Minors and the SAP

ON PROBLEMS
Our choicest plans
have fallen through,
our airiest castles
tumbled over,
because of lines
we neatly drew
and later neatly
stumbled over.

–Piet Hein

Graph minors

Three basic operations on a graph H.

- Removing an edge of H.
- Removing an isolated vertex of H.
- A contraction of an i–j is an edge of G where $N(i) \cap N(j) = \emptyset$. This is the graph obtained from H by identifying vertex i with vertex j.

The graph G is a graph minor of H provided G can be obtained from H by a sequence of basic operations.

Note: When only edge removals are used, G is a minor of H corresponds to H is a supergraph of G.

When no contractions are used, G being a minor of H corresponds to H having G as a subgraph (not necessarily spanning nor induced).

Note: a graph is a tree if and only if it is connected and does not contain K_3 as a minor.

Important results about graph minors.

Theorem (Wagner's Theorem, 1937)

The graph G is planar if and only if G contains neither $K_{3,3}$ nor K_5 as a minor.

Theorem (Robertson-Seymour Graph Minors Theorem, 2003)

Let $\mathcal{F}$ be a family of graphs that is closed under taking minors. Then there exists a finite collection $\mathcal{M}$ of graphs such that a graph G is in $\mathcal{F}$ if and only if G does not contain any graph in $\mathcal{M}$ as a minor.

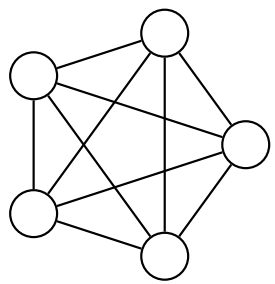
Example: The family of planar graphs is closed under taking minors. Wagner's theorem asserts that $\mathcal{M} = \{K_{3,3}, K_5\}$ is a finite set of excluded minors that characterize planar graphs.

In an upcoming lecture, we'll relate the SAP to minors, as was first done by Colin de Verdiére.

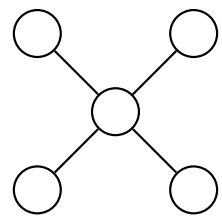
Graph Minors and the SAP

In this lecture, we relate the maximum nullity, graph minors, and the Strong Arnol'd property.

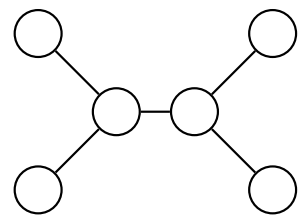
In general the maximum nullity does not behave well with respect to graph minors:



$M(K_5) = 4$



$M(K_{1,3}) = 3$



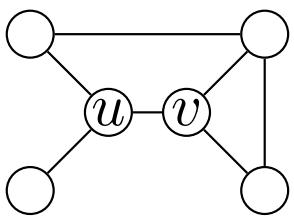
$M(H) = 2$

However, we'll see that the SAP gives a related parameter ξ that is minor monotone

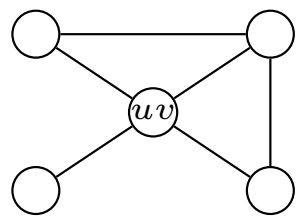
i.e., $\xi(G) \leq \xi(H)$ whenever G is a minor of H.

A graph H is a basic decontraction of the graph G provided $\exists$ vertices u and v such that uv is an edge in H, $N(u) \cap N(v) = \emptyset$ and G is the graph obtained from H by contracting the edge joining uv.

Example



H



G

For convenience, when H has $n+1$ vertices, we typically take $u = n$ and $v = n+1$.

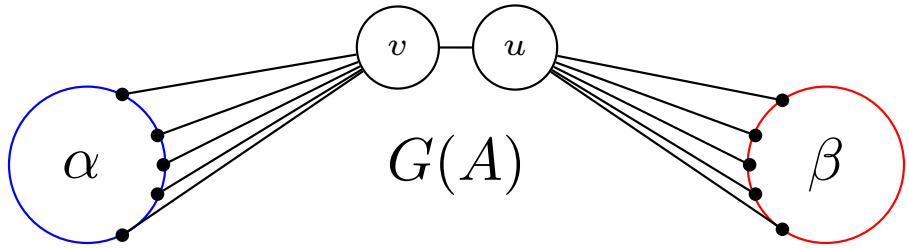
At the matrix level, let $A \in \mathcal{S}(G)$ have the form

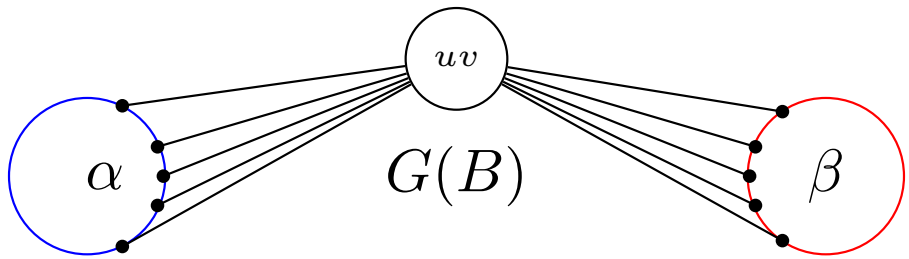
$$\left[\begin{array}{ccc|c} & A(n) & & \mathbf{0} \\ & & & \mathbf{x} \\ & & & \mathbf{y} \\ \hline \mathbf{0}^\top & \mathbf{x}^\top & \mathbf{y}^\top & a_{n,n} \end{array}\right],$$

where $\mathbf{x}$ and $\mathbf{y}$ are nonzero; so they correspond to the neighbors of n in G.
The graph of the matrix

$$B = \left[\begin{array}{ccc|c|c} & A(\{n\}) & & \mathbf{0} & \mathbf{0} \\ & & & \mathbf{x} & \mathbf{0} \\ & & & \mathbf{0} & \mathbf{y} \\ \hline \mathbf{0}^\top & \mathbf{x}^\top & \mathbf{0}^\top & a_{n,n}+2 & -1 \\ \hline \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{y}^\top & -1 & 1 \end{array}\right]$$

is that of the decontraction of G according to the indices corresponding to $\mathbf{x}$ and $\mathbf{y}$.


v
u
α
G(A)
β


uv
α
G(B)
β

Theorem (Decontraction theorem for Inertia (Barioli, Fallat, Hogben (2005)))

Let $A \in \mathcal{S}(G)$ have the SAP, and let H be a graph such that G is a simple contraction of H. Then there exist matrices B^+ and B^- with graph H with the SAP and inertia $i_A + (1,0,0)$ and $i_A + (0,1,0)$, In particular, there exists B with graph H, the SAP and the same nullity as A. respectively.

Proof to come.

This suggests the following definition.

For the graph G, $\xi(G)$ is the largest nullity of a matrix $A \in \mathcal{S}(G)$ that has the SAP.

In general $\xi(G) \leq M(G)$.

E.g., $\xi(K_{1,5}) = 2 < 4 = M(K_{1,5})$.

Why?

$P(K_{1,5}) = 4$. So, $M(K_{1,5}) = 4$.

The adjacency matrix matrix of $K_{1,4}$ has the SAP and nullity 2. $K_{1,4}$ is a minor of $K_{1,5}$. So by the decontraction theorem, $2 \leq \xi(K_{1,5})$.

By Parter-Weiner, $A \in \mathcal{S}(K_{1,5}) \geq 3$ requires

$$A = \begin{bmatrix} a & b & c & d & e & f \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & g \end{bmatrix}.$$

But then

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is an SAP-spoiler for A. Hence $\xi(K_{1,5}) \leq 2$. □

Corollary (Minor Monotonicity) (Barioli, Fallat, Hogben (2005))

ξ is a minor monotone graph parameter; that is,

$$G \text{ is a minor of } H \Rightarrow \ \xi(G) \leq \xi(H).$$

Proof.

It suffices to consider each basic operation in the definition of minors.

Supergraph theorem for Inertia handles removal of an edge.

Extended Supergraph theorem for SAP handles removal of a vertex.

Decontraction Theorem handles a basic contraction. □

Proof of the Decontraction Theorem for Inertia

Take $A \in \mathcal{S}(G)$ with the SAP, and of the form

$$A = \left[\begin{array}{ccc|c} & A(n) & & \begin{matrix}\mathbf{0}\\ \mathbf{x}\\ \mathbf{y}\end{matrix} \\ \hline \mathbf{0}^\top & \mathbf{x}^\top & \mathbf{y}^\top & a_{n,n} \end{array}\right],$$

where $\mathbf{x}$ and $\mathbf{y}$ are nowhere zero; so they correspond to the neighbors $\alpha \cup \beta$ of n in G.

Take $\mathbf{x}$ to be $|\alpha| \times 1$, and $\mathbf{y}$ to be $|\beta| \times 1$.

Choose $\lambda \neq 0$, and set $A_\lambda = A \oplus [\lambda]$.

So A_λ has the SAP.

We are going to slightly modify the proof used for the supergraph theorem.

- Define function like $\mathfrak{f}$, but now $\mathfrak{f}$ is a sum of **three** functions.
- Determine the image of the linearization of each of the functions.
- Show that the sum of the images is all of $\mathrm{Sym}(n+1)$
- Apply the Local surjectivity Theorem to show that there exists a matrix N of the form

$$\left[\begin{array}{ccc|c|c} & \hat{A}(n) & & \begin{matrix}\mathbf{0}\\ \hat{\mathbf{x}}+\mathbf{z}\\ \hat{\mathbf{y}}\end{matrix} & \begin{matrix}\mathbf{0}\\ t(\hat{\mathbf{x}}+\mathbf{z})\\ \mathbf{0}\end{matrix} \\ \hline \mathbf{0}^\top & (\hat{\mathbf{x}}+\mathbf{z})^\top & \hat{\mathbf{y}}^\top & \hat{a}_{n,n} & b_{n,n+1} \\ \hline \mathbf{0}^\top & t(\hat{\mathbf{x}}+\mathbf{z})^\top & \mathbf{0}^\top & b_{n,n+1} & \hat{\lambda} \end{array}\right]$$

close to A_λ and with the same nullity as A.

Finally, we apply a congruence to

$$N = \left[\begin{array}{ccc|c|c} & \hat{A}(n) & & \begin{matrix}\mathbf{0}\\ \hat{\mathbf{x}}+\mathbf{z}\\ \hat{\mathbf{y}}\end{matrix} & \begin{matrix}\mathbf{0}\\ t(\hat{\mathbf{x}}+\mathbf{z})\\ \mathbf{0}\end{matrix} \\ \hline \mathbf{0}^\top & (\hat{\mathbf{x}}+\mathbf{z})^\top & \hat{\mathbf{y}}^\top & \hat{a}_{n,n} & b_{n,n+1} \\ \hline \mathbf{0}^\top & t(\hat{\mathbf{x}}+\mathbf{z})^\top & \mathbf{0}^\top & b_{n,n+1} & \hat{\lambda} \end{array}\right],$$

to obtain the matrix

$$\left[\begin{array}{ccc|c|c} & \hat{A}(n) & & \begin{matrix}\mathbf{0}\\ \mathbf{0}\\ \hat{\mathbf{y}}\end{matrix} & \begin{matrix}\mathbf{0}\\ t(\hat{\mathbf{x}}+\mathbf{z})\\ \mathbf{0}\end{matrix} \\ \hline \mathbf{0}^\top & \mathbf{0}^\top & \hat{\mathbf{y}}^\top & \hat{a}_{n,n} & \hat{b}_{n,n+1} \\ \hline \mathbf{0}^\top & t(\hat{\mathbf{x}}+\mathbf{z})^\top & \mathbf{0}^\top & \hat{b}_{n,n+1} & \hat{\lambda} \end{array}\right],$$

whose graph is H, the decontraction of G.

Defining our functions: f_1

f_1 takes small $(n+1)\times(n+1)$ matrices of the form $S = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{q}^\top & s \end{bmatrix}$ to $(I+S)^\top A_\lambda (I+S)$.

Each matrix in $\mathrm{Im}(f_1)$ has the same inertia as A_λ.

The total derivative of f_1 at the origin is

$$S^\top A_\lambda + A_\lambda S = \begin{bmatrix} R^\top A + AR & \lambda\mathbf{q} + A\mathbf{p} \\ \lambda\mathbf{q}^\top + \mathbf{p}^\top A & 2\lambda s \end{bmatrix}$$

The Jacobian of f_1 at the origin has the form

$$\begin{bmatrix} Df_{\mathsf{old}} & O & O \\ O & A & \lambda I \end{bmatrix}.$$

Since $\lambda \neq 0$, $\mathrm{Im}(Df_1)$ consists of all symmetric $(n+1)\times(n+1)$ matrices whose $(1,1)$-block entry is in $\mathrm{Im}(Df_{\mathsf{old}})$.

Defining our functions: f_2

f_2 embeds $\langle \mathcal{S}(H) \rangle$ into $\mathrm{Sym}(n+1)$

So its total derivative is just the identity map, and

$$\mathsf{Im}(Df_2) = \langle \mathcal{S}(H) \rangle.$$

Since A has the SAP, $\mathsf{Im}(Df_1) + \mathsf{Im}(Df_2)$ contains all the basic symmetric matrices

$$\begin{array}{ll} E_{ii} & (i = 1, \ldots, n+1); \\ E_{ij} + E_{ji} & \{i, j\} \notin \{\{n+1, k\} : k \in \alpha\}. \end{array}$$

So we need to find a f_3 whose tangent space picks up the remaining basic symmetric matrices.

Defining our functions: f_3

f_3 maps $(\mathbf{z}, \mathbf{t}) \in \mathbb{R}^{|\alpha|+1}$ to

$$\left[\begin{array}{ccc|cc} & & & \mathbf{0} & \mathbf{0} \\ & O & & \mathbf{z} & t(\mathbf{x}+\mathbf{z}) \\ & & & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}^\top & \mathbf{z}^\top & \mathbf{0} & 0 & 0 \\ \mathbf{0}^\top & t(\mathbf{x}+\mathbf{z})^\top & \mathbf{0} & 0 & 0 \end{array}\right]$$

The image of Df_3 contains all basic symmetric matrices of the form

$$E_{n,i} + E_{n,i}^\top \quad (i \in \alpha).$$

Thus if we define $\mathfrak{f}$ to be $f_1 + f_2 + f_3$, then the image of $D\mathfrak{f}(O)$ is $\mathrm{Sym}(n+1)$, and the Local Surjectivity Theorem applies!

Jacobian matrix of $\mathfrak{f}$

In terms of the Jacobian matrix we have the following matrix.

The columns are indexed by the variables entries in R, $\mathbf{p}$, $\mathbf{q}$, s; edges of $\mathcal{S}(H)$; $\mathbf{z}$ and t.

The rows are indexed by the (i, j) with $1 \leq i, j \leq n+1$.

$$
\begin{array}{c}
 \\
(i,j);\\ 1 \leq i < j \leq n \\
(j, n+1);\\ j = 1, \ldots, n \\
(n+1, n+1)
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} R & \mathbf{p} & \mathbf{q} & s & \mathcal{S}(H) & \mathbf{z} & t \end{array} \\
\begin{bmatrix}
Df_{\mathsf{Old}} & O & O & \mathbf{0} & \chi_H & \chi_\alpha & O \\
O & A & \lambda I & \mathbf{0} & \mathbf{0}^\top & \mathbf{0}^\top & \chi_\alpha \\
\mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & 2\lambda & 1 & \mathbf{0}^\top & 0
\end{bmatrix}
\end{array}
$$

Since the linearization of $\mathfrak{f}$ at O is surjective, the Local Surjectivity theorem applies.

Thus, we can perturb A_λ by any $O(\epsilon)$ matrix to get a matrix M that is in the image under $\mathfrak{f}$ of some $O(\epsilon)$ perturbation of O.

We can take $M = A_\lambda + \epsilon(E_{n,n+1} + E_{n+1,n})$.
So, M equals $S^\top A_\lambda S + B + C$ where $B \in \mathcal{S}(H)$, and C has the form

$$\left[\begin{array}{ccc|cc} & & & \mathbf{0} & \mathbf{0} \\ & O_{n-1} & & \mathbf{z} & t(\mathbf{x}+\mathbf{z}) \\ & & & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}^\top & \mathbf{z}^\top & \mathbf{0}^\top & 0 & 0 \\ \mathbf{0}^\top & t(\mathbf{x}+\mathbf{z})^\top & \mathbf{0}^\top & 0 & 0 \end{array}\right]$$

Thus $N := M - B - C = S^\top A_\lambda S$ is congruent to A_λ and has the form

$$\left[\begin{array}{ccc|c|c} & \hat{A}(n) & & \begin{matrix}\mathbf{0}\\ \hat{\mathbf{x}}+\mathbf{z}\\ \hat{\mathbf{y}}\end{matrix} & \begin{matrix}\mathbf{0}\\ t(\hat{\mathbf{x}}+\mathbf{z})\\ \mathbf{0}\end{matrix} \\ \hline \mathbf{0}^\top & (\hat{\mathbf{x}}+\mathbf{z})^\top & \hat{\mathbf{y}}^\top & \hat{a}_{n,n} & b_{n,n+1} \\ \hline \mathbf{0}^\top & t(\hat{\mathbf{x}}+\mathbf{z})^\top & \mathbf{0}^\top & b_{n,n+1} & \hat{\lambda} \end{array}\right],$$

By a congruence involving rows n and $(n+1)$, we obtain a matrix congruent to A_λ whose graph is the decontraction H of G. □

Consequences of Fundamental Theorems for the SAP

1. $1 \leq \xi(G) \leq n-1$ for each connected graph on n vertices.

Proof.

The Laplacian matrix A of G has nullity 1. Hence A has the SAP. So $1 \leq \xi(G)$ for all G.

Also, $\xi(G) \leq M(G) \leq n-1$ for connected G. □

2. If G has K_p as a minor, then $\xi(G) \geq p-1$.

Consequences

3. If G has n vertices and an induced subgraph on m vertices that is the disjoint union of paths, then $\xi(G) \leq n - m + 1$.

Proof.

Take $A \in \mathcal{S}(G)$ with the SAP and $\operatorname{nul} A = \xi(G)$.

By the Supergraph theorem, $\exists\ B \in \mathcal{S}(G'))$ with $\operatorname{nul} B = \xi(G)$, where G' is obtained from G by inserting edges to make the induced paths into a single induced path.

This implies that $m - 1 \leq \operatorname{rank}(B)$, and hence

$$\xi(G) = \operatorname{nul} B \leq n - m + 1.$$



Consequences

Note: a co-clique of m vertices is an induced graph that is the union of m vertex disjoint paths of length 0.

Hence, we have the following.

4. If G has order n and has an independent set of size m, then $\xi(G) \leq n - m + 1$.

Consequences

5. $\xi(\sqcup_{i=1}^{c} G_i) = \max_{i=1}^{c} \xi(G_i)$

   Exercise. □

6. $M(C_n) = 2$ for all $n \geq 3$.

Proof. C_3 allows a matrix of nullity 2 with the SAP and C_3 is a minor of C_n for $n \geq 3$. By minor-monotonicity, $\xi(C_n) \geq 2$ for $n \geq 3$.

Also, deleting a vertex of C_n results in P_n, and $M(P_n) = 1$. Hence

$$2 \leq \xi(C_n) \leq M(C_n) \leq 1 + M(P_n) = 2 \quad \square$$

Consequences

7. If G is a connected graph on at least 2 vertices, then

$$\xi(G \vee K_1) \geq \xi(G) + 1.$$

Proof.

Assume G is connected, $A \in \mathcal{S}(G))$, A has the SAP, and $\operatorname{nul} A = \xi(G)$. G connected implies there is a vector $\mathbf{u}$ such that $A\mathbf{u}$ is nowhere zero. Set

$$B = \begin{bmatrix} -\mathbf{u}^\top A\mathbf{u} & \mathbf{u}^\top A \\ A\mathbf{u} & A \end{bmatrix}$$

A spoiler of B has the form

$$X = \begin{bmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & Y \end{bmatrix}.$$

where X is a spoiler for A. A has SAP $\implies X = O \implies B$ has the SAP. Finally, note $\operatorname{nul} B = \operatorname{nul}(A) + 1 = \xi(A) + 1$. $\square$

Consequences

8. For $q \geq p \geq 3$, $\xi(K_{p,q}) = p + 1$.

Proof. $K_{p,q}$ contains $p + q$ vertices, and an independent set of size q. By Consequence 4,

$$\xi(K_{p,q}) \leq p + q - q + 1 = p + 1.$$

Next, we give a lower bound. If $p > 4$, then contract an edge of $K_{p,q}$ to get $K_{p-1,q-1} \vee K_1$. By Consequence 7,

$$\xi(K_{p-1,q-1} \vee K_1) \geq \xi(K_{p-1,q-1}) + 1.$$

Hence, if we can establish the result for $p = 3$, then induction on p will give the full result.

For $q \geq p = 3$,

$$\xi(K_{3,q}) \geq \xi(K_{3,3}).$$

The adjacency matrix of $K_{3,3}$ has the SAP and nullity $4 = p + 1$. So $\xi(K_{3,q}) \geq 4$. □

Consequences

9. If T is a tree other than the path, then $\xi(T) = 2$.

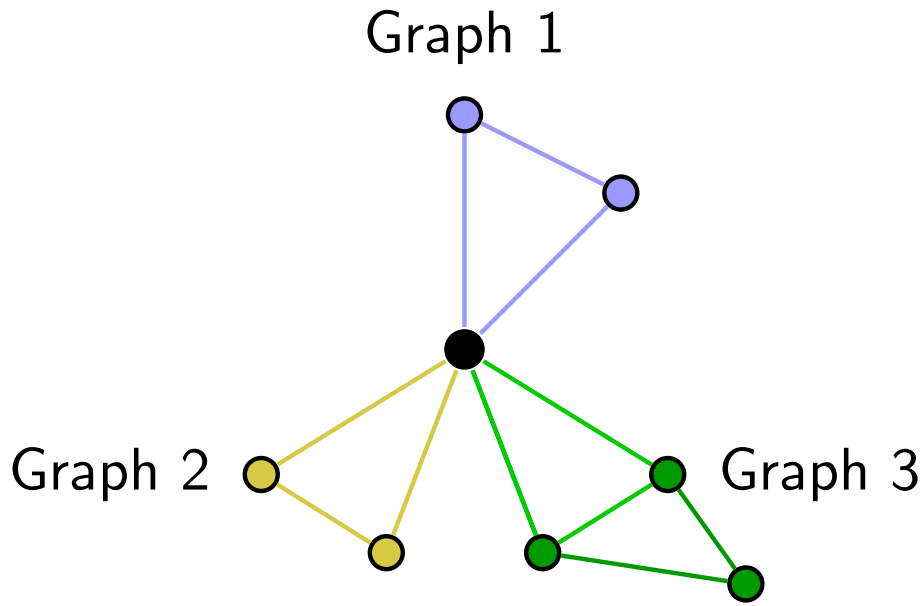
Let T be a tree other than the path. Then T contains $K_{1,3}$ as a minor. Since $\xi(K_{1,3}) = 2$, $\xi(T) \geq 2$.

Now suppose that $A \in \mathcal{S}(G)$ with $\operatorname{nul} A \geq 3$. By the Parter-Wiener theorem, there exists a vertex v of G and subsets $\alpha_1, \alpha_2, \ldots, \alpha_k$ of vertices such that $k \geq 3$, each $A[\alpha_i]$ is not invertible, each α_i is the vertex set of a connected component of $G(v)$ and $\sum_{i=1}^{k} \operatorname{nul} A[\alpha_i] = k + 1$.

If $k \geq 4$ or some $A[\alpha_i]$ has nullity greater than 1, then one can find two null vectors of A with disjoint support and no edge in T with a vertex in one support and a vertex in another. Hence, by van der Holst's null vector support theorem, either A does not have the SAP. Therefore $\xi(T) \leq 2$. $\square$

Consequences

The graph G is a vertex-sum of the graphs $G_1, \ldots, G_s$ provided the G_i are vertex disjoint, one vertex v is identified from each G_i, and then these vertices are collapsed into a single vertex. We denoted G by $G_1 \oplus_v G_2 \oplus_v \cdots \oplus_v G_s$.



10. If G is the vertex sum of connected $G_1, \ldots, G_s$, with $s \geq 2$, then

$$\max \xi(G_i) \leq \xi(G) \leq 1 + \max \xi(G_i)$$

Will explore this in excercises.

Consequences

11. Forbidden characterizations for bounded ξ exist!

Fix a positive integer k. By monotonicity of ξ the set of all G with $\xi(G) \leq k$ is closed under minors.

Proof.

Robertson-Seymour Graph Minors theorem $\implies$ $\exists$ a finite family of graphs $\mathcal{F}$ such that $\xi(G) \leq k$ if and only if no graph in $\mathcal{F}$ is a minor of G. $\square$

Forbidden subgraph characterizations

Characterization of G with $M(G) = 1$ (recall Fiedler's theorem).

First let's find a forbidden minor char'n for G with $\xi(G) \leq 1$.

We know $\xi(C_3) = 2$ and $\xi(K_{1,3}) = 2$.

Now assume $M(G) = 1$.
Then G is connected (lest we use the Laplacian matrix).

Since ξ is minor monotone and $M(G) \geq \xi(G)$, neither C_3 nor $K_{1,3}$ is a minor of G.

Thus, G is a path. □

This gives a strong property proof of Fiedler's result: $M(G) = 1$ if and only if G is a path.

Exercise: Characterize the (not necessarily connected) graphs G with $\xi(G) = 1$.

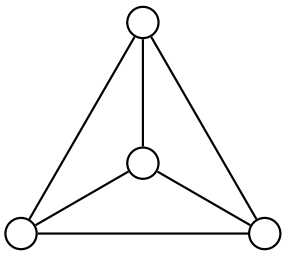
Characterization of connected graphs G with $M(G) = 2$.

Since $M(\sqcup G_i) = \sum M(G_i)$, we can easily reduce to case G is connected.

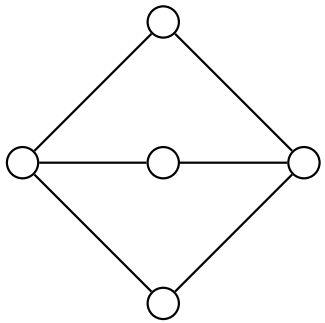
If G is the vertex some of connected $G_1, \ldots, g_k$, then $M(G) = k - 1 + \sum_{i=1}^{k} M(G_i)$.
This allows one to reduce to the case that G is 2-connected.

As $\xi(G) \leq M(G)$, we begin by characterizing the 2-connected G with $\xi(G) = 2$.

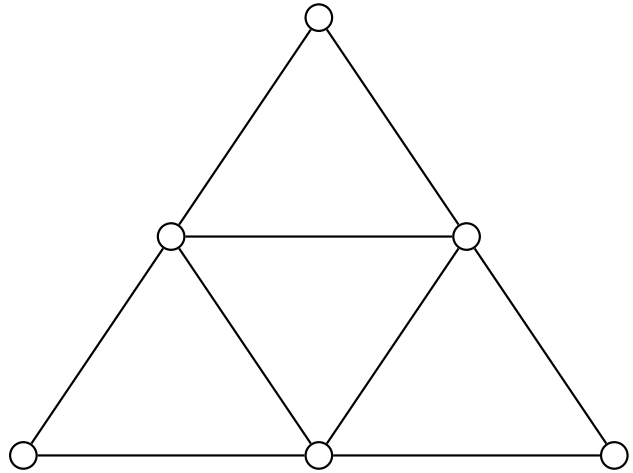
Small graphs with $\xi(G) = 3$: are K_4, $K_{2,3}$ and T_3.



K_4



$K_{2,3}$



The supertriangle

For K_4 use J_4; for $K_{2,3}$ use the adjacency matrix. For T_3, use the sum of three rank 1 matrices corresponding to the edge triangles.

So these are forbidden minors for graphs G with $\xi(G) \leq 2$

A graph is outerplanar if it is planar and can be embedded in the plane so that all vertices lie on the boundary of the outer face.

Neither K_4 nor $K_{2,3}$ is outer planar. T_3 is outer planar.

A graph is a linear singly edge-articulated cycle graph (LSEAC graph) provided it is a "path" of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle. The supertriangle is not a LSEAC.

A graph is a linear 2-tree provided it is a LSEAC, where each appended cycle is a 3-cycle.



A linear 2-tree and an LSEAC graph that is not a linear 2-tree

Characterization of graphs with $M(G) = 2$

Theorem (Barioli, Fallat, Hogben 2006; Johnson, Loewy, Smith 2007)

Let G be a 2-connected graph. Then the following are equivalent:

(a) $M(G) = 2$.

(b) $\xi(G) = 2$.

(c) *G is K_4-, $K_{2,3}$- and T_3-minor free.*

(d) *G is a partial linear 2-tree.*

Comments on the BFH proof.
Since G is 2-connected, $2 \leq \xi(G) \leq M(G)$. Thus (a) $\implies$ (b).

Since $\xi(K_4) = \xi(K_{2,3}) = \xi(T_3) = 3$, the minor monotonicity of ξ gives that (b) $\implies$ (c).

Chartrand and Harary (1967) show:

if G is K_4- and $K_{2,3}$-minor free, then G is outerplanar.

Now show outplanar, 2-connected and T_3-free implies, G is a linear 2-tree.

Can use a technique known as Zero-forcing to show (d) implies (a). Lon will discuss zero-forcing, its variants and consequences in an upcoming lecture.

Characterization of G with $M(G) \leq 3$

The adjacency matrices of K_5 and $K_{3,3}$ both have the SAP and nullity 4.

So, the graphs G that we are seek to avoid K_5 and $K_{3,3}$ as a minors!

Wagner's theorem: The graphs that avoid K_5 and $K_{3,3}$ as minors are precisely the planar graphs.

Consequently,

If $\xi(G) \leq 3$, then G is a planar graph.

Partial k-paths

The set of k-trees is defined recursiveley as follows:

- A complete graph on $(k+1)$-vertices is a k-tree.
- If $G = (V, E)$ is a k-tree and $v_1, \ldots, v_k$ form a clique in G, then the graph obtained from G by inserting a new vertex v and edges from v to each of $v_1, \ldots, v_k$ is a k-tree.

A partial k-tree is a subgraph of a k-tree.

A k-path is a k-tree with either at most $k+1$ vertices or exactly two vertices of degree k. A partial k-path is a subgraph of a k-path.

Hein van der Holst (2007) characterized the 3-connected graphs with $M(G) = 3$ in the next result.

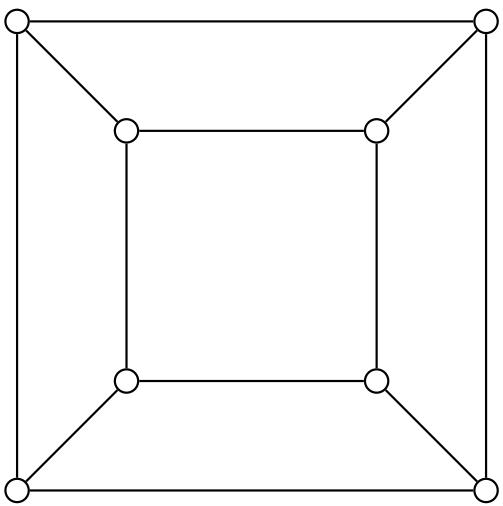
Theorem

For a 3-connected graph G, the following are equivalent:

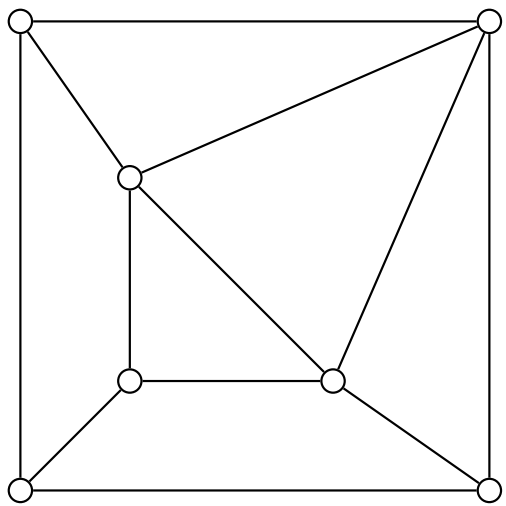
(a) *G is a partial 3-path;*

(b) *$M(G) = \xi(G) = 3$;*

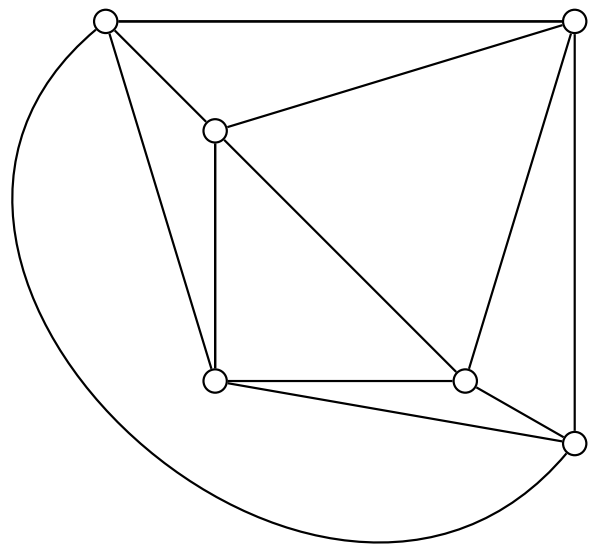
(c) *G is K_5-, $K_{2,2,2}$-, $K_{3,3}$-, Q_3-, and $Q_3Y\Delta$-minor free.*



Q_3



$Q_3Y\Delta$



$K_{2,2,2}$

$\xi(G)$, Hadwiger number and chromatic number $\chi(G)$.

The Hadwiger number $\eta(G)$, of a graph is the largest m such G has K_m as a minor.

By minor-monotonicity, for each graph G, $\eta(G) - 1 \leq \xi(G)$.

Hadwiger's conjecture (which some say is the most famous, open problem in graph theory) is

$$\chi(G) \leq \eta(G).$$

Thus, if Hadwiger's conjecture is true, then

$$\chi(G) - 1 \leq \xi(G).$$

Hadwiger's conjecture

Posed by Hadwiger (1943). He showed

$\chi(G) \geq 1 \implies G$ has K_1 as a minor;

$\chi(G) \geq 2 \implies G$ has K_2 as a minor;

$\chi(G) \geq 3 \implies G$ has $K_3 = C_3$ as a minor;

and studied the case of $\chi(G) \geq 4$.

Dirac (1952) proved the case $\chi(G) \geq 4$.

Wagner's work (1937) shows the case $\chi(G) \geq 5$ is equivalent to the 4-color theorem. (This was proven in Appel-Haken (1976))

$k = 6$ established by Roberston, Seymour, Thomassen (1993)

$k \geq 7$ open!.

Almost all and asymptotics

Mader (1968) proved a rather spectacular result: in loose terms, if a graph has a sufficiently high average degree, then the graph has a K_t as a minor.

Theorem (Thomason, 2001)

There exists a constant $\alpha = 0.319\ldots$ such that every graph G with average degree

$$d(G) \geq (\alpha + o(1))t\sqrt{\log_2 t}$$

contains a K_t-minor. Furthermore, this bound is asymptotically tight.

Theorem (Bollobás, Catlin, and Erdős, 1980)

For almost every graph G of order n, the Hadwiger number $h(G)$ satisfies:

$$\frac{n}{\sqrt{\log_2 n + 4}} \leq \eta(G) \leq \frac{n\sqrt{2}}{\sqrt{\log_2 n - \log_2 \log_2 n}}$$

Consequently, since $\chi(G) \sim \frac{n}{2\log_2 n}$ for almost every graph, Hadwiger's conjecture $\chi(G) \leq \eta(G)$ holds with probability tending to 1 as $n \to \infty$.

Theorem (Delcourt and Postle)

If $\eta(G) < k$, then

$$\chi(G) = O(klog(log(k)).$$

Nordhaus-Gaddum for ξ.

Theorem (Nordhaus-Gaddum (1956))

$$2\sqrt{|G|} \leq \omega(G) + \omega(\overline{G}) \leq \chi(G) + \chi(\overline{G}) \leq |G| + 1.$$

(Here, $\omega(G)$ is the clique number of G)

It is natural to ask for bounds on $\xi(G) + \xi(\overline{G})$ (or $M(G) + M(\overline{G})$).

Theorem (Hall, Hogben et al (2010); Barrett, Fallat et.al (2013))

For each graph G with at least 2 vertices:

$$(4/3)|G| \leq \xi(G) + \xi(\overline{G}) \leq \sqrt{2}|G|$$

This is related to some open problems.

Note the last result is equivalent to:

$$(2-\sqrt{2})|G| \leq \mathsf{mr}_{\xi}(G) + \mathsf{mr}_{\xi}(\bar{G}) \leq (2/3)|G|,$$

where $\mathsf{mr}_{\xi}(H)$ is the smallest rank of a matrix with graph H having the SAP.

GCC and weak GCC

Two conjectures that arose from the 2006 AIM Workshop.

The graph complement conjecture: there exists a c such that $\mathrm{mr}(G) + \mathrm{mr}(\bar{G}) \leq |G| + c$. for all graphs G.
Strong belief that can take $c = 2$.

The weak graph complement conjecture: there exists a constant b such that

$$\mathrm{mr}(G) + \mathrm{mr}(\bar{G}) \leq b|G| + 2.$$

for all graphs G.
(Strong belief that $b = 1$ works)

Can replace the above conjectures with mr_ξ or other strong variants. Lon has some nice work on these conjectures (see Mitchell [2023]).

Recent result

Theorem (Barioli, Fallat, Gupta, Li 2026)

For the Colin de Verdière-type parameter $\nu(G)$, there exists a constant $b \approx 1.708$ such that for every graph G on n vertices:

$$mr_\nu(G) + mr_\nu(G^c) \leq bn$$

Equivalently, the sum of the maximum nullities satisfies:

$$M(G) + M(G^c) \geq (2 - b)n.$$

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