Consequences of the Strong Arnol'd Property

T.T.T.

Put up in a place
where it's easy to see
the cryptic admonishment
T.T.T.

When you feel how depressingly slowly you climb,
it's well to remember that
Things Take Time.

–Piet Hein

Primer on orthogonal complements

Fix V, $\langle \cdot, \cdot \rangle$ a finite dimensional, real inner product space.
The orthogonal complement of the subspace U of V is

$$U^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

Facts about $\perp$'s.

(a) $(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$

(b) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

(c) $(U^{\perp})^{\perp} = U$

(d) $\dim U + \dim U^{\perp} = \dim V$

(e) $U + W = V \iff U^{\perp} \cap W^{\perp} = \{\mathbf{0}\}$.

$\mathbb{R}^{m\times n}$ is a real inner product space endowed with the inner product

$$\langle A, B\rangle = \mathrm{tr}(A^\top B).$$

$\mathrm{Sym}(n)$ is a subspace of $\mathbb{R}^{n\times n}$; so is an inner product space.

In this setting,we take $\perp$'s with respect to $\mathrm{Sym}(n)$.

So for $U \subseteq \mathrm{Sym}(n)$, $U \oplus U^\perp = \mathrm{Sym}(n)$.

$\perp$s to have as friends #1: Matrices with the given supergraph space.

Let G be a graph, and $\langle \mathcal{S}(G) \rangle$ be the span of symmetric matrices with graph G. Thus $A \in \langle \mathcal{S}(G) \rangle$ if and only if $a_{ij} \neq 0 \implies i = j$ or ij is an edge of G.

$$\langle \mathcal{S}(G) \rangle^{\perp} = \{X \in \mathrm{Sym}(n) : \mathrm{tr}(XA) = 0 \text{ for all } A \in \mathcal{S}(G)\}$$

$$= \{X \in \mathrm{Sym}(n) : I \circ X = O \text{ and } A \circ X = O\}.$$

Here $\circ$ is the entrywise product.

Note: $\dim\langle \mathcal{S}(G) \rangle = n + |E(G)|$; $\dim\langle \mathcal{S}(G) \rangle^{\perp} = |E(\overline{G}|$.

$\perp$s to have as friends #2: Tangent space at A to the inertia manifold

$$U = \{R^\top A + AR : R \in \mathbb{R}^{n\times n}\}$$

$$U^\perp = \{X \in \mathrm{Sym}(n) : \mathrm{tr}(X(R^\top A + AR)) = 0 \text{ for all } R \in \mathbb{R}^{n\times n}\}.$$

But $\mathrm{tr}(XR^\top A) = \mathrm{tr}(ARX) = \mathrm{tr}(XAR)$. Hence

$$U^\perp = \{X \in \mathrm{Sym}(n) : 2\mathrm{tr}((XAR) = 0 \text{ for all } R \in \mathbb{R}^{n\times n}\}$$

Therefore,
$U^\perp = \{X \in \mathrm{Sym}(n) : XA = O\} = \{X \in \mathrm{Sym}(n) : AX = O\}$.

Dimensions of tangent space and its $\perp$

Notes

- If $B = Q^\top A Q$ where Q is invertible, then the constant inertia manifolds of B and A have the same dimension.
- If $B = I_r \oplus O$, then

$$\left\{ R^\top B + B^\top R : R \in \mathbb{R}^{n \times n} \right\} = \left\{ \begin{bmatrix} U & V \\ V^\top & O \end{bmatrix} : U \in \mathrm{Sym}(r) \right\}$$

and its $\perp$ equals

$$\left\{ \begin{bmatrix} O & O \\ O & W \end{bmatrix} : W \in \mathrm{Sym}(n-r) \right\}$$

- The dimension of the tangent space to the constant inertia manifold for each $n \times n$ matrix of nullity k is $\binom{n+1}{2} - \binom{k+1}{2}$. The dimension of the $\perp$ is $\binom{k+1}{2}$.

Theorem (Strong Arnol'd Property equivalents)

Let A be an $n \times n$ matrix with graph G. Then the following are equivalent:

(a) *The linearization of the map $\mathfrak{f}$ that sends (R, B) to $(I+R)^\top A(I+R)+B$ is onto.*

(b) *The linearization of the map $\mathfrak{g}$ that maps (R, B) to $(I+R)^\top (A+B)(I+R)$ is onto.*

(c) $\{R^\top A + AR : R \in \mathbb{R}^{n\times n}\} + \langle \mathcal{S}(G)\rangle = \mathrm{Sym}(n)$

(d) *$X = O$ is the only symmetric matrix such that $I \circ X = O$, $A \circ X = O$ and $AX = O$.*

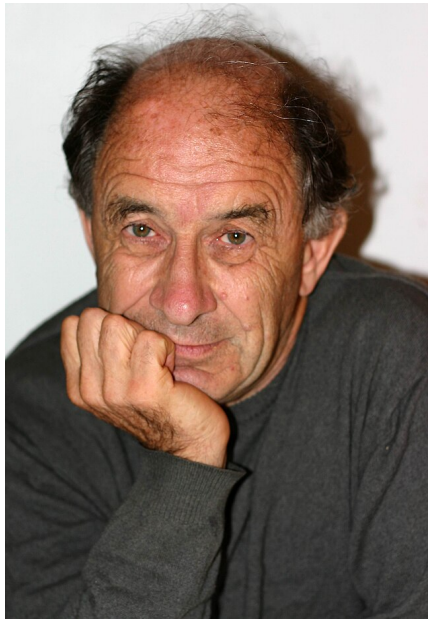
Statement (d) is known as the Strong Arnol'd Property.

Note: we know (b) implies that $\mathfrak{f}$ is locally surjective. $\mathfrak{f}$ maybe locally surjective, even if its linearization is not surjective. When $\mathfrak{f}$ is locally surjective, we say $\mathfrak{f}$ has the weak Arnol'd property.

Arnol'd

"Vladimir Arnol'd (1937–2010) was a towering figure in 20th-century mathematics, best known for the

- Kolmogorov–Arnold–Moser (KAM) theory,
- "Arnold's cat map," and
- contributions to singularity theory, symplectic geometry, and classical mechanics."



From https://mathshistory.st-andrews.ac.uk/Biographies/Arnold/

SAP and supports of null vectors

Proposition (van der Holst (1995,1996))

Let $A \in \mathcal{S}(G)$. Then

(a) *Let A be a matrix with graph G and the SAP. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero null vectors of A. If $\mathbf{x}$ and $\mathbf{y}$ have disjoint support, then there is an edge i–j of G such that $x_i \neq 0$ and $y_j \neq 0$.*

(b) *If $A = A_1 \oplus \cdots \oplus A_k$, then A has the SAP if and only if there exists an i such that each A_j with $j \neq i$ is invertible, and A_i has the SAP.*

Proof of (a) Assume A has the SAP and $\mathbf{x}$ and $\mathbf{y}$ are nonzero null vectors of A with disjoint support. Without loss, we can take

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^\top & A_{22} & A_{23} \\ A_{13}^\top & A_{23}^\top & A_{33} \end{bmatrix},$$

where the columns of A_{11} correspond to the support of $\mathbf{x}$, the columns of A_{22} correspond to the support of $\mathbf{y}$. Then $A(\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top) = O$. Disjoint supports $\implies \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top \neq 0$. Hence $A \circ (\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top) \neq 0$. Thus there is an i such that each of a_{ij}, x_i and y_j is nonzero. □

Sparsity, SAP and nullity

Assume that G be a connected graph and $A \in \mathcal{S}(G)$.
Define θ to be the map that takes $\mathbf{d} \in \mathbb{R}^n$ to

$$\mathrm{diag}(d_1, \ldots, d_n)A + A\mathrm{diag}(d_1, \ldots, d_n).$$

Then

(a) θ is linear,

(b) its image is contained in $\langle \mathcal{S}(G) \rangle$,

(c) if G is not bipartite or some $a_{ii} \neq 0$, then $\ker(\theta) = \{\mathbf{0}\}$, and

(d) if G is bipartite, then $\ker(\theta)$ is at most one dimensional.

It follows that $\{R^\top A + AR \colon R \in \mathbb{R}^n\} \cap \langle \mathcal{S}(G) \rangle$ has dimension at least n if G is not bipartite or some $a_{ii} \neq 0$, and at least $n-1$ if G is bipartite and each $a_{ii} = 0$.

Since

$$\{R^\top A + AR \colon R \in \mathbb{R}^n\}$$

has dimension $\binom{n+1}{2} - \binom{\mathsf{nul}(A)+1}{2}$, we have:

Corollary (SAP and sparsity, Hall, Hogben, Martin, S (2010))

If A is connected with graph G, then $|E(G)| \geq \binom{\mathit{nul}\, A+1}{2} - 1$ edges. The bound is strict if G is either non-bipartite or A has a nonzero diagonal entry.
For every graph G, the maximum nullity of a matrix $A \in \mathcal{S}(G)$ with the SAP is at most

$$\frac{1}{2}\left(-1 + \sqrt{9 + 8|E(G)|}\right).$$

The graph $K_{3,3}$ is an example of a bipartite graph for which the first bound is tight. It has 9 edges, an adjacency matrix having the SAP and nullity 4.

Problem: Characterize the bipartite graphs for which equality holds

Topological properties of SAP

A has the SAP $\iff$ $D\mathfrak{f}$ being surjective.

For an $r \times s$ matrix M, the linear map $\mathbf{x} \to M\mathbf{x}$ is surjective if and only if M has an $r \times r$ invertible submatrix.

Since matrices sufficiently near an invertible matrix are invertible, we have

Theorem (SAP-iness is an open property)

$$A \in \mathrm{Sym}(n) \textit{ has the SAP}$$
$$\Downarrow$$
$$\exists \textit{ an open neighborhood } U \textit{ of } \mathrm{Sym}(n) \textit{ containing } A \textit{ such that every matrix in } U \textit{ has the SAP.}$$

Computational method for determining SAP-iness.

Fix a $n \times n$ symmetric matrix A with graph G.
Consider our friend

$$\mathfrak{f} : \mathcal{B}(O, 1/2) \times \langle \mathcal{S}(G) \rangle \to \mathrm{Sym}(n)$$
$$(R, B) \mapsto (I + R)^\top A(I + R) + B$$

We've seen that $D\mathfrak{f} = R^\top A + AR + B$. So we can determine the partials of $\mathfrak{f}$.

$$\frac{\partial \mathfrak{f}}{\partial r_{ij}} = E_{ji}A + AE_{ij} \qquad (1 \leq i, j \leq n)$$
$$\frac{\partial \mathfrak{f}}{\partial b_{ii}} = 2E_{ii} \qquad (i = 1, \ldots, n)$$
$$\frac{\partial \mathfrak{f}}{\partial b_{ij}} = E_{ij} + E_{ji} \qquad ij \text{ and edge } G.$$

Of course, these partials are **symmetric** $n \times n$ matrices.

For convenience, we can make them into $\binom{n+1}{2} \times 1$ vectors with the $\mathrm{vec}_\wedge$ map, which stacks the above diagonal portion of the columns of a matrix (from left to right) on top of each other.

Example.

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ & m_{22} & m_{23} \\ & & m_{33} \end{bmatrix}$$

$$\mathrm{vec}_\wedge(M) = \begin{bmatrix} m_{11} \\ m_{12} \\ m_{22} \\ m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}$$

The full SAP verification matrix for A is the $(\binom{n+1}{2} \times (\binom{n+1}{2} + n + |E(G)|)$ matrix Φ_A whose columns are

$$\text{the } \mathrm{vec}(\frac{\partial \mathsf{f}}{\partial r_{ij}}), \text{ then the } \mathrm{vec}(\frac{\partial \mathsf{f}}{\partial b_{ii}}), \text{ and finally the } \mathrm{vec}(\frac{\partial \mathsf{f}}{\partial b_{ij}})(ij \text{ an edge}).$$

The rows of Φ_A are indexed by the (i, j) where $1 \leq i \leq j$.

The columns of Φ_A are indexed by the r_{ij} $(1 \leq i, j \leq n)$; the b_{ii} $(i = 1, \ldots, n)$ and the b_{ij} (ij is an edge of G).

The matrices $\mathrm{vec}_\wedge{}^{-1}(\mathbf{v})$ where $\mathbf{v} \in \mathsf{CS}(\Phi(A))$ is the image of $D\mathsf{f}(A)$.

$\mathrm{vec}_\wedge{}^{-1}(\mathsf{LNS}(\Phi(A))$ is just the spoiler space of A!

Column space of $\Phi_A = \mathbb{R}^{\binom{n+1}{2}} \iff$ rows of $\Phi(A)$ are linearly independent $\iff$ A has the SAP.

For

$$A = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Φ_A is

	r_{11}	r_{12}	r_{13}	r_{14}	r_{21}	r_{22}	r_{23}	r_{24}	r_{31}	r_{32}	r_{33}	r_{34}	r_{41}	r_{42}	r_{43}	r_{44}	b_{11}	b_{22}	b_{33}	b_{44}	b_{13}	b_{14}	b_{23}	b_{24}
11	4	0	0	0	0	0	0	0	-2	0	0	0	-2	0	0	0	1	0	0	0	0	0	0	0
12	0	2	0	0	2	0	0	0	-1	-1	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	4	0	0	0	-2	0	0	0	-2	0	0	0	1	0	0	0	0	0	0
13	-1	0	2	0	-1	0	2	0	2	0	-1	0	0	0	-1	0	0	0	0	0	1	0	0	0
23	0	-1	0	0	0	-1	0	0	0	2	-1	0	0	0	-1	0	0	0	0	0	0	0	1	0
33	0	0	-2	0	0	0	-2	0	0	0	4	0	0	0	0	0	0	0	1	0	0	0	0	0
41	-1	0	0	2	-1	0	0	0	0	0	0	-1	2	0	0	-1	0	0	0	0	0	1	0	0
42	0	-1	0	0	0	-1	0	2	0	0	0	-1	0	2	0	-1	0	0	0	0	0	0	0	1
43	0	0	-1	-1	0	0	-1	-1	0	0	0	-2	0	0	2	0	0	0	0	0	0	0	0	0
44	0	0	0	-2	0	0	0	-2	0	0	0	0	0	0	0	4	0	0	0	1	0	0	0	0

Φ_A is

	r_{11}	r_{12}	r_{13}	r_{14}	r_{21}	r_{22}	r_{23}	r_{24}	r_{31}	r_{32}	r_{33}	r_{34}	r_{41}	r_{42}	r_{43}	r_{44}	b_{11}	b_{22}	b_{33}	b_{44}	b_{13}	b_{14}	b_{23}	b_{24}
11	4	0	0	0	0	0	0	0	-2	0	0	0	-2	0	0	0	1	0	0	0	0	0	0	0
12	0	2	0	0	2	0	0	0	-1	-1	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	4	0	0	0	-2	0	0	0	-2	0	0	0	1	0	0	0	0	0	0
13	-1	0	2	0	-1	0	2	0	2	0	-1	0	0	0	-1	0	0	0	0	0	1	0	0	0
23	0	-1	0	0	0	-1	0	0	0	2	-1	0	0	0	-1	0	0	0	0	0	0	0	1	0
33	0	0	-2	0	0	0	-2	0	0	0	4	0	0	0	0	0	0	0	1	0	0	0	0	0
41	-1	0	0	2	-1	0	0	0	0	0	0	-1	2	0	0	-1	0	0	0	0	0	1	0	0
42	0	-1	0	0	0	-1	0	2	0	0	0	-1	0	2	0	-1	0	0	0	0	0	0	0	1
43	0	0	-1	-1	0	0	-1	-1	0	0	0	-2	0	0	2	0	0	0	0	0	0	0	0	0
44	0	0	0	-2	0	0	0	-2	0	0	0	0	0	0	0	4	0	0	0	1	0	0	0	0

Columns corresponding to b_{ij} each have exactly one nonzero entry. So rows of Φ_A are linear independent if and only if the submatrix

$$\Psi_A = \Phi[\alpha, \beta],$$

where α are the rows with index $i \neq j$ and ij not edge of G and β is the set of column indices corresponding to the r_{ij}

Ψ_A is

	r_{11}	r_{12}	r_{13}	r_{14}	r_{21}	r_{22}	r_{23}	r_{24}	r_{31}	r_{32}	r_{33}	r_{34}	r_{41}	r_{42}	r_{43}	r_{44}
23	0	-1	0	0	0	-1	0	0	0	2	-1	0	0	0	-1	0
43	0	0	-1	-1	0	0	-1	-1	0	0	0	-2	0	0	2	0

The SAP verification matrix for A is the submatrix Ψ_A of Φ_A corresponding to

- rows indexed by (i, j) ($i \neq j$ and ij is not an edge of G); and
- columns are indexed by the $r_{i,j}$.

The rows of Ψ_A are indexed by i, j where $i \neq j$ and ij is not an edge of G

The columns of Ψ_A are indexed by variables r_{ij}.

Theorem

The symmetric matrix A has the SAP if and only if its SAP verification matrix, Ψ_A, has linear independent rows.

The SAP spoiler space

The (SAP) spoiler space of an $n \times n$ is

$$\text{Spoiler}(A) = \{X \in \text{Sym}(n) : \ : I \circ X = O, A \circ X = O, AX = O.\}$$

Thus, A has the SAP $\iff$ $\text{Spoiler}(A) = \{0\}$.

A nonzero matrix in $\text{Spoiler}(A)$ is an (SAP) spoiler of A.

Example: If $A = A_1 \oplus A_2$, then

$$\text{Spoiler}(A) = \left\{ \begin{bmatrix} U & V \\ V^\top & W \end{bmatrix} : U \in \text{Spoil}(A_1), W \in \text{Spoil}(A_2), \right.$$
$$\left. V \in \langle \mathbf{w}_1 \mathbf{w}_2^\top : \mathbf{w_i} \in \text{NS}(A_i) \rangle. \ \right\}.$$

SAP with respect to a supergraph

If A is a matrix whose graph is a subgraph of H, we say that A has the SAP with respect to H provided $X = O$ is the only symmetric matrix such that

$$I \circ X = O, A_H \circ X = O \text{ and } AX = O,$$

where A_H is the adjacency matrix of H.

Examples:

A has the SAP w.r.t to $G(A)$ if and only if A has the SAP.

The matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

has the SAP w.r.t. H, where H the path 1–2–3–4.

To verify this, take X of the form

$$X = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ a & 0 & 0 & 0 \\ b & c & 0 & 0 \end{bmatrix}$$

such that $AX = O$.

Then $a = 0$, $-b + c = 0$, $a - b = 0$ and $c = 0$. So $X = O$.

Note that A has the SAP w.r.t. H provided

$$\text{Spoiler Space} \cap \{E_{ij} + E_{ji} : ij \in H - G\}^{\perp} = \{0\}.$$

In particular, A has the SAP w.r.t $H \Rightarrow H$ has at least dim Spoil(A) edges not in $G(A)$.

The spoiler space of A in the last example is 1-dimensional and spanned by

$$\begin{bmatrix} O & J_2 \\ J_2 & O \end{bmatrix}.$$

Hence, A has the SAP with respect to any H obtained from $G(A)$ by inserting at least one edge.

$\exists$ trouble in paradise!

Suppose we wish to study the maximum multiplicity for an n-cycle C_n. Since $M(\text{Path}) = 1$, $M(C_n) \leq 2$.

We get all excited to use the Supergraph Theorem. Each proper subgraph H of C_n is either disconnected or a path. So the maximum multiplicity of a matrix with the **SAP** and graph H is at most 1.

So the Supergraph Theorem can't be used to tell us anything about $M(C_n)$.

Not all is lost (thanks to linear algebra)!

The road to wisdom? –
Well, it's plain
and simple to express:
Err
and err
and err again
but less
and less
and less.

– Piet Hein

Linear algebra to the rescue!

$$M \in \mathbb{R}^{m \times n} \text{ of rank } m \text{ and } \mathbf{b} \in \mathbb{R}^m$$
$$\Downarrow$$
$$\exists \mathbf{x} \text{ such that } M\mathbf{x} = \mathbf{b} \text{ and } \|\mathbf{x}\| \leq \frac{\|\mathbf{b}\|}{\sigma_m},$$
where σ_m is the smallest singular value of A.

Consequently, we have the next result.

Lemma

If the $m \times n$ matrix M has rank m, then $\exists\ k > 0$ such that for all $\epsilon > 0$ the image under M of $\mathcal{B}(\mathbf{0}, k\epsilon)$ contains $\mathcal{B}(\mathbf{0}, \epsilon)$

This means, when $\text{rk}(M) = m$:

> *If $M\mathbf{x} = \mathbf{b}$ has a solution, and we perturb $\mathbf{b}$ by ϵ to get $\hat{\mathbf{b}}$, then $M\mathbf{x} = \hat{\mathbf{b}}$ has a solution that is an $O(\epsilon)$ perturbation of the original solution.*

Not surprisingly, there is an analog of this for functions of several variables that are differentiable.

Theorem (Quantitative Local Surjectivity Theorem, see Section 2.5A of Marsden, Ratiu and Abraham)

Let $\mathbf{p} \in \mathbb{R}^m$, *and* U *be an open neighborhood of* $\mathbf{p}$ *such that* $h : U \to \mathbb{R}^n$ *is continuously differentiable on* U.

Then $\exists$ *a* $k > 0$ *such that* $\forall \epsilon > 0$ *sufficiently small and* $\mathbf{u}_0 \in U$,

$$\mathcal{B}(h(\mathbf{u}_0), \epsilon) \subseteq h(\mathcal{B}(\mathbf{u}_0, k\epsilon))$$

In other words, a perturbation of $\mathbf{v} = h(\mathbf{u}_0)$ *by* ϵ *is the image of a perturbation of* $\mathbf{u}$ *by at most* $k\epsilon$.

"Err less and less and less"

Assume that A has the SAP w.r.t. H. Then
$\mathfrak{f}\colon \mathcal{B}(0,1/2) \times \langle \mathcal{S}(H) \rangle \to \mathrm{Sym}(n)$ by

$$\mathfrak{f}(R,B) \mapsto (I+R)^\top A(I+R) + B$$

is locally surjective.

Let $\mathbf{d}$ be a unit vector in the image of $Df(O,O)$, say $\mathbf{d} = Df\mathbf{u}$
Then

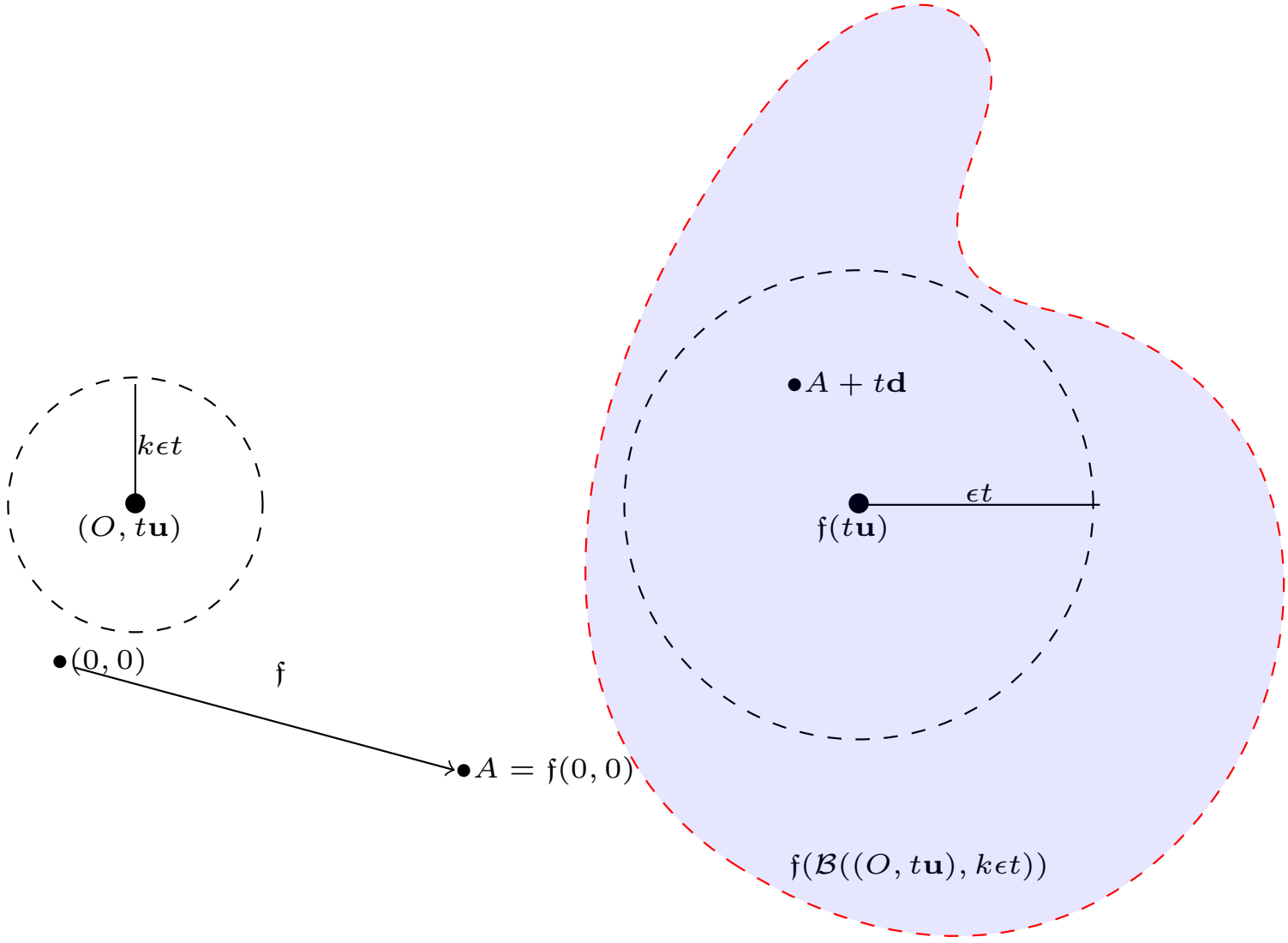
$$\mathbf{0} = \lim_{t\to 0} \frac{f(t\mathbf{u}) - A - Df(t\mathbf{u})}{\|t\mathbf{u}\|} = \lim_{t\to 0} \frac{f(t\mathbf{u}) - A - t\mathbf{d}}{t}.$$

So we can choose t of order ϵ so that $A + t\mathbf{d}$ is within $t\epsilon$ of $f(t\mathbf{u})$.

By the quantitative Local Surjectivity Theorem,

$$A + t\mathbf{d} = (I+R)^\top A(I+R) + B$$

for some (R,B) that is $O(t\epsilon)$ from $(O,t\mathbf{u})$.


$k\epsilon t$
$(O, t\mathbf{u})$
$(0, 0)$
$\mathfrak{f}$
$A = \mathfrak{f}(0, 0)$
$A + t\mathbf{d}$
ϵt
$\mathfrak{f}(t\mathbf{u})$
$\mathfrak{f}(\mathcal{B}((O, t\mathbf{u}), k\epsilon t))$

So $A + t\mathbf{d} - B = (I + R)^\top A(I + R)$.

$A + t\mathbf{d} - B$ is congruent to A.

Choosing ϵ sufficiently small, the graph of $A + t\mathbf{d}$ is $G(A) \cup G(\mathbf{d})$.

As B is $\mathsf{O}(t\epsilon)$ and $t\mathbf{d}$ is $\mathsf{O}(t)$, $A + t\mathbf{d} - B$ has the same graph. $\square$

In general, this gives the next result.

Theorem (Matrix Liberation for Inertia)

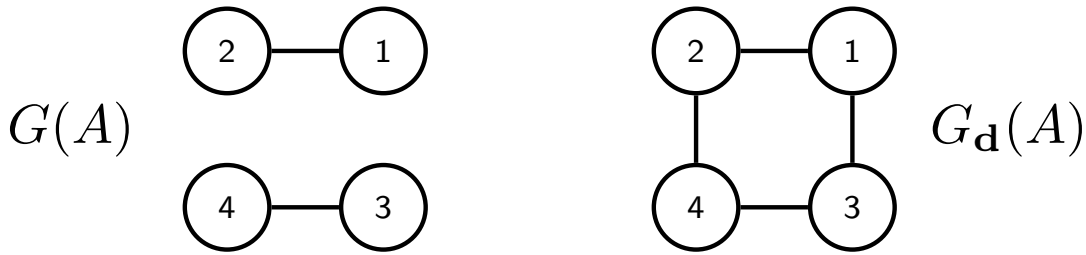
Let $A \in \mathcal{S}(G)$, and $D = R^\top A + AR$ for some $R \in \mathbb{R}^{n\times n}$. If A has the SAP with respect to $G(A) \cup G(D)$, then there is a matrix with graph $G(A) \cup G(D)$ having the SAP, and the same inertia as A.

Example of application.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \text{ and } R = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{d} = R^\top A + AR = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



So, there exists a matrix with graph C_4 having inertia $(2, 0, 2)$.

Note: we can't find an R such that $R^\top A + AR = E_{23} + E_{32}$, lest we'd assert there is a matrix whose graph is P_4 having nullity 2!

Applying the Matrix Liberation Lemma to get a result about $\mathcal{S}(H)$ seems to require two things:

(a) Finding an R such that $G(R^\top A + AR) \cup G(A) = H$; that is, find vector in the column space of Φ_A that corresponds to a matrix with graph H.

(b) Showing A has the SAP w.r.t. H.

We can convert this into checking SAP-iness with respect to several graphs.

Let $A \in \mathcal{S}(G)$. A liberation set of A is a collection α of edges not in G such that for each edge $e \in \alpha$, A has the SAP w.r.t. the graph obtained from G by inserting the edges in $\alpha \setminus \{e\}$.

Example:

With

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

We saw that A has the SAP w.r.t any graph obtained from $G(A)$ by inserting at least one edge.

Hence, each two element subset of the edges 1–3, 1–4, 2–3, 2–4 not in A is a liberation set.

Also, no subset of a single edge is a liberation set.

Theorem (Liberation Sets, Lin, Oblak, Smigoc (2023))

Let $A \in \mathcal{S}(G)$ and H be a supergraph of A. Then the hypotheses of the Matrix Liberation theorem are satisfied if and only if $E(H) \setminus E(G)$ is a liberation set for A.

This implies, using our 4×4 example, any supergraph of $2K_2$ with at least 4 edges (in particular C_4 and Paw supports a matrix with inertia $(2, 0, 2)$).

The theorem follows from a basic (real) linear algebra fact.

Let M be an $m \times n$ real matrix. Let β be a subset of $\{1, \ldots, m\}$. Then the rows of M with index in $\bar{\beta}$ are linearly independent and there exists a vector in the column space of M with support β if and only if the rows of $\bar{\beta} \cup \{i\}$ of M are linearly independent for each $i \in \beta$.

We'll explore this in an exercise, and Helena will discuss more in a later talk.

The Supergraph Theorem does show that if there is a nice needle (A with given inertia and SAP) in one haystack ($A \in \mathcal{S}(G)$), then every "nearby" haystack ($\mathcal{S}(H)$, H a supergraph of G) has a needle ($B \in \mathcal{S}(H)$) with the same inertia as A.

But we are restricted to haystacks with the same order! Ouch!

We'd like to have results that say: if you can find a nice needle N in a haystack, then nearby haystacks (even of larger size) have needles that are akin to N.

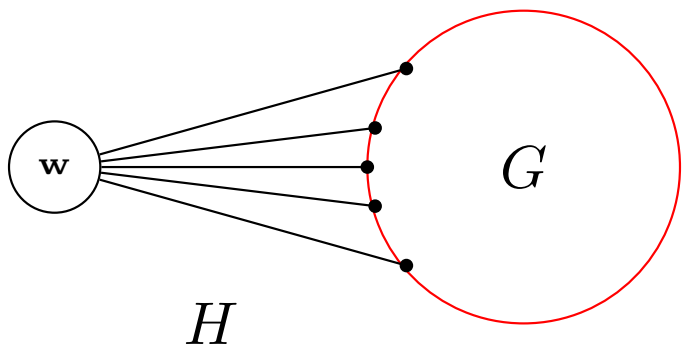
We now discuss two such results.

Theorem (Augmentation Theorem)

Let

- A *be a symmetric* $n \times n$ *with SAP and graph* G*;*
- β *be a set of vertices of* G *such that* $\forall$ *nonzero null vectors* $\mathbf{v}$ *of* A $\exists$ *at least two* $i, j \in \beta$ *with* $v_i, v_j \neq 0$*; and*
- H *be obtained from* G *by inserting a new vertex* w *along with edges joining* w *to the elements of* β*.*

Then $\exists$ B *with graph* H *and* $i(B) = i(A) + (0, 0, 1)$*.*



Proof.

Consider the matrix $C = A \oplus [0]$. C has the desired inertia.

Since A has the SAP, the spoiler space of C consists of matrices of the form

$$\begin{bmatrix} O & \mathbf{u} \\ \mathbf{u}^\top & 0 \end{bmatrix},$$

where $\mathbf{u} \in \mathsf{NS}(A)$

For each $\mathbf{u} \in \mathsf{NS}(A)$ and each $i \in \alpha$, there is at least one $j \in \beta$ with $i \neq j$ and $\mathbf{u}_j \neq 0$.

Hence, C has the SAP w.r.t. $H \setminus \{ni\}$ for each $i \in \beta$.

This $\alpha := \{ni : i \in \beta\}$ is a liberation set. □

Not necessarily symmetric matrices

It is natural to study not necessarily symmetric matrices

The zero-nonzero pattern of an $m \times n$ matrix $A = [a_{ij}]$ is the $(0, *)$-matrix obtained from A by replacing each nonzero by a $*$. The set of real matrices whose zero-nonzero pattern is P is denoted by $\mathcal{Z}(P)$.

Given an $m \times n$ pattern P. The bigraph, $\mathcal{B}(P)$, of $P = [ij]$ has vertices $1, \ldots, m$ and $1', \ldots, n'$ and edges i–j' whenever $p_{ij} \neq 0$.

When $m = n$ we can also identify P with a digraph $\mathcal{D}(P)$ that has vertices $1, \ldots, n$ and arcs $i \to j$ if and only if $p_{ij} \neq 0$. Often in this case, we will require each diagonal of P to be nonzero.

What's the analog of the SAP for zero/nonzero patterns?

It depends (when the matrix is square)!

For A square, you can either study congruence ($A \mapsto M^\top AM$, M invertible) or equivalence $A \mapsto MAN$ (M, N) that are invertible.

For equivalence (including non-square), we have the ASAP.
A has the Asymmetric Strong Arnol'd Property (ASAP) provided

$$A \circ X = O,\ X^\top A = O \ \text{ and } \ AX^\top = O.$$
$$\Downarrow$$
$$\mathsf{X{=}O}$$

For square and congruence, we have the $nSAP$:
$A = [a_{ij}]$ (with each a_{ii} nonzero) has the non-symmetric Strong Arnol'd Property (nSAP) provided

$$I \circ X = O, A \circ X = O, \text{ and } AX^\top \text{ is skew-symmetric}$$
$$\Downarrow$$
$$X = O.$$

Derivation of nSAP

Here we use the same map

$$\mathfrak{f}(R, B) = (I + R)^\top A(I + R) + B$$

as for the SAP, except $B \in \langle Z(P_A) \rangle$, and our inner product is now $\langle X, A \rangle = \mathrm{tr}(X^\top A)$.

The same calculations of $D\mathfrak{f}(O, O)$ and $\perp$s give the nSAP.

Derivation of ASAP

For R and S in sufficiently small neighborhoods of O,
$f : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathbb{R}^{m\times n}$ by $f(R,S) = (I+R)A(I+S)$.

One can check that $df_{O,O} = RA + AS$.

As before, but using $\mathfrak{f}(R,S,B) = (I+R)A(I+S)+B$, we get $\mathfrak{f}$ is locally surjective if and only if
$\{RA + AS : R \in \mathbb{R}^{m\times m}, S \in \mathbb{R}^{n\times n}\} + \langle \mathcal{Z}(P)\rangle = \mathbb{R}^{m\times n}$,

Taking $\perp$'s gives the ASAP condition.

Theorem (Arav, van der Holst (2024))

Let A have the ASAP and zero-nonzero pattern P. Then there exists a matrix $\widehat{P}$, the nSAP and nullity that of A in each of the following instances:

(a) *$\widehat{P}$ is a superpattern of P;*

(b) *P is an induced pattern of $\widehat{P}$;*

(c) *$\hat{P}$ contains P as a minor.*

The stable-nullity of an $n \times n$ zero-nonzero pattern P with $P \geq I$ is the largest nullity of a matrix in $\mathcal{Z}(P)$ with the ASAP.

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