# Strong Spectral/Multiplicity Property
# &
# Applications

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- Donald Duck: Mathematics!
- The True Spirit of Adventure: Right. Mathematics. The boundless treasures of science are locked behind those doors. In time, they will be opened by the curious and inquiring minds of future generations. In the words of Galileo, "Mathematics is the alphabet with which God has written the universe."

# The many doors of strong properties

| Name | Matrix Family | Invariants | Condition | Established Results | Associated Parameters |
|---|---|---|---|---|---|
| Strong Arnol'd (SAP) | $A \in \mathcal{S}(G)$ | nul $A$; $i(A)$ | $X$ symmetric,<br>$A \circ X = O$, $I \circ X = O$, $AX = O$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | $M(G), \xi(G)$ |
| Strong Spectral (SSP) | $A \in \mathcal{S}(G)$ | spec$(A)$ | $X$ symmetric,<br>$A \circ X = O$, $I \circ X = O$, $[A, X] = O$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | hap$(G)$, $\overline{\text{hap}}$(G) |
| Strong Multiplicity (SMP) | $A \in \mathcal{S}(G)$ | multiplicity lists | $X$ symmetric,<br>$A \circ X = O$, $I \circ X = O$, $[A, X] = O$,<br>tr$(A^i X) = 0$ for $i = 1, \ldots, n-1$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | hap$(G)$ |
| Strong Null Interlacing (SNIP) | $A \in \mathcal{S}(G)$<br>$r$ fixed index | (nul $A$, nul $A(v)$) | $X$ symmetric,<br>$A \circ X = O$, $I \circ X = O$, $AX(r,:] = O$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | (nul$(A)$, nul$(A(r))$, $\xi\xi$ |
| Duarte Interlacing | $A \in \mathcal{S}(G)$<br>$r$ fixed vertex | (spec$(A)$, spec$(A(r))$<br>Interlaced spectra | $X$ symmetric<br>$A \circ X = O$, $I \circ X = O$, $[A, X](r,:] = O$<br>$\Downarrow$<br>$X = O$ | | |
| Strong symplectic spectral | $N \in \mathcal{S}(G)^L$ | spec$_S$ | $Y$ symmetric<br>$Y \circ N = O$<br>$\Omega N Y = Y N \Omega$<br>$\Downarrow$<br>$Y = O$ | O,S,E,P,B,L,D | $M_S(G^L)$ |

# Additional doors

| Name | Matrix Family | Invariants | Condition | Established Results | Associated Parameters |
|---|---|---|---|---|---|
| Asymmetric Strong Arnol'd (ASAP) | $A_{n\times n} \in \mathcal{Z}(P)$ | nullity<br>algebraic multipicity of $0$ | $A \circ X = O$,<br>$AX^\top$ skew-symmetric,<br>$A^\top X$ skew-symmetric<br>$\Downarrow$<br>$X = O$ | O,S,E,P,,D | stable nullity |
| Non-symmetric<br>Strong Arnol'd (nSAP) | $A_{n\times n} \in \mathcal{Z}(P)$<br>$a_{ii} = 0$ for all $i$ | real congruence<br>invariants | $AX^\top = O$ and $X^\top A = O$<br>$\Downarrow$<br>$X = O$ | $A \circ X = O$,<br>O,S,E,P,L,M,D | |
| Non-symmetric<br>Strong Spectral (nSSP) | $A \in \mathcal{Z}(P)$ | Jordan form $A$<br>gmult of $0$ | $A \circ X = O$,<br>$[A, X^\top] = O$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | |
| Non-symmetric<br>Strong Multiplicity (nSMP) | $A \in \mathcal{Z}(P)$ | Jordan form $A$<br>gmult of $0$ | $A \circ X = O$,<br>$[A, X^\top] = O$,<br>$\mathrm{tr}(A^i X^\top) = O$ for $i = 1, \ldots, n$<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,M,D | |
| Strong Inner Product (SIPP)<br>$M^\top M$ | $M$ $m \times n$ of rank $m$ | nullity pairs | $X$ $m \times n$<br>$(XM) \circ M = O$<br>$\Downarrow$<br>$X = O$ | | |
| Strong Singular Value (SSVP) | $A \in \mathcal{Z}(P)$ | $\sigma$-values | $X$ $m \times n$<br>$A \circ X = O$,<br>$A^\top X$ symmetric,<br>$XA^\top$ symmetric,<br>$\Downarrow$<br>$X = O$ | O,S,E,P,B,L,D | |

# Keys

| Family description | Abbreviation |
|---|---|
| Symmetric matrices with graph $G$ | $\mathcal{S}(G)$ |
| Positive semidefinite matrices with graph $G$ | $\mathcal{S}_+(G)$ |
| Symmetric matrices with labeled graph $G^L$ | $\mathcal{S}(G)^L$ |
| Matrices with zero-nonzero pattern $P$ | $\mathcal{Z}(P)$ |
| Matrices with sign-pattern $S$ | $\mathcal{Q}(S)$ |

| Name of result | Abbreviation |
|---|---|
| Openness | O |
| Super-theorem | S |
| Extended Super-theorem | E |
| Proximity property | P |
| Bifurcation property | B |
| Matrix Liberation | L |
| Minors | M |
| Density | D |

- Openness means that if the property is true for a matrix, then it is true for all matrices in some open neigbhorhood.
- The Super-theorem for a matrix $A$ asserts that if $A$ has the given strong property with the given structure (e.g., graph, zero-nonzero pattern), then for each structure on the same vertex set that contains that of $A$ has a matrix with the same pertinent spectral property as $A$.
- The Extended Super-theorem is like the super-theorem, but allows the addition of new vertices in the structure.
- The Proximity property asserts that there is an $\epsilon > 0$ such that if $\|B - A\| < \epsilon$ and $B$ has a pertinent spectral invariant but different structure than $A$, then there is a matrix with structure that of $A$ with the same spectral invariant as $B$.
- The bifurcation property asserts that each refinement of the multiplicities of eigenvalues or singular values of a matrix
  $A$ with the SSP can be achieved with the same structure as $A$.
- Minors refers to the strong property gives rise to graph parameters that are minor monotone.
- Density refers to lower bounds on the size (e.g. number of edges, or nonzero entries) of the structure needed to achieve a certain spectral invariant with the given strong property.

# Doors for Jacobian methods

| name | pattern classes | replace entries with variables | Jacobian | contexts/properties |
|---|---|---|---|---|
| square | $n \times n$, $Z(P), Q(P)$ | $n$ entries | char. poly. coeff. | $S, I, q$ |
| rectangular | $n \times n$ $Z(P), Q(P)$ | all nonzero entries | char. poly. coeff. | $S, I, q$ |
| structured | $2n \times 2n$ $C = \begin{bmatrix} A & D \\ I & O \end{bmatrix}$ | $n$ entries of $A$ and of $D$ | char. poly. coeff. | $S, I, q$ |
| derivative | $n \times n$, $Z(P), Q(P)$, repeat eigenvalue $\lambda$ | $n$ entries | char. poly and derivative at $\lambda$ | $q$ |
| deficient rank | $n \times n$, $Z(P), Q(P)$, $P_{ii} = 0$ | $n$ entries, rank $n-1$ | char. poly. coeff. | $q$ |
| trace | $Z(P), Q(P)$ | $n$ entries | $f_k = \frac{1}{k}\mathrm{trace}(X^k)$ | $S$ |

S - spectrum
I - inertia
q - distinct eigenvalues

# Strong Multiplicity Property[1]

The strong multiplicity property (SMP) is the SSP with the following additional constraints on certain traces:

$$\text{tr}(A^i X) = O \text{ for } i = 1, \ldots, n.$$

That is, $A$ has the SMP provided:

*$X = O$ is the only symmetric matrix such that $I \circ X = O$, $A \circ X = O$, $[A, X] = O$ and tr$(A^i X) = O$ for $i = 1, \ldots, n$.*

Certainly SSP $\implies$ SMP. SMP is a more powerful tool when all we care about is multiplicities.

[1] Unless otherwise stated resulst in this lecture are from Barret et al

# Example

Consider the matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The symmetric $X$ such that $I \circ X = O$, $I \circ A = O$ and $AX = XA$ are $X$ of the form

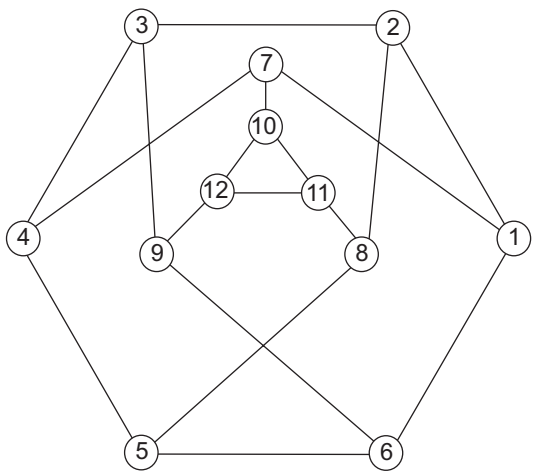
$$X = \begin{bmatrix} 0 & 0 & a & 0 & -a & 0 \\ 0 & 0 & 0 & a & 0 & -a \\ a & 0 & 0 & 0 & a & 0 \\ 0 & a & 0 & 0 & 0 & a \\ -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 \end{bmatrix}$$

For such $X$, $\mathrm{tr}(N^2X) = 4a$. So the additional trace conditions require $X = O$. So $A$ has the SMP but not the SSP.

# Interesting example

There does exist $A \in \mathcal{S}(C_6)$ with the SSP with the same multiplicity list as $N$; namely, $(1, 2, 2, 1)$

The graph below has the property that it realizes the multiplicity list $(3, 5, 4)$ with the SMP. However, such realization has the SSP!




Can show (exercise) if $G$ has the SSP and multiplicity list $(m_1, \ldots, m_q)$, then $G$ has at least $\sum_{i=1}^{q} \binom{m_i}{2}$ edges. Hunt for a multiplicity list and graph that barely violate this condition. Then hope to find a realization with the SMP and multiplicity list $(3, 5, 4)$. This $G$ has $18 < \binom{3}{2} + \binom{5}{2} + \binom{4}{2}$. So no matrix in $S(G)$ has multiplicity list $(3, 5, 4)$.

The SMP is based on the following observations. An $n \times n$ symmetric $A$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_q$ can be expressed as:

$$A = \sum_{k=1}^{q} \lambda_j E_j,$$

where $E_j$ is the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace corresponding to $\lambda_j$. The $E_j$ are called the Frobenius covariants of $A$.

Can show:

- $I = E_1 + \cdots + E_q$
- $AE_k = \lambda_k E_k$
- $E_i E_j = E_i E_j = \delta_{i,j} E_i$, and
- each $E_k$ is a polynomial in $A$.

## Lemma

*Let $A$ be a symmetric $n \times n$ matrix with distinct eigenvalues $\lambda_1 < \cdots < \lambda_q$. Let $A = \sum_{i=1}^{k} \lambda_i E_i$, where $E_i$ are the Frobenius covariants of $A$. Then $\{E_1, \ldots, E_q\}^{\perp} = \{I, \ldots, A^{q-1}\}^{\perp}$.*

Proof. It suffices to show that $\langle E_1, \ldots, E_q \rangle = \langle A^0, \ldots, A^{q-1} \rangle$.

As $A = \lambda_i E_i$ and the $E_i$ are mutually orthogonal idempotents, $A^k = \sum_{i=1}^{q} \lambda_i^k E_i$, and we have the containment $\supseteq$.

Since each $E_i$ is a polynomial in $A$, and each $A^m$ $(m \geq q)$ is a polynomial in $A$ of degree at most $q$. Hence we have the containment $\subseteq$. □

For a given $A$, can now define a map $h\colon \mathbb{R}^q \to \mathrm{Sym}(n)$ by

$$(x_1, \ldots, x_q) \mapsto \sum_{i=1}^{q} x_i E_i$$

Then $Dh_{\mathbf{0}}(\mathbf{x}) = \sum_{i=1}^{q} x_i E_i$.
Next, define $\mathfrak{h}\colon \mathcal{O} \times \langle \mathcal{S}(G) \rangle \times \mathbb{R}^q \to \mathrm{Sym}(n)$ by

$$(K, B, \mathbf{c}) \mapsto e^{-K}(A + \sum_{j=1}^{q} c_i E_i)e^{K} + B.$$

Then $D\mathfrak{h}_{(O,O,0)} = -KA + AK + \sum_{j=1}^{q} c_i E_i + B$.
So $D\mathfrak{h}$ is locally surjective at the origin exactly when

$$\{-KA+AK : K \in \mathrm{Skew}(n)\}+\langle \mathcal{S}(G) \rangle+\langle E_i : i = 1, \ldots, q \rangle = \mathrm{Sym}(n).$$

This is equivalent to (via $\perp$'s to): the definition of $A$ having the SMP.

# Theoretical consequences of the SMP

If $H$ is a supergraph of $G$, let $C$ be the adjacency matrix of $H - G$. Then for $\epsilon$ sufficiently small, there exist $K$, $L$, $\mathbf{c}$ of order $\epsilon^2$ such that

$$e^{-K}(A + \sum_{i=1}^{k} c_i E_i)e^K + B = A + \epsilon C.$$

So the matrix $e^{-K}(A + \sum_{i=1}^{k} c_i E_i)e^K = A + \epsilon C$ has graph $H$, and is similar to $A + \sum_{i=1}^{k} c_i E_i$.

This equals

$$\sum_{i=1}^{q} (\lambda_i + c_i) E_i,$$

and has ordered multiplicity list $\mathbf{m}$.

## Theorem (Supergraph, extended supergaph, bifurcation, minors theorem for multiplicity lists)

*Let $A \in \mathcal{S}(G)$ have the SMP and ordered multiplicity list $\mathbf{m}$. Then*

(a) *If $H$ contains $G$ as an induced subgraph, then there is a matrix in $\mathcal{S}(H)$ with the SMP whose ordered multiplicity list can be obtained from $\mathbf{m}'$ by inserting any number of ones in any location.*

(b) *if $\mathbf{m}'$ is refinement of $\mathbf{m}$, then there is a matrix in $\mathcal{S}(G)$ with the SMP and ordered multiplicity list $\mathbf{m}'$.*

(c) *If $H$ contains $G$ as a minor, then there is a matrix in $\mathcal{S}(H)$ whose ordered multiplicity list can be obtained by preceding $\mathbf{m}$ by a number of $1$s and succeeding $\mathbf{m}$ by ones.*

# Consequences of the SSP and SMP

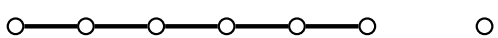
## Theorem (Ahamadi et al 2013, Barrett et al 2016)

*Let $G$ be a graph. Then there is a matrix $A \in \mathcal{S}(G)$ such that $A$ has an eigenvalue of multiplicity at least $3$ or $A$ has a pair of eigenvalues of multiplicity at least $2$ if and only if $G$ is not one of*

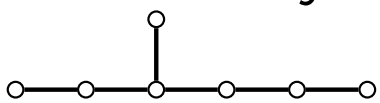
(a) *a path,*



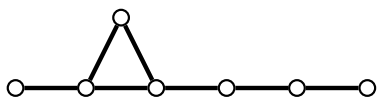
(b) *the disjoint union of a path and an isolated vertex,*



(c) *a path with an additional leaf attached to an interior vertex, item the disjoint union of a path and an isolated vertex,*



(d) *a path, along with an additional vertex joined to two consecutive internal vertices of the path*



# Sketch of proof that uses SSP and SMP

Can see that each graph in the families listed have $M(G) \leq 2$, and don't allow a pair of multiple eigenvalues.

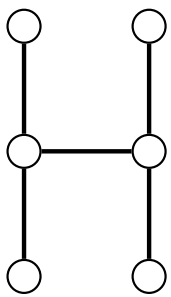
For the other direction, we look for minor minimal examples of graphs that allow a matrix with the SMP of nullilty 3 or a pair of multiple eigenvalues.

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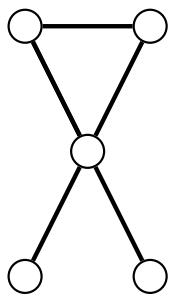
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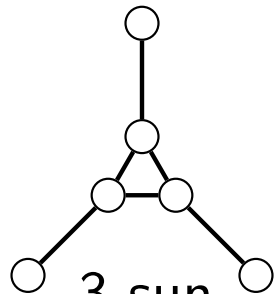
We find these along with the four cycle and $K_3 \sqcup K_3$.



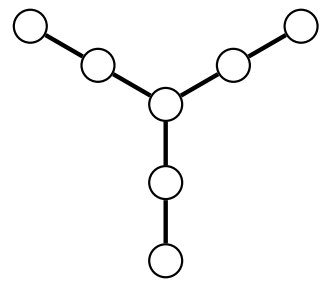
H-graph



Campstool



3-sun



Long $Y$

Now assume $G$ is a graph with $M(G) \leq 2$ that doesn't allow a pair of multiple eigenvalues. Then $G$ does not contain the $H$, Campstool, 3-sun, Long $Y$, $C_4$, $K_3 \sqcup K_3$ as a minor.

If $G$ is disconnected, then $G$ consists of two connected components, one is path and the other is an isolated vertex. This gives us a desired graph.

Now assume $G$ is connected.

Since $K_3 \sqcup K_3$ is not a minor, $G$ does not contain a pair of disjoint cycles.

If $G$ contains two cycles, then their union either has $C_4$, or the Campstool at a minor.

Thus $G$ contains at most one cycle, and that cycle is of length 3.

Suppose $G$ has cycle.

To avoid the 3-sun, one of the vertices of the 3-cycle has degree 2.

Next, to avoid the 3-sun, and the campstool, $G$ is a path along with a vertex joined to two consecutive vertices.

We are left with the case that $G$ is a tree.

Since $M(G) \leq 3$. The maximum degree is $3$.

If there is a vertex of maximum degree 3, is a generalized star. Avoiding the Long Y, implies $G$ is a path along with a leaf incident to an interior vertex.

Otherwise, $G$ is a (very acceptable) path. $\square$

# Small order graphs

## Minimal order 4 subgraphs for attainment of ordered multiplicity list with SSP

| | |
|---|---|
| (3,1), (1,3) | $K_4$ |
| (2,2) | $C_4$ |
| (1,2,1) | $C_4$, $K_3 \sqcup K_1$, $K_{1,3}$ |
| (2,1,1),(1,1,2) | $C_4 \sqcup K_1$, $K_3 \sqcup K_1$ |
| (1,1,1,1) | $4K_1$ |

Example of necessary work for $(1, 2, 1)$.

- Must find realizations with SSP, multiplicity list $(1, 2, 1)$ and graph $C_4$ ($K_3 \sqcup K_1$, and $K_{1,3}$).
- Identify the graphs that avoid $C_4$, $K_3 \sqcup K_1$, $K_{1,3}$ (these disjoint unions of paths)
- Can't get SSP and multiple eigenvalue with disjoint union of paths.

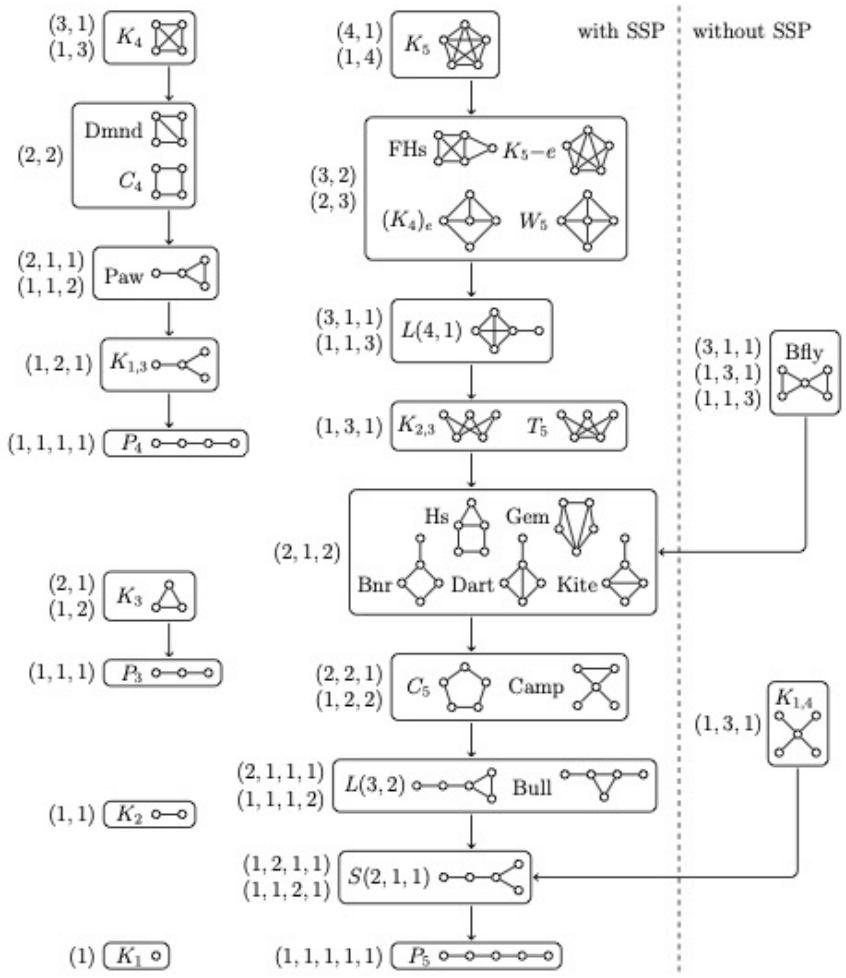
# Minimal order 5 subgraphs for attainment of ordered multiplicity list with SSP.

| | |
|---|---|
| (4,1), (1,4) | $K_5$ |
| (3,2),(2,3) | $(K_4)_e$ |
| (3,1,1),(1,1,3) | $K_4 \sqcup K_1$, $(K_4)_e$ |
| (1,3,1) | $K_4 \sqcup K_2$, $K_{2,3}$ |
| (2,1,2) | $C_4 \sqcup K_1$, Bowtie |
| (2,2,1) | $C_4 \sqcup K_1$, $C_5$, Campstool |
| (2,1,1,1),(1,1,1,2) | $C_4 \sqcup K_1, C_5, K_3 \sqcup 2K_1$ |
| (1,2,1,1), (1,1,2,1) | $C_4 \sqcup K_1, C_5, K_3 \sqcup 2K_1, K_{1,3} \sqcup K_1$ |
| (1,1,1,1,1) | $5K_1$ |

Minor monotonicity makes it reasonable to completely resolve the the multiplicity list problem for small order graphs (under the assumption of SMP/SSP) .

## Basic technique

1. Identify low lying fruit (extremal multiplicity list)
2. Use that the resolution for $G$ and $\mathbf{m}$ resolves for all supergraph of $G$ and refinements of $\mathbf{m}$, and all extended supergraphs of $G$ and augmentations.
3. Use Parter-Weiner for trees.
4. Use interlacing.
5. Compute when necessary.




The connected graphs of order at most 5 with their ordered multiplicity lists. If a box is joined to another box by a line then the graphs in the upper box can realize every ordered multiplicity list of the graphs in the lower box.

# Unpacking this chart

- One can understand disconnected graphs from information about each of their components; the chart only lists connected graphs.
- An arrow from $H$ to $G$ indicates that the multiplicity lists achievable for $H$ are contained in those for $G$. Often this comes from the Superpattern theorem, But not always; e.g., one can't use the Superpattern theorem on the Paw and (2,1,1)) to say something about $C_4$.
- All but four of the multiplicity list-graph pairs can be realized with the SSP.
- Bifurcation for multiplicities is very useful; e.g.; One example for $W_5$ and $(3, 2)$ handles $W_5$ and all refinements of $(3, 2)$.
- Minor monotonicity allows one to pass information for graphs on $n$ vertices to graphs on more vertices (e.g.; $K_{1,3}$ $(1, 2, 1)$, and $S(2, 1, 1)$ $(1, 2, 1, 1)$).

Fortunately, for $n \leq 5$, every ordered multiplicity list is spectrally arbitrary for the graphs that attain it.

That is, for every multiplicity list $\mathbf{b}$ if $G$ realizes the multiplicity list $\mathbf{m} = (m_1, \ldots, m_q)$, then for all $\lambda_1 < \ldots, \lambda_k$, $G$ realizes a matrix with spectrum $\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}$.

Graphs of order 6 are studied in Ahn et al (2021). Some are not spectrally arbitrary; some have spectrally arbitrary properties that are un-resolved.

Research catalyzed includes;

- Kenter and Lin (2022)
  Developed methods to bound sums of multiplicities, and to automate the process.
- Fallat, Lin, Vishwakarma (2022)
  Use order $n \leq 6$ classification to prove theorems about the existence of nowhere-zero eigenspaces.
  This is connected to Helena and her co-author's work. Stay tuned.
- Barrett, Butler, Lin (2023)
  Optimizes the use of the Matrix Liberation theorem. and uses order $\leq 6$ char'n to resolve many inverse multiplicity list problems.
- Allred, Curlee, Fallat, Olivas, Vishwakarma (2023). Extend the finding of the 6-vertex paper to larger composite graphs.

# Aside: example of not spectrally arbitrary list

Barioli and Fallat found an example of a graph $G$ and a multiplicity list $\mathbf{m} = (m_1, \ldots, m_q)$ for which there are algebraic constraints on the multisets of real numbers of these multiplicities that are the spectrum of some matrix $A \in \mathcal{S}(G)$.

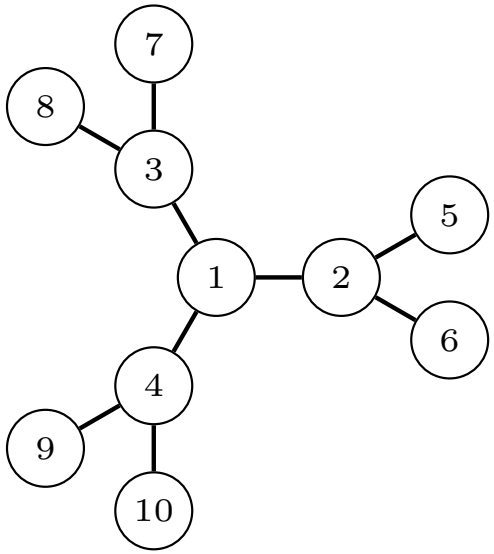
One way to verify this is via a nice result of Godsil.

## Theorem

*Let $A \in \mathcal{S}(T)$, $T$ a tree. Let $\gamma$ be the set of vertices of a path in $T$, and $\lambda$ be an eigenvalue of $A$ or multiplicity $m$. Then $\lambda$ is an eigenvalue of $A(\alpha)$ of multiplicity at least $m - 1$.*

## Corollary

*If $A \in \mathcal{S}(T)$, $T$ is a tree, and $\alpha$ is the vertex set of $t$ vertex disjoint paths of $T$, and $\lambda$ is an eigenvalue of $A$ of multiplicity $m$, then $\lambda$ is an eigenvalue of $A(\alpha)$ for of multiplicity at least $\max\{m - |t|, 0\}$.*




The Barioli-Fallat Tree

$\mathbf{m} = (1, 2, 4, 2, 1)$

and eigenvalues

$\lambda_1 < \cdots < \lambda_5$

Paths $9$–$8$; $8$–$8$ and $5$–$6$ imply that $A[\{10\}] = [\lambda_3]$.

Similarly, $A[\{i\}] = [\lambda_3]$ for $i = 5, \ldots, 9$.

Paths $5$–$6$; $7$–$8$ and $9$–$10$ imply $A[\{1\}] = [\lambda_3]$

Path $10$–$8$ implies $\lambda_2$ and $\lambda_4$ are eigenvalues of $A[\{2, 5, 6\}]$

Similarly, $\lambda_2$ and $\lambda_4$ are eigenvalues of $A[\{3, 7, 8\}]$ and of $A[\{4, 9, 10\}]$.

$\lambda_3$ also eigenvalues of these $3 \times 3$.

Now calculate trace in two ways to get $\lambda_1 + \lambda_5 = \lambda_2 + \lambda_4$.

Recent work "Spectrally arbitrariness for trees fails spectacularly" (Fallat et al 2023) extends this.
They give

- infinitely many families of graphs and multiplicity lists that are achievable only by a five variable parametrized set of real numbers are given.
- an example of an unordered unordered multiplicity list $m_1, \ldots, m_q$ can be realized by $G$, then one can replace any $m_j$ by $m_j - 1$ and $1$ and the list is still realizable.
- An unordered multiplicity list and a graph that can be achieved but only with a matrix whose off-diagonal, nonzero entries are all of the same absolute value.

# Characteriation of multiple multiples

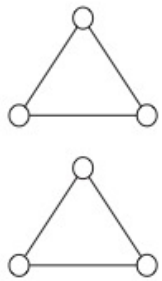
Question: which graphs $G$ allow a matrix $A$ with at least two multiple eigenvalues?

Note: We already solved the analogous problem for at least one multiple eigenvalue.

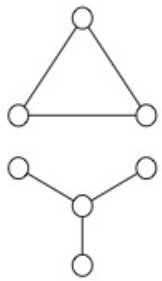
General approach:

1. Identify the family $\mathcal{F}$ of minor minimal graphs that allow multiple multiples with the SMP.
   Repeat until exhausted:
   (a) Identify a graph $G$ that allows multiple multiples and the SSP.
   (b) Find a minor minimal subgraph of $G$ with respect to these properties, and include in $\mathcal{F}$ if not already present.
   (c) Determine the structure of graphs $G$ that avoid those minors in $\mathcal{F}$.
   (d) Try to show that the structure prevents multiple multiples with the SSP; or if one encounters something that allows multiple multiples with the SSP and return to (a).
2. Solve the problem without the SMP assumption. (First, pray that this is not necessary!)

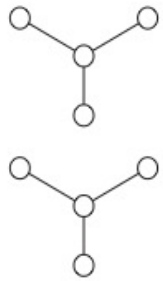
Looking at graphs of small order that allow the SMP and two multiple eigenvalues on arrives the following 8 graphs that are minor minimal with respect to this property.



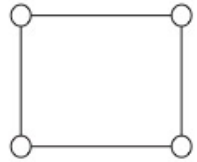
$K_3 \dot{\cup} K_3$



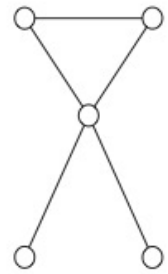
$K_3 \dot{\cup} K_{1,3}$



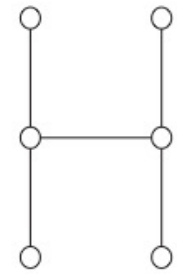
$K_{1,3} \dot{\cup} K_{1,3}$



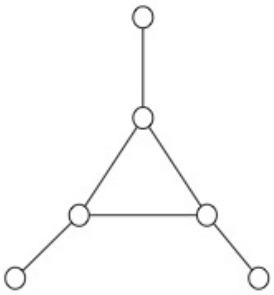
$C_4$



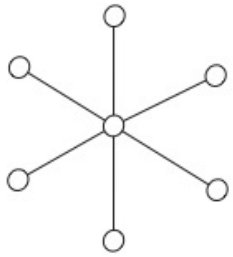
Campstool



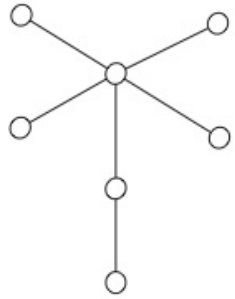
$H$-tree



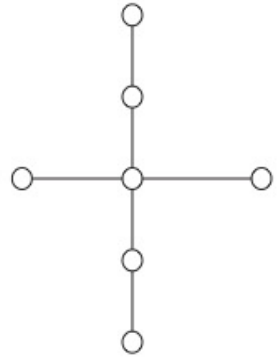
3-sun



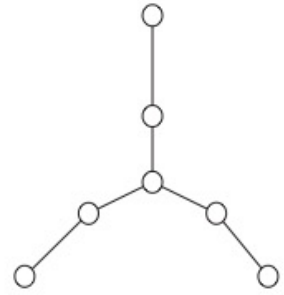
$K_{1,6}$



$S(2,1,1,1,1)$



$S(2,2,1,1)$



$S(2,2,2)$

**Theorem**

*Let $G$ be a graph.*

(1) *If $G$ is a connected graph and none of the eleven graphs shown in the figure is a minor of $G$, then any matrix $A \in \mathcal{S}(G)$ has at most one multiple eigenvalue.*

(2) *There exists a matrix $A \in \mathcal{S}(G)$ with SSP (or with the SMP) having more than one multiple eigenvalue if and only if one of the eleven graphs shown in the figure is a minor of $G$.*

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- J. Ahn et al., Ordered multiplicity inverse eigenvalue problem for graphs on six vertices, Electron. J. Linear Algebra **37** (2021), 316–358.
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- F. H. J. Kenter and J. C.-H. Lin, A zero forcing technique for bounding sums of eigenvalue multiplicities, Linear Algebra Appl. **629** (2021), 138–167