 0 \\ 0 & 0 & i & d & j \\ 0 & 0 & 0 & j & e \end{bmatrix}, g, h, i, j \neq 0$$

Not just the paths!

As we saw, no matrix with labeled graph



has $(\Omega N)^2 = -I$. As $p = 2$, this labeled graph requires distinct symplectic eigenvalues. Here's a another way to show this that is more generalizable.

Consider

$$N + x\Omega = \begin{bmatrix} a & e & f+x & g \\ e & b & 0 & h+x \\ f-x & 0 & c & 0 \\ g & x-h & 0 & d \end{bmatrix}$$

If N has a multiple symplectic eigenvalue, then there exist a nonzero real λ and a nonzero vector $\mathbf{z}$ such that $(N + i\lambda x\Omega)\mathbf{z} = \mathbf{0}$ and $z_3 = 0$.

This requires z_1 to be 0, which in turn requires the columns of $\begin{bmatrix} e & g \\ b & h+x \\ 0 & 0 \\ x-h & 0 \end{bmatrix}$ to be linearly dependent. This doesn't occur because of the real/non-real pattern.

This leads to the notion of Quantum zero forcing.

Can use to show that the path 1–$(p+1)$–2—$(p+1)$– $\cdots$ – p–$(2p)$ requires all simple symplectic eigenvalues.

Some labeled paths fail!

The positive definite matrix

$$N = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{24}{25} & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

has labeled graph P_6 with the coupling below:

P_6

and 1 is a multiple symplectic eigenvalue!

The positive definite matrix M has the strong symplectic spectral property provided

$$X \text{ symmetric, } X \circ M = O, \text{ and } M(X\Omega) = (X\Omega)M$$
$$\Downarrow$$
$$X = O.$$

Exercise:
Let D be a positive definite diagonal $p \times p$ matrix. Show that the symplectic eigenvalues of $I_p \oplus D$ are the diagonal entries of D, and that $I_p \oplus D$ has the SSSP if and only if the diagonal entries of D are distinct.

Theorem (Supergraph Theorem)

Let N be a $2p \times 2p$ positive definite matrix having the SSSP and labeled graph G, and let H be a labeled supergraph of G on $2p$ vertices.

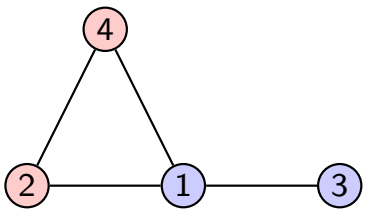
Then there exists a matrix with graph H having the same symplectic eigenvalues as N and also having the SSSP.

Corollary

If S is a set of p distinct postive real numbers and G^L is a labeled graph, then there is a positive definite matrix with labeled graph G^L with symplectic eigenvalues S.

Can show each of the following allows a a matrix with a multiple singular value and the SSSP:

Any coupling of a 4-cycle.



Would be nice to show each coupling of $n \geq 4$ cycle.
Would be nice to understand P_n and its couplings.

Theorem (Proximity Theorem for SSSP)

Let N be a $2p \times 2p$ positive definite matrix having the SSSP and labeled graph G.

For some $\epsilon > 0$, whenever $\hat{N}$ is a $2n \times 2n$ positive definite matrix with $\|\hat{N} - N\| < \epsilon$, there exists a matrix with labeled graph G having the same symplectic spectrum as $\hat{N}$.

Theorem (Bifurcation Theorem for SSSP)

Let N be a $2p \times 2p$ positive definite matrix having the SSSP and labeled graph G whose symplectic eigenvalues have the multiplicity list $\mathbf{m} = (m_1, \ldots, m_q)$.

Then for each refinement $\mathbf{m}'$ of $\mathbf{m}$, there exists a matrix with labeled graph G whose symplectic eigenvalues have the multiplicity list $\mathbf{m}'$.

Conjecture: for each G allowing a pure Gaussian state N, and each multiplicity list $\mathbf{m}$, there exists a positive definite matrix $N \in \mathcal{S}(G)$ with a multiplicity list $\mathbf{m}$.

The Bifurcation theorem resolves the conjecture provided N has the SSSP.

Sketch of proof of the Superpattern theorem

The symplectic Lie Algebra $\mathfrak{sp}(2p)$ consists of the matrices of the form

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where H_{12} and H_{21} are symmetric $p \times p$ matrices, and $H_{22} = -H_{11}^{\top}$,

Such matrices are known as Hamiltonian matrices. Their defining property is: H is a Hamiltonian matrix if and only if $H\Omega$ is symmetric.

It is also known that if H is a Hamiltonian matrix, then e^H is a symplectic matrix.

Fix a $2p \times 2p$ positive definite matrix N.

We now define a function $f : \mathfrak{sp}(2p) \to \mathrm{Mat}^+_{2p\times 2p}$ from the Hamiltonian matrices to the positive definite matrices by $H \to e^{H^\top} N e^H$.

Note

- Each matrix in the image of f is positive definite and has the same symplectic eigenvalues as N,
- $Df_O = H^\top A + A^\top H$.

Now define $g : \mathcal{O} \to \langle \mathcal{S}^L_G \rangle$ by $g(B) = B$.

Finally, define $\mathfrak{f} : \mathfrak{sp}(2p) \times \mathcal{O} \to \mathrm{Mat}^+_{2p\times 2p}$ by

$$\mathfrak{f}(H, B) = e^{H^\top} N e^H + B.$$

The image of $D\mathfrak{f}_{(O,O)}$ is

$$\{H^\top N + N^\top H : H \text{ is Hamiltonian}\} + \langle \mathcal{S}_G^L \rangle.$$

So $D\mathfrak{f}_{(O,O}$ is onto (and $\mathfrak{f}$ is locally onto near (O,O)) if and only if

$$\{H^\top N + N^\top H : H \text{ is Hamiltonian}\} + \langle \mathcal{S}_G^L \rangle = \mathrm{Sym}(n) \qquad (1)$$

We can take this as the definition of the SSVP: N has the strong symplectic spectral property provided (1) holds.

Taking $\perp$'s an equivalent statement is (exercise):

> *N has the SSVP provided $Y = O$ is the only symmetric matrix such that $Y \circ N = O$ and $\Omega N Y = Y N \Omega$.*

The Supergraph theorem follows the Local Surjectivity theorem as for all the other strong properties.

Many unresolved questions in this setting

(Ongoing work with Gupta, Hogben, Hunnell, Wong)

1. Characterization of labeled graphs that requires all simple.

2. Determination if labeled graph G allows pure Gaussian state implies that G is spectrally arbitrary.

3. Understanding of symplectic eigenvalues for labelings for trees.

4. Study of the maximum multiplicity of a symplectic eigenvalue of a matrix with labeled graph G.

5. Establishing the analog of the decontraction theorem in the symplectic setting.

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