The Strong Spectral Property I

THE ONLY SOLUTION

We shall have to evolve

problem-solvers galore–

for each problem they solve

creates ten problems more!

–Piet Hein

The short-story: the symmetric matrix A has the SSP provided

$$X \text{ symmetric, } A \circ X = O,\ I \circ X = O \text{ and } AX - XA = O$$
$$\Downarrow$$
$$X = O.$$

Note that the only difference between the SAP and the SSP is that $AX = O$ is replaced by $AX - XA = O$.

Often use $[A, X] = O$ rather than $AX - XA = O$ where $[A, X]$ is the commutator of A and X.

Derivation of the SSP

For the SAP, we use congruence ($A \mapsto P^\top AP$, P invertible), because congruence preserves nullity.

Orthogonal similarity (that is, $A \mapsto Q^{-1}AQ$, Q orthogonal) preserves symmetry and the spectrum.

Let $\mathrm{Skew}(n)$ denote the set of $n \times n$ orthogonal matrices.

It is known that if $K \in \mathrm{Skew}(n)$, then

- e^K is orthogonal,
- $(e^K)^\top = e^{-K}$; and
- every orthogonal matrix near I is of this form.

This suggests studying the map

$$f : \mathcal{O}_1 \to \mathrm{Sym}(n) \text{ by } f(K) = e^{-K} A e^K,$$

where $\mathcal{O}_1$ is a small enough neighborhood of O in $\mathrm{Skew}(n)$ that $K \in \mathcal{O}_1 \implies e^{-K} A e^K$ has nonzero entries where A's entries are nonzero.

So the map

$$f\colon \mathcal{O}_1 \to \mathrm{Sym}(n) \text{ by } f(K) = e^{-K} A e^{K},$$

f is differentiable at O, and

$$Df_O(K) = -KA + AK.$$

We'll use the familiar map $g\colon \langle \mathcal{S}(G) \rangle \to \mathrm{Sym}(n)$ by $g(B) = B$ where $G = G(A)$. We also choose a sufficiently small neighborhood $\mathcal{O}_2$ of O in $\langle \mathcal{S}(G) \rangle$ such that $B \in \mathcal{O}_2 \implies A + g(B) \in \mathcal{S}(G)$. As before

$$dg_O(B) = B$$

Next, we set $\mathfrak{f}\colon \mathcal{O}_1 \times \mathcal{O}_2 \to \mathrm{Sym}(n)$ by

$$\mathfrak{f}(K, B) = e^{-K} A e^{K} + B.$$

Following the development of the SAP, we desire $\mathfrak{f}$ to be locally surjective at (O, O); that is, we want $D\mathfrak{f}_{(O,O)}$ to have $\mathrm{Sym}(n)$ as its image.

Since $D\mathfrak{f}_{(O,O)} = -KA + AK + B$, the linearization of $\mathfrak{f}$ is surjective if and only if

$$\{-KA + AK : K \in \mathrm{Skew}(n)\} + \langle \mathcal{S}(G) \rangle = \mathrm{Sym}(n) \qquad (1)$$

We can take (1) as the definition of A having the SSP.

Or, as in the SAP case, we can use orthogonal complements to get another formulation.

Recall, we are thinking of $\mathrm{Sym}(n)$ as an inner product space, where $\langle C, D \rangle = \mathrm{tr}(CD)$.

Given a subspace V of $\mathrm{Sym}(n)$, $V^\perp$ is its orthogonal complement of V relative to $\mathrm{Sym}(n)$; that is

$$V^\perp = \{M \in \mathrm{Sym}(n) : \mathrm{tr}(MN) = 0 \text{ for all } N \in U.\}.$$

As before $\langle \mathcal{S}(G) \rangle^\perp = \{Y \in \mathrm{Sym}(n) : I \circ Y = O \text{ and } A \circ Y = O\}$.

Now

$$\begin{aligned}\mathrm{tr}((-KA+AK)X) &= \mathrm{tr}(-KAX+AKX)\\ &= \mathrm{tr}(-AXK+XAK)\\ &= \mathrm{tr}(-AX+XA)K).\end{aligned}$$

So the symmetric X is in $\{-KA+AK : K \in \mathrm{Skew}(n)\}^{\perp}$ if and only if $-AX+XA$ is orthogonal to every skew-symmetric matrix; or equivalently if and only if $-AX+XA$ is symmetric.
But $-AX+XA$ is skew-symmetric. Hence, the above is equivalent to $-AX+XA=O$.

So $\mathrm{im}(Df_O)^{\perp} = \{X \in \mathrm{Sym}(n) : AX = XA\}$.

Putting this all together,

$$\{-RA + AR : R \in \mathbb{R}^{n\times n}\} + \langle \mathcal{S}(G)\rangle = \mathrm{Sym}(n)$$

(via taking $\perp$'s) is equivalent to the subspaces

$$\begin{gathered}\{X \in \mathrm{Sym}(n) : AX = XA\} \\ \{Y \in \mathrm{Sym}(n) : A \circ I = O \text{ and } A \circ X = O\}\end{gathered} \tag{2}$$

having trivial intersection.

(2) is what we take for the definition of SSP: A has the Strong Spectral Property provided:

$$\begin{gathered}X \text{ symmetric, } A \circ X = O,\ I \circ X = O \text{ and } AX = XA \\ \Downarrow \\ X = O.\end{gathered}$$

Examples

- As $AX = O \implies [A - \lambda I, X] = O$, if A has the SSP, then $A - \lambda I$ has the SAP for each λ.
- $C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ has eigenvalues $0, 0, 2, -2$.

  For $X = \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \end{bmatrix}$ $(A - \lambda I)X = O \implies X = O$. But $[A, X] = O$ does not imply $X = O$. Thus $A + \lambda I$ has the SAP for each λ; but A does not have the SAP.

Spoiler space

The SSP spoiler space for A is

$$\text{spoil}(A) = \{X \in \text{Sym}(n) : I \circ X = O, A \circ X = O, AX = XA\}.$$

An SSP spoiler for A is a nonzero matrix in the spoiler space of A. Thus A has the SSP if and only if it does not have a spoiler.

Example (exercise). The spoiler space for

$$\begin{bmatrix} J_2 & O \\ O & J_2 \end{bmatrix}$$

is the set of all matrices of the form

$$\begin{bmatrix} 0 & 0 & a+b & a-b \\ 0 & 0 & a-b & a+b \\ a+b & a-b & 0 & 0 \\ a-b & a+b & 0 & 0 \end{bmatrix}.$$

Given $A \in \mathcal{S}(G)$ and H a supergraph of G, we say A has the SSP with respect to H provided $X = O$ is the only symmetric matrix such that $I \circ X = O$, $A_H \circ X = O$ and $[A, X] = O$, where A_H is the adjacency matrix of H.

$J_2 \oplus J_2$ does not have the SSP. It does have the SSP, with respect to any graph obtained by inserting two edges (exercise).

Theoretical underpinnings of SSP

Theorem

Verification Matrix for SSP Let $A \in \mathcal{S}(G)$ be an $n \times n$ matrix. Then there is an $\left(\binom{n+1}{2} - |E(G)|\right) \times \binom{n+1}{2}$ matrix Ψ_A such that A has the SSP if and only if the rows of Ψ_A are linearly independent.

Proof.

Let Φ be the $\binom{n+1}{2} \times \binom{n+1}{2}$ matrix whose rows are indexed by (r, s) $(1 \leq r \leq s)$, columns by (i, j) $(1 \leq i < j \leq n)$, and (i, j)th column is $\mathrm{vec}_\wedge((A(E_{ij} - E_{ji}))$. Then Φ is the submatrix of Ψ with rows (i, j) where $i \neq j$ and $ij \notin E(G)$.

□

There are some concrete problems in the homework that will deepen one's understanding of this result.

Openness property for the SSP

Theorem (SSP is an open property)

Let $A \in \mathrm{Sym}(n)$ has the SSP. Then each matrix in $\mathrm{Sym}(n)$ sufficiently close to A has the SSP.

Proof. A has the SSP if and only if its verification matrix has linearly independent rows. The verification matrix is a continuous function of the entries of A.

The result now follows from the continuity of the determinant function. $\square$

The Superpattern for SSP

SSP $\implies$ $\mathfrak{f}$ is locally surjective.

Let H be a superpattern of G, and let C be the adjacency matrix of $H - G$.

Then for ϵ sufficiently small, there are matrices $K \in \mathcal{O}_1 \subseteq \mathrm{Skew}(n)$ and $B \in \mathcal{O}_2 \subseteq \langle \mathcal{S}(G) \rangle$ with

$$e^{-K} A e^{K} = A - B + \epsilon C.$$

The LHS is similar to A; so has the same spectra as A.
The RHS is a matrix with graph H.

Theorem (Superpattern Theorem for the SSP)

If A has graph G, H is a supergraph of G and A has the SSP, then there exists a matrix $B \in \mathcal{S}(H)$ that has the same spectrum as A.

Theorem (Direct Sum Theorem for SSP)

Let A and B be symmetric matrices. Then $A \oplus B$ has the SSP if and only if both A and B have the SSP and share no common eigenvalue.

Proof.

X nonzero SSP spoiler for $A \implies X \oplus O$ is nonzero spoiler for $A \oplus B \implies A \oplus B$ does not have SSP. Similarly, B does not have SSP $\implies A \oplus B$ does not have SSP.

If A and B have a common eigenvalue λ, then

$$X = \begin{bmatrix} O & \mathbf{u}\mathbf{v}^\top \\ \mathbf{v}\mathbf{u}^\top & O \end{bmatrix}$$

is an SSP spoiler for $A \oplus B$, where $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors corresponding to λ for A and B, respectively. Hence, in this case, $A \oplus B$ does not have the SSP.

Can also argue (exercise!) that if $\begin{bmatrix} X_1 & X_2 \\ X_2^\top & X_3 \end{bmatrix}$ is a nonzero SSP spoiler for $A \oplus B$, then either X_1 is a nonzero spoiler for A, X_3 is a nonzero spoiler for B, or X_2 is a nonzero matrix with $AX_2 = X_2B$. This last implies that A and B has a common eigenvalue. □

Corollary (Distinct Spectra Realized by every graph)

Let G be a graph on n vertices, and Λ be a set of n distinct real numbers. Then there exists $A \in \mathcal{S}(G)$ with spectrum equal to Λ.

Proof.

Take D to be the diagonal matrix whose diagonal entries are the elements of Λ.

By last theorem, D has the SSP.

Now apply the Supergraph Theorem. □

RIP: Monfared-Shader proof.

SSP and Chromatic Number

$q(G)$ is the minimum of distinct eigenvalues of a matrix with graph G.

Theorem

Let G be a graph. Then $q(G) \leq \chi(\bar{G})/2$.

Proof.

Take an optimal coloring of the complement, $\bar{G}$ of G. Its color classes correspond to cliques of G.
Hence, G is the supergraph of a disjoint union, H, of $\chi(\bar{G})$ cliques.
Form a matrix M whose graph is H by using a matrix of the form $a_i I + b_i J$ for the ith connected component.
By choosing appropriate a_i and b_i we can make the eigenvalues of the different components distinct. Such M has the SSP. Also, M has $\chi(\bar{G})$ distinct eigenvalues and the SSP.
Now apply the Supergraph theorem. □

Theorem (Extended Superpattern theorem for SSP)

Let $A \in \mathcal{S}(G)$, and let H be a graph obtained for G by inserting k new vertices and then inserting any number of edges.
If S is a set of distinct real numbers disjoint from the spectrum of A, and A has the SSP, then there is a matrix with graph H whose spectrum consists of S along with the spectrum of A.

Example:

Let G be any graph of n vertices containing K_m. Let α be any multiset of n real numbers with at least $n - m + 1$ distinct elements. Then there is a matrix in $\mathcal{S}(G)$ with spectrum α.

Theorem (Matrix Liberation for the SSP)

Let A be an $n \times n$ matrix with the SSP, and let D be a matrix of the form $-KA + AK$ for some $K \in \mathrm{Skew}(n)$. Let G_D be the graph of A in the direction of D. If A has the SSP with respect to G_D, then there is a matrix with graph G_D, the SSP and the same spectrum as A.

Corollary (Liberating Sets Theorem)

Let $A \in \mathcal{S}(G)$, and α be a subset of edges not in G. If A has the SSP with respect to each $G + \alpha'$ where α' is obtained from G by deleting one element of α, then there is matrix in $(G + \alpha)$ having the SSP and the same spectrum as A.

Example. For $A = J_2 \oplus J_2 \in \mathcal{S}(2K_2)$, we know that A has the SSP with respect to any graph obtained from $2K_2$ by inserting any two edges. So by the Liberation Sets theorem, $K_4 -$ edge allows the spectrum $0, 0, 2, -2$.

We also could look at the function $\mathfrak{g} : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathrm{Sym}(n)$ by

$$\mathfrak{g}(K, B) = e^{-K}(A + B)e^{K}.$$

Since total derivative of $\mathfrak{g}$ at (O, O) is the same as that of $\mathfrak{f}$ at (O, O), g is locally surjective at (O, O) if and only if A has the SSP.

If g is locally surjective, then for some small $\epsilon > 0$, if $\widehat{C}$ is a symmetric matrix within ϵ of A, then there exist matrices K and B such that

$$e^{-K}(A + B)e^{K} = \widehat{A}.$$

In this case $A + B$ has the same spectrum as A, and since B is small, $A + B$ and A have the same graph.

Theorem (Proximity Theorem for SSP)

Let A be a matrix in $\mathcal{S}(G)$ with the SSP. Then $\exists\ \epsilon > 0$ such that if Λ is the spectrum of a symmetric matrix within ϵ of A, then Λ is the spectrum of matrix in $\mathcal{S}(G)$.

Let A be an $n \times n$ matrix. We can list its distinct eigenvalues by

$$\lambda_1 < \lambda_2 < \cdots < \lambda_q.$$

The ordered multiplicity list of A is

$$m_A = (m_1, m_2, \ldots, m_q),$$

where m_i is the multiplicity of λ_i as an eigenvalue of A.

A refinement of m_A is a tuple of positive integers obtained from m_A by replacing each m_i with a partition of m_i.

For example. $(2, 1, 2)$ is a refinement of $(3, 2)$, $(1, 3, 1, 1)$ is not a refinement of $(2, 2, 2)$.

Corollary (Bifurcation Theorem for Multiplicities)

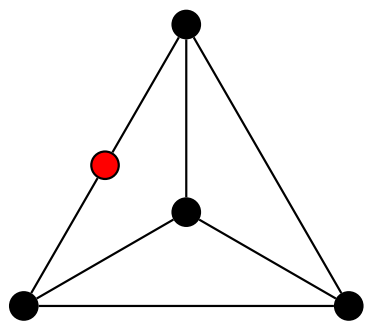
Let $A \in \mathcal{S}(G)$ have the SSP and ordered multiplicity list $m = (m_1, m_2, \ldots, m_q)$. Then for each refinement of $\widehat{m}$ of m, there exists a matrix in $\mathcal{S}(G)$ with ordered multiplicity list $\hat{m}$.

Examples.

- For K_n, J_n gives an ordered multiplicity list $(1, n-1)$ with the SSP. So we can get any $(m_1, m_2, \ldots, m_q)$ where $q \geq 2$, $m_1 = 1$, each m_i is positive and $\sum_i^q m_i = n$.
- For $a > 1$ the matrix

$$M_5 = \begin{bmatrix} a^2 & 0 & \sqrt{2}a & a & a \\ 0 & 1 & -\sqrt{2} & 1 & 1 \\ \sqrt{2}a & -\sqrt{2} & 4 & 0 & 0 \\ a & 1 & 0 & 2 & 2 \\ a & 1 & 0 & 2 & 2 \end{bmatrix}$$

has multiplicity $(3, 1, 1)$, the SSP and graph



$(K_4)_e$

Thus any extended graph of the $(K_4)_e$ allows any multiplicity with a single eigenvalue of multiplicity two for 3.

For a symmetric matrix A, $q(A)$ denotes the number of distinct eigenvalues of A.

$q(G)$ denotes the minimum of $q(A)$ over all $A \in \mathcal{S}(G)$.

For G of order n,

(a) $1 \leq q(G) \leq n-1$

(b) $q(G) = n-1$ if and only if G is a path;

(c) $q(G) = 1$ if and only if G has no edges.

(d) $q(G) = 2$ if and only if there is a symmetric orthogonal matrix Q with graph G.

Proof of (d). $q(G) = 2$ if and only if there is a matrix A in $\mathcal{S}(G)$ with exactly 2 distinct eigenvalues. By shift and scale we can assume these are ± 1.

So $q(G) = 2$ if and only if there is a matrix $A \in \mathcal{S}(G)$ with $A^\top A = A^2 = I$. The latter is equivalent to the existence of an orthogonal matrix with graph G.

The bifurcation Lemma for Multiplicities implies that if H is a supergraph of G, then $q(H) \leq q(G)$.

This leads to questions of on-going interest such as:

- Determine the set of ordered multiplicity lists achieved by $A \in \mathcal{S}(G)$.
- For which valid multiplicity lists $m = (m_1, \ldots, m_q)$ for $\mathcal{S}(G)$ is it the case that every $\lambda_1 < \cdots < \lambda_q$ there is a matrix $A \in \mathcal{S}(G)$ with these eigenvalue and multiplicity list m?
- Determine or bound $q(G)$ for various families of graphs.

We'll disucss the first two in the next lecture.

Helena will the third questions later this week.

Theorem (Decontraction Lemma)

Let G be a graph on n vertices that is a simple contraction of H. Let $A \in \mathcal{S}(G)$ have the SSP. Then for λ of sufficiently large absolute value, there is a matrix B in $\mathcal{S}(H)$ such that $\operatorname{spec}(B) = \{\lambda\} \cup \operatorname{spec}(A)$.

Example: constraints on λ may be needed!

C_4 is a minor of C_5. C_4 allows a matrix A with the SSP and $\mathbf{m}_A = (2, 2)$.

However, it is known that if $B \in \mathcal{S}(C_5)$, then either the largest or smallest eigenvalue of B is simple.

Hence, $\mathcal{S}(C_5)$ does not allow the ordered multiplicity list $(2, 1, 2)$.

But, the decontraction theorem does imply that C_5 allows the ordered multiplicity lists $(1, 2, 2)$ and $(2, 2, 1)$.

Theorem

Let $m = (m_1, \ldots, m_q)$ be the ordered multiplicity list of a matrix $A \in \mathcal{S}(G)$ having the SSP. If G is a minor H, then there exists a matrix $B \in \mathcal{S}(H)$ with the SSP whose ordered multiplicity list ends with m.

Corollary

Let $\mathbf{m} = (m_1, \ldots, m_q)$ be a list of q positive integers. Then there exists a finite list $\mathcal{L}$ of graphs, such that a graph G allows a symmetric matrix with the SSP whose ordered multiplicity list ends with $\mathbf{m}$ if and only if M does not contain any graph in $\mathcal{L}$ as a minor.

Can replace "ends with" any one of:

(a) "starts with"

(b) "is a consecutive string in"

(c) "agrees with m starting at position k" k fixed.

(d) "agrees with m starting at position $|H| - k$.

For an $n \times n$ symmetric matrix A, define $\mathrm{hap}(A)$ to be $n - q(A)$; that is, $\mathrm{hap}(A)$ is the number of coincidences among the eigenvalues of A.

Define $\mathrm{hap}(G)$ to be $\max_{A \in \mathcal{S}(G)} \mathrm{hap}(A)$.

We've already seen that hap doesn't behave well with respect to subgraphs (or more generally minors)

However, the parameter

$$\overline{\mathrm{hap}(G)} := \max_{\substack{A \in \mathcal{S}(G) \\ \text{with SSP}}} \mathrm{hap}(A)$$

is much nicer.

Theorem (Hap happens)

$\overline{\mathrm{hap}}$ *is minor monotone; that is,*

$$G \text{ a minor of } H \implies \overline{\mathrm{hap}(G)} \leq \overline{\mathrm{hap}(H)}.$$

$\overline{hap}$ will be discussed more in a forthcoming lecture.

Sketch of proof of SSP Decontraction Theorem

A key idea here is a simple observation about matrix equations. Consider the matrix equation

$$\begin{bmatrix} M & N \\ O & \lambda I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}, \tag{3}$$

where M is invertible, M, N, $\mathbf{b}$ and $\mathbf{c}$ are fixed; and λ is nonzero but can vary.
The solution to (3) is:

$$\mathbf{x} = M^{-1}\mathbf{b} - \frac{1}{\lambda} M^{-1} N \mathbf{c} \qquad \mathbf{y} = \frac{1}{\lambda}\mathbf{c}.$$

So as $\lambda \to \infty$, $\mathbf{x}$ is $O(\mathbf{b})$, and $\mathbf{y}$ of negligible order. If we perturb $\mathbf{b}$ to $\mathbf{b}'$ by order ϵ, and $\mathbf{c}$ to $\mathbf{c}'$ by order $\lambda\epsilon$, then the solution to the new system changes by $O(\epsilon)$.
Upshot: by choosing λ sufficiently large, we can still have control of solutions to systems where the $\mathbf{c}$ is significantly perturbed.

Use in proof of the SSP Decontraction theorem

We use $\|E\|_{\max}$ to denote the maximum absolute value of an entry in the matrix E. For $\epsilon > 0$ and a graph G, $\mathcal{S}(G)_\epsilon$ denotes all symmetric matrices E such that $\|E\|_{\max} \leq \epsilon$ and $e_{ij} \neq 0 \implies i = j$ or ij is an edge of G.

Setting.

H a graph with

vertices $1, \ldots, n+1$;

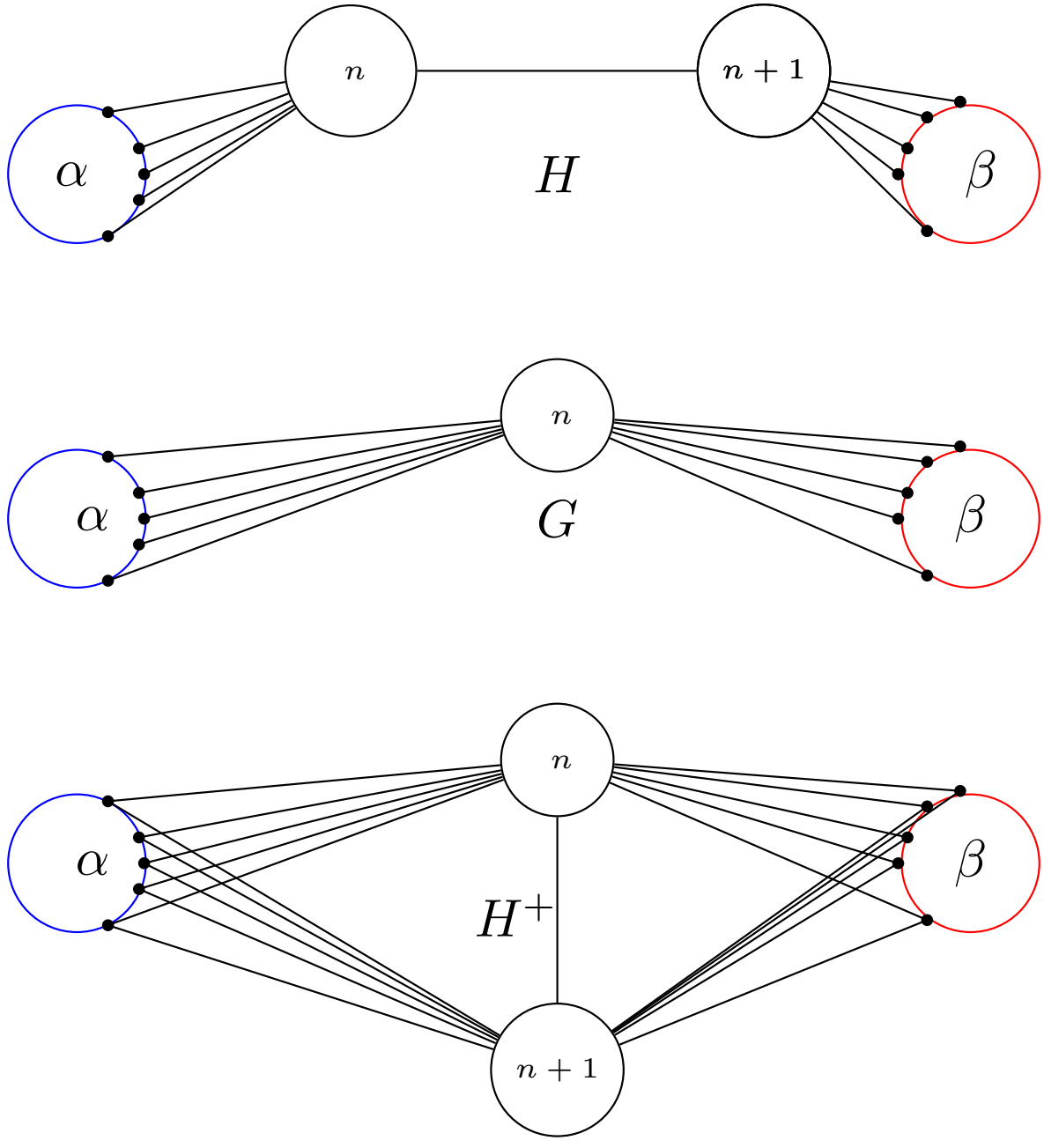
an edge between n and $n+1$;

n and $n+1$ have no common neighbors.

G is the contraction of H along the edge n–$(n+1)$.

The set of neighbors of $n+1$ in H other than n is denoted by β and the set of neighbors of n, other than $n+1$ is denoted by α.

H^+ is the graph obtained from G by inserting vertex $n+1$ and joining it to n and all neighbors of n.


n
n + 1
α
H
β
n
α
G
β
n
α
H+
β
n + 1

We have a matrix $A \in \mathcal{S}(G)$ with the SSP.

$$A = \left[\begin{array}{ccc|c} & A(1) & & \begin{matrix}\mathbf{0}\\ \mathbf{u}\\ \mathbf{v}\end{matrix} \\ \hline \mathbf{0}^\top & \mathbf{u}^\top & \mathbf{v}^\top & a_{nn} \end{array}\right]$$

The rows of $\mathbf{u}$ correspond to α, and the rows of $\mathbf{v}$ to β. We wish to use A to find a matrix in $\mathcal{S}(G)$ whose spectrum consists of that of A along with one additional eigenvalue λ.
A good place to start is

$$A_\lambda := \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^\top & \lambda \end{bmatrix}$$

As long as λ is not in the spectrum of A, A_λ has the SSVP. So any supergraph, in particular, H^+, of $G(A_\lambda)$ allows a matrix with the same eigenvalues as A_λ. But a matrix in $\mathcal{S}(H^+)$ has more edges than H. But...we need to be able to "eliminate" some nonzero entries!

A reasonable idea is to try to construct a matrix in $\mathcal{S}(H^+)$ of the form

$$C = \left[\begin{array}{ccc|c|c} & \hat{A}(1) & & \mathbf{0} & \mathbf{0} \\ & & & \hat{\mathbf{u}} & -\hat{\mathbf{u}} \\ & & & \hat{\mathbf{v}} & \hat{\mathbf{v}} \\ \hline \mathbf{0}^\top & \hat{\mathbf{u}}^\top & \hat{\mathbf{v}}^\top & d & w \\ \hline \mathbf{0}^\top & -\hat{\mathbf{u}}^\top & \hat{\mathbf{v}}^\top & w & \hat{\lambda} \end{array}\right] \tag{4}$$

with the same spectrum as A_λ, and then use a Given's similarity to get the desired matrix in $\mathcal{S}(H)$.

More precisely, using the matrix where $Q = (I \oplus \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ we obtain

$$Q^\top C Q = \left[\begin{array}{ccc|c|c} & \hat{A}(1) & & \mathbf{0} & \mathbf{0} \\ & & & \hat{\mathbf{0}} & -\sqrt{2}\hat{\mathbf{u}} \\ & & & \sqrt{2}\hat{\mathbf{v}} & \hat{\mathbf{0}} \\ \hline \mathbf{0}^\top & \hat{\mathbf{0}}^\top & \sqrt{2}\hat{\mathbf{v}}^\top & * & \neq \\ \hline \mathbf{0}^\top & -\sqrt{2}\hat{\mathbf{u}}^\top & \hat{\mathbf{0}}^\top & \neq & * \end{array}\right].$$

$Q^\top C Q$ has graph H, and spectrum that of A_λ.

The trouble is: we want $C(1)$ to be close to A, but then it seems that in the last row and column of C we are doing "large" perturbations. But here's where the above observation comes into play.

If E is a matrix whose graph is a subgraph of that of $G \sqcup K_1$ with $\|E\|_{\max} \leq \epsilon$, then the SSP verification matrix of $A_\lambda + E$ has the form

$$\begin{bmatrix} \Psi(A) & R \\ O & \lambda I - A \\ 0 & \mathbf{r}^\top \end{bmatrix} + O(E),$$

where each entry of R and $\mathbf{r}$ has modulus at most 1.

So, for λ sufficiently large and ϵ small, $A_\lambda + \epsilon E$ will have the SSP, and the Local Surjectivity theorem can handle perturbations of $A_\lambda + \epsilon E$ of "large order".

For $E \in \mathcal{S}(G)_\epsilon$,

$$\hat{E} = \left[\begin{array}{cccc|c} & & E & & \mathbf{0} \\ & & & & -\mathbf{u} \\ & & & & \mathbf{v} \\ & & & & 1 \\ \hline \mathbf{0}^\top & -\mathbf{u}^\top & \mathbf{v}^\top & 1 & 0 \end{array}\right]$$

Observations from the previous slide imply that $\exists\ \Delta < 1$ such that for each $E \in \mathcal{S}(G)_\epsilon$, $\exists\ F \in \mathcal{S}(H^+)$ such that

- $\|F\|_{\max} \leq \Delta \|E\|_{\max}$;
- the graph of F is a subgraph of H^+; and
- the matrix $B_E := A_\lambda + \hat{E} + F$ has the same spectrum as A_λ.

In other words: B_E is "nearly" of the desired form

$$C = \left[\begin{array}{ccc|c|c} & \hat{A}(1) & & \mathbf{0} & \mathbf{0} \\ & & & \hat{\mathbf{u}} & -\hat{\mathbf{u}} \\ & & & \hat{\mathbf{v}} & \hat{\mathbf{v}} \\ \hline \mathbf{0}^\top & \hat{\mathbf{u}}^\top & \hat{\mathbf{v}}^\top & d & w \\ \hline \mathbf{0}^\top & -\hat{\mathbf{u}}^\top & \hat{\mathbf{v}}^\top & w & \hat{\lambda} \end{array}\right]$$

This gives us a map $\omega\colon \mathcal{S}(G)_\epsilon \to \langle \mathcal{S}(G) \rangle$ by

$$\omega(E) = (Q^\top B_E Q)(n+1).$$

We claim there is an E such that $\omega(E) = E$.

For such an E, B_E has the form (4), and $Q^\top B_E Q$ is matrix with graph H and their spectra are the same as that of A_λ.

And we're done!

Suppose not $\omega(E) \neq O$ for all $E \in \mathcal{S}(G)_\epsilon$.

Then for each E, $\|(\omega(E))\|_{\max} > 0$, and we define a function $\tau : = \mathcal{S}(G)_\epsilon \to \mathcal{S}(G)_\epsilon$ by

$$\tau(E) = -\epsilon \frac{E - \omega(E)}{\|(E - \omega(E)\|_{\max}}.$$

τ maps the "closed ball" $\mathcal{S}(G)_\epsilon$ to itself.

We claim that τ has no fixed point.

To see this consider (i, j) where the (i, j)-entry of $\tau(E)$ has largest absolute value. Note for this (i, j) the (i, j)-entry of $\tau(E)$ equals $\pm\epsilon$.

If $i = j$ or i—j is an edge of G and neither i nor j is n, then the (i, j)-entry of $\tau(E)$ has absolute value at most $\Delta\epsilon < \epsilon$. So $\tau(E) \neq E$ is this case.

The n-th row of B_E looks like

$$\begin{bmatrix} \mathbf{0} & \mathbf{u}^\top + \mathbf{e}_1^\top + \mathbf{f}_1^\top & \mathbf{v}^\top + \mathbf{e}_2^\top + \mathbf{f}_2^\top & * & * \end{bmatrix}.$$

The $(n+1)$-st row of B_E looks like

$$\begin{bmatrix} \mathbf{0} & -\mathbf{u}^\top & \mathbf{v}^\top & * & * \end{bmatrix}.$$

As $\|F\|_{\max} < \Delta\|E\|_{\max}$, In this case, $E_{n,j}$ and $\tau(E_{n+1,j})$ have opposite signs.

So $\tau(E) \neq E$.

But ...

Brouwer (1911) now stands up and shouts: this is impossible!
Every continuous map from a closed ball to itself has a fixed point!

Therefore, there is an $E \in \mathcal{S}(G)_\epsilon$ such that $\omega(B_E) = E$.

One can verify that $\omega(E)$ has graph H, is co-spectral with A_λ, and has the SSP.

This gives the sketch of a proof of the Decontraction theorem!
Note it is significantly more complicated than that for the SAH.
There we could choose λ arbitrarily close to 0.

The Strong Properties Template

Hopefully, you are beginning to see a pattern.

We wish to study some spectral property P of matrices in some family $\mathcal{F}$.

We find a continuous group G that acts on $\mathcal{F}$ and preserves property P.

For fixed $A \in \mathcal{F}$, we define a map $\mathfrak{f}(G, B) = A^G + B$.

We compute $D\mathfrak{f}_(O, O)$ and define the Strong (P, F) property to be the condition that the linearization is onto.

Using $\perp$'s we can get an equivalent formulation of the strong property.

Once we have the strong (P, F)-property we use analysis to get.

- (P, F) verification matrix
- The strong (P, F)-property is an open property.
- The Superfamily theorem for the strong (P, F)-property.
- The Proximity theorem for the strong (P, F)-property.
- When possible, the Extended Superfamily theorem.
- When possible, a Decontraction theorem.
- When possible, minor monotone parameters related to P.

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