Jacobian methods and strong properties

Inverse eigenvalue problems for sign patterns or zero-nonzero patterns

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix} \qquad \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & * & * \end{bmatrix}$$

A matrix pattern $\mathcal{A}$ is spectrally arbitrary if the pattern puts no restrictions on the the eigenvalues of a matrix.

A pattern $\mathcal{A}$ is inertially arbitrary if $i(\mathcal{A})$ contains every ordered triple of nonnegative integers (i_+, i_-, i_0) with $i_+ + i_- + i_0 = n$

$q(A)$ is the number of distinct eigenvalues of A.
$q(\mathcal{A})$ is the minimum number of distinct eigenvalues allowed by $\mathcal{A}$.

$\mathcal{A}$ spectrally arbitrary $\implies$ $\mathcal{A}$ is inertially arbitrary
$\mathcal{A}$ spectrally arbitrary $\implies$ $\mathcal{A}$ allows a nilpotent matrix
$\implies$ $q(\mathcal{A}) = 1$.

Small orders

Order 2: [Drew, Johnson, Olesky, vd Driessche. 2000]
$\mathcal{A}$ is inertially arbitrary $\iff$ $\mathcal{A}$ is spectrally arbitrary $\iff$ $\mathcal{A}$ is equivalent to

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix}$$

Order 3: [Britz, McDonald, Olesky, vd Driessche. 2004] $\mathcal{A}$ is spectrally arbitrary $\iff$ up to equivalence, $\mathcal{A}$ is a superpattern of

$$\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, or \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ - & + & + \end{bmatrix}.$$

Order 3: [Cavers, VM. 2005] $\mathcal{A}$ is inertially arbitrary $\iff$ $\mathcal{A}$ is spectrally arbitrary.

Small orders, and a necessary condition

Order 4: [Cavers, VM. 2005]

$$\mathcal{N} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & - & - \\ + & + & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}.$$

$\mathcal{N}$ is inertially but not spectrally arbitrary.

Theorem (Cavers, VM. 2005)

If $\mathcal{A}$ is inertially arbitrary, then $\mathcal{A}$ must have at least one negative diagonal entry and at least one positive diagonal entry and at least one negative two-cycle ($a_{ij}a_{ji} < 0$).

Historical example: Jacobian method

$$\mathcal{A} = \begin{bmatrix} - & + \\ - & + \end{bmatrix}. \quad A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \qquad \text{Set } X = \begin{bmatrix} x_1 & 1 \\ -4 & x_2 \end{bmatrix}.$$

$c_X(z) = z^2 + f_1 z + f_2$, with

$$f_1 = -x_1 - x_2 \qquad \text{and} \qquad f_2 = x_1 x_2 + 4$$

$$Df(\mathbf{x}) = \left[\frac{\partial f_i}{\partial x_j}\right] = \begin{bmatrix} -1 & -1 \\ x_2 & x_1 \end{bmatrix}$$

$$\operatorname{rank}\left(Df(-2,2)\right) = \operatorname{rank}\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} = 2$$

We say $\mathcal{A}$ (or A) allows a full rank Jacobian.

Theorem (Drew, Johnson, Olesky, van den Driessche. 2000)
A is nilpotent and allows a full-rank Jacobian $\Rightarrow$ $\mathcal{A}$ is spectrally arbitrary.

Nipotent Jacobian method

Given $\mathcal{A}$ is $n \times n$. Suppose there exists a nilpotent $A \in \mathcal{A}$ with $m \geq n$ nonzero entries. Let X denote the matrix obtained from A by replacing the m nonzero entries with variables $x_1, \ldots, x_m$. Let

$$c_X(z) = z^n + f_1 z^{n-1} + f_2 z^{n-2} + \cdots + f_{n-1} z + f_n$$

where each $f_i = f_i(x_1, \ldots, x_m)$ is a polynomial in $x_1, \ldots, x_m$.
If rank$(Df|_{X=A}) = n$, then we say A allows a full-rank Jacobian.

Theorem (Drew, Johnson, Olesky, van den Driessche. 2000)

If $A \in Q(\mathcal{A})$ is nilpotent and A allows a full-rank Jacobian, then every superpattern of $\mathcal{A}$ is a spectrally arbitrary pattern.

Examples

Theorem (Britz, McDonald, Olesky, van den Driessche 2004)

For $n \geq 3$,

$$\mathcal{V}_n = \begin{bmatrix} + & - & 0 & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & 0 & - \\ + & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

is spectrally arbitrary.

Theorem (Pereira 2007)

Every full pattern that allows a nilpotent matrix is spectrally arbitrary.

Open and Superpattern Theorem for full-rank Jacobian

Theorem (Cavers, Garnett, Kim, Olesky, van den Driessche, VM 2013)
Suppose $A \in \mathcal{A}$ allows a full-rank Jacobian. If

$$c_A(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n$$

then for any $(b_1, \ldots, b_n)$ sufficiently close to $(c_1, \ldots, c_n)$, each superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$, $\exists B \in Q(\hat{\mathcal{A}})$ that has a full-rank Jacobian and

$$c_B(z) = z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_{n-1} z + b_n.$$

Proximity theorem for full-rank Jacobian

Corollary (Cavers, Garnett, Kim, Olesky, van den Driessche, VM 2013)
Suppose $A \in Q(\mathcal{A})$ allows a nonzero Jacobian. If A has spectrum

$$\operatorname{spec} A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

then for any multiset $\{\mu_1, \mu_2, \ldots, \mu_n\}$ closed under complex conjugation, with each μ_k sufficiently close to λ_k, each superpattern $\hat{\mathcal{A}}$ of pattern $\mathcal{A}$ has a matrix $B \in Q(\hat{\mathcal{A}})$ with

$$\operatorname{spec} B = \{\mu_1, \mu_2, \ldots, \mu_n\}.$$

Bifurcation and inertia

Recall $i(A) = (i_+, i_-, i_0)$.
The refined inertia of A is $ri(A) = (i_+, i_-, i_z, 2i_p)$ where i_z is the number of zero eigenvalues and $2i_p = i_0 - i_z$.

Lemma (Cavers, Garnett, et al. 2013)

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ has a full-rank Jacobian, and let $ri(A) = (i_+, i_-, i_z, 2i_p)$.

(i) *If $i_z \geq 1$, then*
$(i_+ + 1, i_-, i_z - 1, 2i_p),\ (i_+, i_- + 1, i_z - 1, 2i_p) \in ri(\hat{\mathcal{A}})$.

(ii) *If $i_z \geq 2$, then $(i_+, i_-, i_z - 2, 2(i_p + 1)) \in ri(\hat{\mathcal{A}})$.*

(iii) *If $i_p \geq 1$, then*
$(i_+ + 2, i_-, i_z, 2(i_p - 1)),\ (i_+, i_- + 2, i_z, 2(i_p - 1)) \in ri(\hat{\mathcal{A}})$.

Furthermore, for each modified refined inertia, there is a matrix in $\hat{\mathcal{A}}$ with this refined inertia that has a full-rank Jacobian.

Example: full-rank Jacobian a necessary condition

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

There is a matrix with pattern $\mathcal{A}$ and refined inertia $(0, 3, 0, 2)$ (make all entries unit), but no matrix with this pattern and refined inertia allows a full-rank Jacobian. (TBD)

The pattern has no matrices having eigenvalues with positive real part. [Jeffries, Klee, van den Driessche 1977]. Thus there is no matrix with pattern $\mathcal{A}$ and refined inertia $(2, 3, 0, 0)$.

Application of bifurcation: a North-East lemma

Corollary (Cavers, Garnett et al. 2013)

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ has

$$ri(A) = (i_+, i_-, i_z, 2i_p)$$

with $i_z \geq 2$. If A has a full-rank Jacobian, then

$$(a, b, c) \in i(\hat{\mathcal{A}})$$

for every $a \geq i_+$, $b \geq i_-$ and $c \geq 0$ with $a + b + c = n$.

Inertially arbitrary patterns

Theorem (Cavers, Fallat, 2012)

Suppose $A \in Q(\mathcal{A})$ has refined inertia $ri(A) = (0, 0, i_z, 2i_p)$ for some $i_z \geq 2$. If A has a full-rank Jacobian, then every superpattern of $\mathcal{A}$ is inertially arbitrary.

Theorem (Cavers, Garnett et al. 2013)

Suppose $A, B \in Q(\mathcal{A})$ have refined inertias $ri(A) = (1, 0, i_z(A), 2i_p(A))$ and $ri(B) = (0, 1, i_z(B), 2i_p(B))$ such that $i_z(A), i_z(B) \geq 2$. If both A and B have a full-rank Jacobian and $(0, 0, n) \in i(\hat{\mathcal{A}})$ for some superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$, then $\hat{\mathcal{A}}$ is inertially arbitrary.

Example: $n \geq 6$

$$\mathcal{W}_n = \begin{bmatrix} + & - & 0 & 0 & 0 & \cdots & 0 \\ + & 0 & - & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & - & 0 & \ddots & \vdots \\ 0 & + & 0 & 0 & - & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & + & 0 & \cdots & \cdots & 0 & - \\ 0 & + & 0 & \cdots & \cdots & 0 & - \end{bmatrix}.$$

There is a matrix $W \in \mathcal{W}_n$ with refined inertia $(0, 0, 4, n-4)$ (that has a full-rank Jacobian). Thus $\mathcal{W}_n$ is inertially arbitrary and so is every superpattern of $\mathcal{W}_n$.

Another Hessenberg example

Theorem (Cavers, Garnett et al. 2013)

Let $\mathcal{A}$ be a full tridiagonal sign pattern. If there exists a matrix $A \in Q(\mathcal{A})$ with refined inertia $(0, 0, i_z, 2i_p)$ with $i_z \geq 2$, then all superpatterns of $\mathcal{A}$ are inertially arbitrary.

More Hessenberg inertially arbitrary pattern

Theorem (Cavers, Garnett, et al. 2013)

Let $\mathcal{A}$ be a proper lower Hessenberg sign pattern with the first column having all entries nonzero. If $A \in Q(\mathcal{A})$ has refined inertia $(0, 0, i_z, 2i_p)$ with $i_z \geq 2$, then all superpatterns of $\mathcal{A}$ are inertially aribitrary.

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 & 0 \\ - & 0 & 0 & + & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & 0 & - & - \end{bmatrix}$$

$\mathcal{A}$ allows inertia $(0, 0, 2, 4)$. Thus $\mathcal{A}$ is inertially aribitrary.
$\mathcal{A}$ does not allow a characteristic polynomial with the coefficient of x^2 negative. Thus $\mathcal{A}$ is not spectrally arbitrary.

Pattern bordering

Given A of order n let B be the *bordered* order $n+1$ matrix:

$$B = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{x}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{z} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} I_n & \mathbf{0} \\ -\mathbf{x}^T & 1 \end{bmatrix} = \left[\begin{array}{c|c} A - \mathbf{z}\mathbf{x}^T & \mathbf{z} \\ \hline \mathbf{x}^T(A - \mathbf{z}\mathbf{x}^T) & \mathbf{x}^T\mathbf{z} \end{array}\right].$$

A nilpotent $\Rightarrow$ B nilpotent.

$$\mathcal{A} = \begin{bmatrix} + & - & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & - \\ 0 & 0 & + & - \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + \\ 0 & 0 & + & - & 0 \\ 0 & 0 & + & - & 0 \\ - & 0 & 0 & 0 & - \end{bmatrix}$$

Bordering extensions

Theorem (Olesky, van den Dreissche, VM 2017)

Suppose $A \in \mathcal{A}$ has refined inertia $(0, 0, r, n-r)$ with $r \geq 2$, and A allows a full-rank Jacobian. Suppose

- $a_{kk} \neq 0$
- $a_{kv} \neq 0$ *for some* $v \neq k$.
- *cofactor* C_{kv} *is nonzero.*

then a bordered matrix B has refined inertia $(0, 0, r+1, n-r)$, B allows a full-rank Jacobian and every superpattern of $\mathcal{B}$ is inertially arbitrary.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 1 & x_2 & 0 & 0 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 1 & x_4 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + \\ 0 & 0 & + & - & 0 \\ 0 & 0 & + & - & 0 \\ - & 0 & 0 & 0 & - \end{bmatrix}$$

Another bordering extension

Theorem (Olesky, van den Dreissche, VM 2017)

Suppose $A \in \mathcal{A}$ with refined inertia $(0, 0, r, n-r)$ for some $r \geq 2$ and A allows a full-rank Jacobian. Suppose

- a_{jk}, $j \neq k$, *is non-variable for some choice of* X.
- $a_{kv} \neq 0$ *for some* v
- *cofactor* C_{jv} *is nonzero*

then a bordered matrix B has refined inertia $(0, 0, r+1, n-r)$, B allows a full-rank Jacobian, and every superpattern of $\mathcal{B}$ is inertially arbitrary.

$$A = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 \\ 0 & - & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & + & 0 \end{bmatrix}.$$

Jacobian of full rank and $q(\mathcal{A}) = 1$

Theorem (Breen, Brouwer, Catral, Cavers, van den Driessche, VM 2022)

Let $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. If $A \in \mathcal{A}$ allows a full-rank Jacobian and $q(A) = 1$ then $q(\hat{\mathcal{A}}) = 1$. Further, either $\hat{\mathcal{A}}$ or $-\hat{\mathcal{A}}$ allows the characteristic polynomial $(z+1)^n$.

Example

$$\mathcal{C} = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & + \\ - & - & \cdots & - & - \end{bmatrix}$$

$\mathcal{C}$ realizes characteristic polynomial $(z+1)^n$ and has a full-rank Jacobian. Thus $q(\hat{\mathcal{C}}) = 1$ for every superpattern $\hat{\mathcal{C}}$ of $\mathcal{C}$.

Close look at Jacobian entries*

Lemma (Garnett, Shader, 2013)

Let X be the matrix obtained by replacing the entries of $A \in \mathbb{R}^{n\times n}$ by variables. If c_k is the coefficient of z^{n-k} in the characteristic polynomial of X, then

$$\left.\frac{\partial c_k}{\partial x_{i,j}}\right|_{Y=A} = \left[\textit{the coefficient of } z^{n-k} \textit{ in } -\operatorname{adj}(zI-A)_{ji}\right]$$

Let $c_X(z) = z^n + c_1 z^{n-1} + \cdots + c_{n-1}z + c_n$. For (i,j) with $x_{ij} \neq 0$,

$$\begin{aligned}
\frac{\partial c_X}{\partial x_{ij}} &= \lim_{h\to 0} \frac{\det(zI-(X+hE_{ij})) - det(zI-X)}{h} \\
&= \lim_{h\to 0} \frac{\det(zI-X) - h(-1)^{i+j}\det((zI-X)(i,j)) - det(zI-X)}{h} \\
&= -\operatorname{adj}(zI-X)_{ji}
\end{aligned}$$

Hence

$$\frac{\partial c_1}{\partial x_{ij}} z^{n-1} + \frac{\partial c_2}{\partial x_{ij}} z^{n-2} + \cdots + \frac{\partial c_{n-1}}{\partial x_{ij}} z^1 + \frac{\partial c_n}{\partial x_{ij}} = -\operatorname{adj}(zI-X)_{ji}$$

*This section: results of [Garnett, Shader, 2013] and [Cavers, Garnett, Kim, Olesky, van den Driessche, VM 2013]

Structure of adjoint

Lemma (Garnett, Shader, 2013)

If A is nilpotent then

$$\operatorname{adj}(zI - A)_{ji} = \sum_{k=0}^{n-1} A_{ji}^k z^{n-1-k}$$

$$(zI - A)^{-1} = \frac{1}{z}\left(I - \frac{1}{z}A\right)^{-1} = \frac{1}{z}\left(I + \frac{1}{z}A + \frac{1}{z^2}A^2 + \cdots + \frac{1}{z^{n-1}}A^{n-1}\right)$$

Recall for any $B \in \mathbb{R}^{n\times n}$, $B\operatorname{adj}(B) = (\det B)I$, so

$$(zI - A)\operatorname{adj}(zI - A) = z^n I$$

hence

$$\operatorname{adj}(zI - A) = z^n (zI - A)^{-1} = \left(z^{n-1}I + z^{n-2}A + z^{n-3}A^2 + \cdots + A^{n-1}\right)$$

Structure of Jacobian: example

Lemma (Garnett, Shader, 2013)

Let X be the matrix obtained by replacing the nonzero entries of nilpotent $A \in \mathbb{R}^{n\times n}$ by variables. If c_k is the coefficient of z^{n-k} in the characteristic polynomial of X, then

$$\left.\frac{\partial c_k}{\partial x_{ij}}\right|_{X=A} = -A_{ji}^{k-1}$$

$$\mathcal{A} = \begin{bmatrix} 0 & + & + \\ - & + & 0 \\ - & 0 & - \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \qquad \text{is nilpotent}$$

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^1 = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -4 & 4 & -4 \\ -4 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$J|_{X=A} = (-1)\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 2 & 2 & -1 & 2 & -1 & -2 \\ 4 & -4 & -4 & 2 & 2 & 2 \end{bmatrix}$$

prerequisite for nilpotent Jacobian method

Theorem (Bergsma, VM 2012)

If A is nilpotent and $A^k = O$ for some $k < n$, then A does not have a full-rank Jacobian.

Interlude

$A^k = O$ for some $k < n$ means the minimal polynomial of A is not equal to its characteristic polynomial.

A matrix is nonderogatory if the minimal polynomial and characteristic polynomial of A are equal.

Facts: the following are equivalent for $A \in \mathbb{R}^{n \times n}$

- A is nonderogatory
- the only matrices which commute with A are polynomials in A.
- each eigenvalue of A appears in exactly one Jordan block of the Jordan canonical form of A
- the geometric multiplicity of each eigenvalue is one
- A is similar to an lower Hessenberg matrix H with $H_{ij} \neq 0$ for $j = i + 1$

see e.g. [Mackey, Mackey, Petrovic, 1999]

rephrasing the prerequisite to the nilpotent-Jacobian method

Theorem (Bergsma, VM 2012)

If A is nilpotent and A has a full-rank Jacobian, then A is nonderogatory.

Corresponding results for a not-necessarily nilpotent matrix

$$\operatorname{adj}(zI - A) = \sum_{k=0}^{n-1} h_k(A) z^{n-1-k}$$

for some polynomials $h_k(z)$ of degree k, $1 \leq k \leq n-1$ and $h_0(A) = I$. (If A nilpotent, then $h_k(z) = z^k$).

Lemma (Cavers, Garnett, et al. 2013)

Let X be the matrix obtained by replacing the nonzero entries of $A \in \mathbb{R}^{n \times n}$ by variables. If c_k is the coefficient of z^{n-k} in the characteristic polynomial of X, then

$$\left.\frac{\partial c_k}{\partial x_{ij}}\right|_{X=A} = -[h_{k-1}(A)]_{ji}$$

Characterization of full-rank Jacobian

Theorem (Cavers, Garnett, et al. 2013)

A does not have a full-rank Jacobian if and only if there exists a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$.

[$\Rightarrow$] If $\mathrm{J}|_A$ has rank less than n, then $\exists$ scalars $c_0, c_1, \ldots, c_{n-1}$ not all zero such that $(c_0 I + c_1 h_1(A) + \cdots + c_{n-1} h_{n-1}(A)) \circ A^T = O$. But this implies that there exists a nonzero polynomial $q(z)$ of degree at most $n-1$ (since each $h_k(z)$ has degree at most $n-1$) such that $q(A) \circ A^T = O$.

[$\Rightarrow$] If $\mathrm{J}|_A$ has full rank, then the set

$$W = \{I \circ A^T, h_1(A) \circ A^T, \ldots, h_{n-1}(A) \circ A^T\}$$

is linearly independent. But $\mathrm{span}(W) \subset \mathrm{span}(W')$ for

$$W' = \{I \circ A^T, A \circ A^T, A^2 \circ A^T, \ldots, A^{n-1} \circ A^T\}.$$

Thus, W' is a linearly independent set and $q(A) \circ A^T \neq O$ for any nonzero polynomial $q(z)$ of degree at most $n-1$.

Nonderogatory condition for full rank Jacobian

Theorem (Cavers, Garnett, et al. 2013)

A does not have a full-rank Jacobian if and only if there exists a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$.

Note: If A is derogatory and $q(z)$ is the minimum polynomial of A, then $\deg(q(z)) \leq n-1$ and $q(A) = O$.

Corollary (Cavers, Garnett, et al. 2013)

If A is a derogatory matrix of order n, then A does not allow a full rank Jacobian.

Lemma (Cavers, Garnett, et al. 2013)

Suppose A is a nonderogatory matrix of order n. Then $\exists$ a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$ if and only if $\exists\ X \neq O$ such that $XA = AX$ and $X \circ A^T = 0$.

Let A be a nonderogatory matrix of order n.

[$\Leftarrow$] Suppose there is a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ with $q(A) \circ A^T = O$. Since A is nonderogatory, the matrices $I, A, A^2, \ldots, A^{n-1}$ are linearly independent. This implies $q(A) \neq 0$, since $q(z) \neq 0$. Furthermore, $q(A)$ commutes with A. Thus, we may take $X = q(A)$.

[$\Rightarrow$] Suppose there is a matrix $X \neq O$ which commutes with A and $X \circ A^T = O$. Since A is nonderogatory, $X = q(A)$ for some nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$.

Established equivalences

Given A is nonderogatory:

A does not have a full-rank Jacobian

$$\Updownarrow$$

$\exists$ a nonzero polynomial $q(z) \in \mathbb{R}[z]$ of degree at most $n-1$ such that $q(A) \circ A^T = O$.

$$\Updownarrow$$

$\exists\ X \neq O$ such that $XA = AX$ and $X \circ A^T = 0$.

$$\Updownarrow$$

$\exists\ X \neq O$ such that $X^T A = AX^T$ and $X \circ A = 0$.

Nilpotent Centralizer method

Theorem (Cavers, Garnett, et al. 2013)

Let A be a nonderogatory. Then A has a full-rank Jacobian if and only if the zero matrix is the only matrix X such that $XA = AX$ and $X \circ A^T = O$.

Theorem (Nilpotent centralizer method)

(Garnett, Shader, 2012) Let $A \in Q(\mathcal{A})$ be a nonderogatory nilpotent matrix. If $X = O$ is the only matrix X such that $XA = AX$ and $X \circ A^T = O$, then $\mathcal{A}$ is spectrally arbitrary.

Corollary (Nilpotent centralizer method)

(Garnett, Shader, 2012) Let $A \in Q(\mathcal{A})$ be a nonderogatory nilpotent matrix. If $X = O$ is the only matrix X such that $XA = AX$ and $X \circ A^T = O$, then $\mathcal{A}$ is spectrally arbitrary.

$$\mathcal{T}_3 = \begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & - & - \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{2} \\ 0 & -1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & x_1 \\ 0 & x_2 & 0 \\ x_3 & 0 & 0 \end{bmatrix}$$

$$AX = \begin{bmatrix} 0 & \frac{1}{2}x_2 & x_1 \\ \frac{1}{2}x_3 & 0 & -x_1 \\ -x_3 & -x_2 & 0 \end{bmatrix} \qquad XA = \begin{bmatrix} 0 & -x_1 & -x_1 \\ -x_2 & 0 & \frac{1}{2}x_2 \\ x_3 & \frac{1}{2}x_3 & 0 \end{bmatrix}$$

$$[AX = XA] \Rightarrow X = 0$$

Thus $\mathcal{T}_3$ is spectrally arbitrary.

Path

[Drew, Johnson, Olesky, van den Driessche. 2000]

$$\mathcal{T}_n = \begin{bmatrix} - & + & 0 & \cdots & \cdots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \cdots & \cdots & 0 & - & + \end{bmatrix}$$

Conjecture: $\mathcal{T}_n$ is spectrally arbitrary for all $n \geq 2$.



[Behn, Driessel, Hentzel, Vander Velden, Wilson. 2011]
$\mathcal{T}_n$ is allows a nilpotent matrix.

Path

[Drew, Johnson, Olesky, van den Driessche. 2000]

$$\mathcal{T}_n = \begin{bmatrix} - & + & 0 & \cdots & \cdots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \cdots & \cdots & 0 & - & + \end{bmatrix}$$

Conjecture: $\mathcal{T}_n$ is spectrally arbitrary for all $n \geq 2$.



[Behn, Driessel, Hentzel, Vander Velden, Wilson. 2011]
$\mathcal{T}_n$ is allows a nilpotent matrix.

Theorem (Garnett, Shader. 2011)

Every superpattern of $\mathcal{T}_n$ is spectrally arbitrary for $n \geq 2$.

Early inertially arbitrary classes

$$\mathcal{S}_{n,r} = \begin{bmatrix} - & + & \cdots & + & 0 & \cdots & \cdots & 0 \\ - & 0 & + & & \ddots & \ddots & & \vdots \\ - & - & 0 & + & \ddots & & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & + \\ - & - & & & \ddots & \ddots & + & + \\ 0 & - & - & & & \ddots & + & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & 0 & + \\ 0 & \cdots & 0 & - & - & \cdots & - & + \end{bmatrix}$$

$\mathcal{S}_{n,n}$ is inertially arbitrary. [Gao, Shao. 2001]
$\mathcal{S}_{n,n-1}$ is inertially arbitrary. [Miao, Li. 2002]

Finishing earlier example with refined inertia $(0, 3, 0, 2)$

Theorem (Cavers, Garnett, et al. 2013)

Let A be a nonderogatory. Then A has a full-rank Jacobian if and only if the zero matrix is the only matrix X such that $XA = AX$ and $X \circ A^T = O$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then A is nonderogatory, $X \circ A^T = O$, and $XA = AX$.
Therefore, A does not allow a full-rank Jacobian.

Other Jacobian contexts...

Structured matrices: quadratic eigenvalue problems*

The structured matrix

$$C(A,B) = \left[\begin{array}{c|c} A & B \\ \hline I & O \end{array}\right]$$

is studied in the context of a linear system of n differential equations

$$\frac{d^2\mathbf{x}}{dt^2} - A\frac{d\mathbf{x}}{dt} - B\mathbf{x} = 0 \tag{1}$$

with $A, B \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

$\lambda \in \operatorname{spec}(C) \iff \lambda$ is a root of the char. equation for (1)
i.e. $\det(\lambda^2 I - \lambda A - B) = 0$

*Structured matrix results from [Berliner, Catral, Cavers, Kim, van den Driessche 2024]

Structured matrices: a Jacobian

Given G, H are vertex-labelled graphs. Let

$$\mathcal{S}(G,H) := \{C(A,B) \in \mathbb{R}^{2n\times 2n} \mid A \in \mathcal{S}(G), B \in \mathcal{S}(H)\}$$

Let X be the matrix obtained from $C(A,B) \in \mathcal{S}(G,H)$ by replacing a_{kk} and b_{kk} with x_k and x_{n+k} resp. and

$$c_X(z) = z^{2n} + f_1 z^{2n-1} + \cdots + f_{2n-1}z + f_{2n}$$

C has a structured full-rank Jacobian if the corresponding polynomial map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$F(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_{2n}(\mathbf{x}))$$

has a nonsingular Jacobian.

A Jacobian technique for structured matrices

$$C(A,B) = \left[\begin{array}{c|c} A & B \\ \hline I & O \end{array}\right]$$

Theorem (Berliner, Catral, et al. 2024)

Suppose $C(A,B) \in \mathcal{S}(G,H)$ has a structured full-rank Jacobian and the coefficients of the characteristic polynomial of $C(A,B)$ are $(c_1,\ldots,c_{2n})$. If $(c_1',\ldots,c_{2n}')$ is sufficiently close to $(c_1,\ldots,c_{2n})$, then for every supergraph G' of G and every supergraph H' of H, $\exists C' \in \mathcal{S}(G',H')$ such that C' has a characteristic polynomial with coefficients $(c_1',\ldots,c_{2n}')$.

Jacobian method for structured matrices

$$C(A,B) = \left[\begin{array}{c|c} A & B \\ \hline I & O \end{array}\right]$$

Corollary (Berliner, Catral, et al. 2024)

Suppose $C \in \mathcal{S}(G,H)$ has a structured full-rank Jacobian. If Λ is set of complex numbers, closed under complex conjugation, that is sufficiently close to $\operatorname{spec}(C)$, then $\forall$ supergraph G' of G and $\forall$ supergraph H' of H, $\exists C' \in \mathcal{S}(G',H')$ with $\operatorname{spec}(C') = \Lambda$.

Corollary (Berliner, Catral, et al. 2024)

Suppose $C \in \mathcal{S}(G,H)$ has a structured full-rank Jacobian and C is nilpotent. Then $S(G,H)$ is spectrally arbitrary. In fact, $S(G',H')$ is spectrally arbitrary $\forall$ supergraph G' of G and $\forall$ supergraph H' of H,

Example: structured Jacobian

G a path on three vertices and H is an empty graph.

$$C = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

is nilpotent, $C \in \mathcal{S}(G, H)$, and C has a structured full-rank Jacobian (with determinant 1). Thus $\mathcal{S}(G, H)$ is spectrally arbitrary.

Structured spectrally arbitrary patterns

Theorem (Berliner, Catral, et al. 2024)

If G, H are vertex-labelled graphs with $n \leq 4$ vertices, then $\mathcal{S}(G, H)$ is spectrally arbitrary.

Theorem (Berliner, Catral, et al. 2024)

If G, H are vertex-labelled graphs with n vertices, and suppose G has Hamiton path. Then $\mathcal{S}(G, H)$ is spectrally arbitrary.

Conjecture: [Berliner, Catral et al. 2024] If G, H are any vertex-labelled graphs then $\mathcal{S}(G, H)$ is spectrally arbitrary.

Variation on polynomial map: trace map

Given a pattern $\mathcal{A}$ and a matrix $X \in Q(\mathcal{A})$ with some of the nonzero positions containing a variable in $\{x_1, \ldots, x_n\}$ and with $c_X(z) = z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n$. The polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ is

$$f(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_n(\mathbf{x})).$$

An alternate mapping, a trace-map is $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$g(\mathbf{x}) = (\mathrm{tr}(X), \mathrm{tr}(X^2), \ldots, \mathrm{tr}(X^n))$$

Newton's identities for elementary symmetric functions implies $\exists$ an infinitely differential function $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = g \circ h$.

Hence the the Jacobian $Df(\mathbf{x})$ has full-rank if and only if the Jacobian $Dg(\mathbf{x})$ has full-rank.

Example with trace Jacobian

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} -x_1 & 1 & 0 \\ -x_2 & 0 & 1 \\ -x_3 & 0 & 1 \end{bmatrix} \qquad X^2 = \begin{bmatrix} x_1^2 - x_2 & x_1 & 1 \\ x_1x_2 - x_3 & -x_2 & 1 \\ x_1x_3 - x_3 & -x_3 & 1 \end{bmatrix}$$

$$X^3 = \begin{bmatrix} -x_1^3 + 2x_1x_2 - x_3 & x_1^2 - x_2 & 1 - x_1 \\ x_3x_1 - x_3 + x_2^2 - x_1^2x_2 & x_1x_2 - x_3 & 1 - x_2 \\ -x_3(x_1^2 - x_1 - x_2 + 1) & x_1x_3 - x_3 & 1 - x_3 \end{bmatrix}$$

$$g(x_1, x_2, x_3) = (1 - x_1, 1 - 2x^2 + x_1^2, 1 - 3x_3 + 3x_1x_2 - x_1^3)$$

$$Dg(\mathbf{x}) = \begin{bmatrix} -1 & 0 & 0 \\ 2x_1 & -2 & 0 \\ -3x_1^2 + 3x_2 & -3x_1 & -3 \end{bmatrix}$$

$Dg((1,1,1))$ has full rank $\implies$ the pattern of A is spectrally arbitrary.

Results that used a trace map

Theorem (Monfared, Shader, 2013)

Let G be a connected grpah on n vertices and real numbers $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n$ then there is a matrix $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ such that $A(i)$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$.

Theorem (Berliner, Catral, Cavers, Kim, van den Driessche, 2024)

Let G and H be vertex-labelled graphs on n vertices and Λ be a multiset of $2n$ complex numbers invariant under conjugation such that real (resp. non-real) elements have multiplicity at most two (resp. one). Then there exists $C \in \mathcal{S}(G, H)$ such that $\operatorname{spec}(C) = \Lambda$.

Another method: Jacobian with deficient rank

Theorem (Breen, Brouwer, Catral, Cavers, vdDriessche, VM 2022)

Let $\mathcal{A}$ be a loopless pattern and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. If $A \in \mathcal{A}$ allows a Jacobian with rank $n-1$ then there exists $B \in \hat{\mathcal{A}}$ such that $q(B) = q(A)$.

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ + & + & + & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 8 & 6 & 0 \end{bmatrix}$$

$q(A) = 2 \quad \Rightarrow \quad q(\mathcal{A}) = 2$ ($\mathcal{A}$ is not potentially nilpotent)

A allows a rank 3 Jacobian $\quad \Rightarrow \quad q(\hat{\mathcal{A}}) \leq 2$ for all superpatterns $\hat{\mathcal{A}}$ of $\mathcal{A}$

Deficient rank Jacobian: $q = 1$

Corollary (Breen, Brouwer, et al. 2022)

Let $\mathcal{A}$ be an $n \times n$ loopless pattern. If $A \in \mathcal{A}$ allows a Jacobian with rank $n-1$ and $q(A) = 1$, then $q(\hat{\mathcal{A}}) = 1$ for every superpattern $\hat{\mathcal{A}}$.

$$\mathcal{Z} = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Z is nilpotent and allows a Jacobian with rank 2.
$\Rightarrow q(\hat{\mathcal{Z}}) = 1$ for any superpattern $\hat{\mathcal{Z}}$ of $\mathcal{Z}$.

Deficient rank Jacobian: $q = 1$

Corollary (Breen, Brouwer, et al. 2022)

Let $\mathcal{A}$ be an $n \times n$ loopless pattern. If $A \in \mathcal{A}$ allows a Jacobian with rank $n-1$ and $q(A) = 1$, then $q(\hat{\mathcal{A}}) = 1$ for every superpattern $\hat{\mathcal{A}}$.

$$\mathcal{Z} = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Z is nilpotent and allows a Jacobian with rank 2.
$\Rightarrow q(\hat{\mathcal{Z}}) = 1$ for any superpattern $\hat{\mathcal{Z}}$ of $\mathcal{Z}$.

Note: $\mathcal{Z}$ does not allow a matrix having $q = 1$ with a rank 3 Jacobian.

Note: allowing $q = 1$ is preserved for superpatterns here, but not allowing nilpotence.

Example: loopless and rank $n-1$ Jacobian

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & + \\ + & + & \cdots & + & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_n & a_{n-1} & \cdots & a_2 & 0 \end{bmatrix}$$

with $a_m = (n-1)\binom{n-1}{m-1} - \binom{n-1}{m}$. Then $q(A) = 2$.

$q(\mathcal{A}) \neq 1$; hence $q(\mathcal{A}) = 2$.
A allows a Jacobian with rank $n-1$ (since the Jacobian is equivalent to a diagonal matrix with exactly one zero diagonal entry) $\Rightarrow$ every superpattern $\hat{\mathcal{A}}$ has $q(\hat{\mathcal{A}}) \leq 2$.
In fact,

$$q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 0, 1, \ldots, 1 \rangle \quad \text{or} \quad \mathrm{q}_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, \ldots, 1 \rangle$$

a derivative-Jacobian method

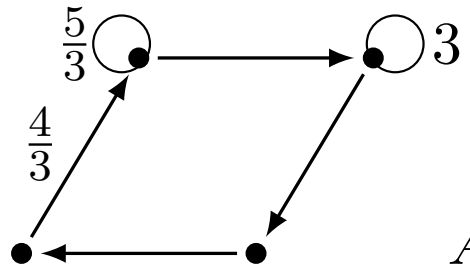
Theorem (Breen, Brouwer, Catral, Cavers, vdDriessche, VM 2024)

Let $\mathcal{D}$ be digraph on n vertices. Suppose $\mathcal{D}$ has a subdigraph G with at least two arcs and $\lambda \in \mathbb{R}$ is an eigenvalue of A having multiplicity two. Suppose X is a matrix obtained by replacing two nonzero arc weights a and b of A with real variables x_1 and x_2 and the characteristic polynomial of X is $c(z)$. If the 2×2 Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial c(\lambda)}{\partial x_1} & \frac{\partial c(\lambda)}{\partial x_2} \\ \frac{\partial c'(\lambda)}{\partial x_1} & \frac{\partial c'(\lambda)}{\partial x_2} \end{bmatrix}$$

has full rank when evaluated at $(x_1, x_2) = (a, b)$, then there exists $B \in \mathcal{A}(\mathcal{D})$ having λ as an eigenvalue with multiplicity two.

Using the derivative-Jacobian method: repeated roots

$\mathcal{D}$ 

$$A = \begin{bmatrix} \frac{5}{3} & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4}{3} & 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_2 & 0 & 0 & 0 \end{bmatrix}$$

A has eigenvalues $1, 1, \frac{4}{3} \pm \frac{2\sqrt{7}}{3}$: $\quad c_A(1) = 0$ and $c'_A(1) = 0$.

$$c_X(z) = z^4 - (x_1 + 3)z^3 + (3x_1)z^2 - x_2$$

$$c_X(1) = -2 + 2x_1 - x_2$$

$$c'_X(1) = -5 + 3x_1$$

$$J = \left.\frac{\partial(c_X(1), c'_X(1))}{\partial(x_1, x_2)}\right|_{(x_1,x_2)=(\frac{5}{3},\frac{4}{3})} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$$

J has rank 2 $\Rightarrow$ every digraph having $\mathcal{D}$ as a subdigraph allows an eigenvalue 1 with multiplicity 2.

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