Liberation Sets

Goals

- Revisit SAP and SSP with respect to a graph,
- The SSP liberation set of a matrix,
- Some applications.

A spoiler is found. Now what?

Recall, the map $\mathfrak{f}$ from Lecture 3 that led to the Supergraph Theorem for Inertia:

$$\mathfrak{f} : \mathcal{B}(O, \epsilon) \times \langle \mathcal{S}(G) \rangle \to \mathrm{Sym}(n)$$
$$\mathfrak{f}(R, B) = (I + R)^\top A (I + R) + B.$$

For our theory to work we need $\mathfrak{f}$ to be locally surjective. If this condition fails, we may be able to obtain a locally surjective function by extending the domain of $\mathfrak{f}$.

For a supergraph H of G on the same vertex set, we define:

$$\mathfrak{f}_H : \mathcal{B}(O, \epsilon) \times \langle \mathcal{S}(H) \rangle \to \mathrm{Sym}(n)$$
$$\mathfrak{f}_H(R, B) = (I + R)^\top A (I + R) + B.$$

SAP with respect to a graph

Let $A \in \mathcal{S}(G)$, H a supergraph of G on the same vertex set with the adjacency matrix A_H, and

$$\mathfrak{f}_H : \mathcal{B}(O, \epsilon) \times \langle \mathcal{S}(H) \rangle \to \mathrm{Sym}(n)$$
$$\mathfrak{f}_H(R, B) = (I + R)^\top A(I + R) + B.$$

If $X = O$ is the only symmetric matrix such that

$$I \circ X = O,\ A_H \circ X = O,\ AX = O,$$

then the map $\mathfrak{f}_H$ is locally surjective.

This implication can be established using the same arguments as the ones we used for $\mathfrak{f}$. See Strong Arnol'd Property equivalents in Lecture 4.

Strong Properties with respect to a graph

A matrix $A \in \mathcal{S}(G)$ has *the SAP with respect to H* if $X = O$ is the only symmetric matrix that satisfies $X \in \langle \mathcal{S}(\overline{H}) \rangle$ and $AX = O$.

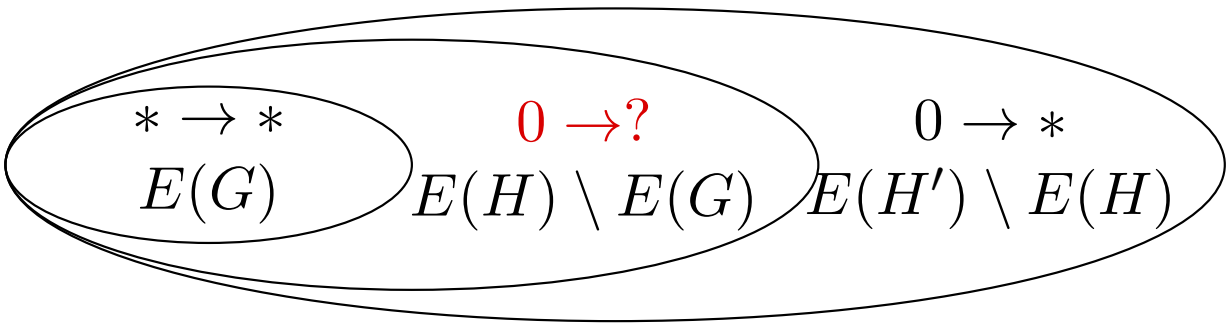
Let G be a spanning subgraph of H. A matrix $A \in \mathcal{S}(G)$ has *the SSP with respect to H* if $X = O$ is the only symmetric matrix that satisfies $X \in \langle \mathcal{S}(\overline{H}) \rangle$ and $[A, X] = O$.

Retracing steps to Supergraph Theorem

If $\mathfrak{f}_H$ is locally surjective, then $\exists$ neighborhood $V \subseteq \mathrm{Sym}(n)$ of A and a neighborhood of U of (O, O) in $\mathcal{B}(O, \epsilon) \times \langle \mathcal{S}(H) \rangle$ such that $\mathfrak{f}_H(U) = V$.

Let G, H, H' be graphs such that $V(G) = V(H) = V(H')$ and $E(G) \subseteq E(H) \subseteq E(H')$. For sufficiently small t:
$A + tA_{H'-H} \in V$. So there exists (R, B) in U with
$A + tA_{H'-H} = \mathfrak{f}_H(R, B) = (I + R)^\top A(I + R) + B$.
Note $A' = A + tA_{H'-H} - B$ is congruent to A, and

$$E(G) \cup E(H' - H) \subseteq E(G(A')) \subseteq E(H').$$



$*$: nonzero
0: zero
$?$: no information

Unsatisfactory Supergraph Theorem

(Lin, Oblak, Š, 2021)

Let G, H, H' be the graphs such that $V(G) = V(H) = V(H')$, $E(G) \subseteq E(H) \subseteq E(H')$, and let $A \in \mathcal{S}(G)$ be a matrix with the SAP (the SSP) with respect H.

Then for any $\epsilon > 0$, there is a matrix $A' \in \langle \mathcal{S}(H') \rangle$ such that

- inertia of A' = inertia of A ($\mathrm{spec}(A') = \mathrm{spec}(A)$),
- $\|A - A'\| < \epsilon$,
- A' has the SAP (the SSP) with respect to H',
- every entry of A' that corresponds to an edge in $E(H') \setminus E(H)$ is nonzero.

The zero-nonzero pattern of A' on $E(H) \setminus E(G)$ is not given by the theorem. Imposing additional conditions may allow us to determine the pattern.

Adding an eigenvalue of high multiplicity

..but at a cost

Theorem (Lin, Oblak, Š., 2021)

Let G be a graph, $A \in \mathcal{S}(G)$ with the SSP, $v \in V(G)$, and $\lambda \in \mathbb{R}$ with $\lambda \notin \operatorname{spec}(A) \cup \operatorname{spec}(A(v))$.
Then for any given positive integer s, there exists a matrix $A' \in \mathcal{S}(G \oplus_v K_{s+1})$ having the SSP such that

$$\operatorname{spec}(A') = \operatorname{spec}(A) \cup \{\lambda^{(s)}\}.$$

Notation: Let G and H be two graphs. If each of G and H has a vertex labeled as v, then the vertex-sum $G \oplus_v H$ of G and H at v is the graph obtained from the disjoint union of G and H by identifying the two vertices labeled by v.

If $\lambda \notin \operatorname{spec}(A)$, then $\hat{A} = A \oplus \lambda I_s \in \mathcal{S}(G \dot{\cup} sK_1)$ has the SSP with respect to $H = G \dot{\cup} K_s$. Additional assumptions: $\lambda \notin \operatorname{spec}(A(v))$, A' nonzero on $\{v, j\}$, $j \in V(K_s)$, allow us to fix the pattern of A'.

Proof

Unsatisfactory Supergraph Theorem gives us $A' \in \langle \mathcal{S}(G \oplus_v H) \rangle$ with:

- $\operatorname{spec}(A') = \operatorname{spec}(\hat{A}) = \operatorname{spec}(A) \dot{\cup} \{\lambda^{(s)}\}$,
- $\|\hat{A} - A'\| < \epsilon$,
- $a'_{i,j} \neq 0$ for all $\{i, j\} \in \alpha$.
- $A'[W] - \lambda I$ is nonsingular for $W = V(G) \setminus \{v\}$.

Left to prove: $a'_{i,j} \neq 0$ **for all** $\{i, j\} \in E(K_s)$**.**
Consider the $(|G| + 1) \times (|G| + 1)$ submatrix B of $A' - \lambda I$ induced on rows $V(G) \cup \{i\}$ and columns $V(G) \cup \{j\}$:

$$B = \begin{bmatrix} A'[W] - \lambda I & \mathbf{a} & \mathbf{0} \\ \mathbf{a}^\top & a'_{vv} - \lambda & b \\ \mathbf{0}^\top & c & a'_{ij} - \delta_{ij}\lambda \end{bmatrix}.$$

If $a'_{ij} - \delta_{ij}\lambda = 0$, then

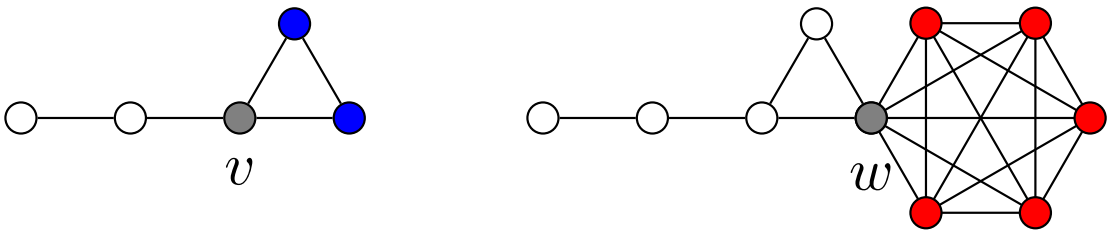
$$\operatorname{rank}(B) = \operatorname{rank}(A'[W] - \lambda I) + 2 = |G| - 1 + 2 = |G| + 1.$$

This contradicts $\operatorname{rank}(B) \leq \operatorname{rank}(A' - \lambda I) = |G|$.

Example

Let λ_i, $i = 1, \ldots, 5$, be distinct real numbers. Then:

- $\exists A_0 \in \mathcal{S}(P_3)$ with the SSP and $\mathrm{spec}(A_0) = \{\lambda_1, \lambda_2, \lambda_3\}$.
- $G := P_3 \oplus_v K_3$, $\exists A_1 \in \mathcal{S}(G)$ with the SSP and $\mathrm{spec}(A_1) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4^{(2)}\}$.
- $H := G \oplus_w K_6$, $\exists A_2 \in \mathcal{S}(H)$ with the SSP and $\mathrm{spec}(A_2) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4^{(2)}, \lambda_5^{(5)}\}$.



To apply our last theorem inductively we keep track of the eigenvalues of A, but not of the eigenvalues of a submatrix $A(v)$. However, if we assume that $\mu \notin \mathrm{spec}(A) \cup \mathrm{spec}(A(v))$ for any $v \in V(G)$ and $\mu \neq \lambda$, then we can guarantee that $\mu \notin \mathrm{spec}(A') \cup \mathrm{spec}(A'(u))$ for any $u \in V(G \oplus_v K_{s+1})$, since A' can be chosen to be arbitrarily close to $A \oplus \lambda I_s$.

Verification Matrix

Let G be a graph on n vertices, $\overline{m} = |E(\overline{G})|$, and $A \in \mathcal{S}(G)$. The *SSP verification matrix* $\Psi_{\mathrm{SSP}}(A)$ is the $\overline{m} \times \binom{n}{2}$ matrix whose rows are $\mathrm{vec}_\wedge([A, E_{i,j} + E_{j,i}])$ for all pairs (i, j) with $1 \leq i < j \leq n$ and $\{i, j\} \in E(\overline{G})$ under the lexicographical order.

The *SAP verification matrix* $\Psi_{\mathrm{SAP}}(A)$ is the $\overline{m} \times n^2$ matrix whose rows are $\mathrm{vec}(A(E_{i,j} + E_{j,i}))$ for all pairs (i, j) with $1 \leq i < j \leq n$ and $\{i, j\} \in E(\overline{G})$ under the lexicographical order.

Notation: For $n \times n$ matrix M let $\mathrm{vec}_\wedge(M)$ be the vector in $\mathbb{R}^{\binom{n}{2}}$ that records the strictly upper triangular part of M, excluding the diagonal entries. (Note: $[A, E_{i,j} + E_{j,i}]$ is a skew-symmetric matrix.)

Matrix Liberation Lemma — vector version

(Barrett et al. 2017)

Let G be a graph, $A \in \mathcal{S}(G)$, and $\Psi = \Psi_{\mathrm{SSP}}(A)$ ($\Psi = \Psi_{\mathrm{SAP}}(A)$). Let $\mathbf{x} \in \mathrm{Col}(\Psi)$ such that A has the SSP (the SAP) with respect to $G + \mathrm{supp}(\mathbf{x})$.

Then there exists a matrix $A' \in \mathcal{S}(G + \mathrm{supp}(\mathbf{x}))$ with the SSP (the SAP) and $\mathrm{spec}(A') = \mathrm{spec}(A)$ ($\mathrm{inertia}(A') = \mathrm{inertia}(A)$).

This is a restatement of Matrix Liberation Lemma for Inertia from Lecture 4. The lemma relies on a vector $\mathbf{x}$ in its assumption while the conclusion only uses the support of $\mathbf{x}$. Our aim is to use linear algebra to us better understand all possible supports of vectors $\mathbf{x}$ that meet the requirements in this lemma.

Example

Let $G = K_4 \dot{\cup} K_1$ and $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \in \mathcal{S}(G)$.

The SSP verification of A is equal to

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & -3 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -3 \end{pmatrix},$$

the rows of Ψ are indexed by the nonedges $\{1,5\}$, $\{2,5\}$, $\{3,5\}$, and $\{4,5\}$. Let $\alpha = \{\{1,5\},\{2,5\}\}$ and $\beta = \{\{3,5\},\{4,5\}\}$.

$\mathbf{x} = (0,0,1,-1)^\top \in \mathrm{Col}(\Psi)$, $\mathrm{supp}(\mathbf{x}) = \beta$, and the rows indexed by α form a linearly independent set. By Matrix Liberation Lemma, there is a matrix $A' \in \mathcal{S}(G + \beta)$ with the SSP and $\mathrm{spec}(A') = \mathrm{spec}(A) = \{0^{(3)}, 4^{(2)}\}$.

Liberation Set

Let G be a graph and $A \in \mathcal{S}(G)$. A nonempty set of edges $\beta \subseteq E(\overline{G})$ is called an *SSP liberation set of A* (an *SAP liberation set of A*) if and only if A has the SSP (the SAP) with respect to $G + \beta'$ for all $\beta' \subset \beta$ with $|\beta'| = |\beta| - 1$.

Matrix Liberation Lemma — set version

(Lin, Oblak, Š., 2023)

Let A be a matrix in $\mathcal{S}(G)$ and β an SSP (an SAP) liberation set of A.

Then there is a matrix $A' \in \mathcal{S}(G + \beta)$ with the SSP (the SAP) such that $\mathrm{spec}(A) = \mathrm{spec}(A')$ $(\mathrm{rank}(A) = \mathrm{rank}(A'))$.

Step-by-step to Liberation

Suppose $A \in \mathcal{S}(G)$ has a liberation set $\beta = \{e_1, \ldots, e_k\}$ and $\beta_i := \beta \setminus \{e_i\}$, $i = 1, \ldots, k$.
Since A has the SSP with respect to $G + \beta_i$ for all $i \in [k]$ one may perturb A to obtain A_1 that satisfies:

- the entries in A_1 corresponding to e_1 become nonzero,
- every nonzero entry in A stays nonzero also in A_1,
- off-diagonal entries of A_1 outside $E(G) \cup \beta$ remain zero,
- A_1 has the same spectrum as A,
- the new matrix still has the SSP with respect to $G + \beta_i$ for each i where A_1 is zero on the entry of e_i.

If A_1 is nonzero on the entry of e_2, then set $A_2 = A_1$. Otherwise, since A_1 has the SSP with respect to $G + \beta_2$, we may perturb A_1 into A_2 in the same way; in particular, the entry of e_2 becomes nonzero. We may continue this process inductively to obtain $A_3, \ldots, A_k$. Note that once an entry turns nonzero, it stays nonzero in this process.

Equivalent assumptions in Matrix Liberation Lemma

Let $A \in \mathcal{S}(G)$, and Ψ the SSP (the SAP) verification matrix for A. For $\beta \subseteq E(\overline{G})$, $\beta \neq \emptyset$, the following are equivalent:

1. β is an SSP (an SAP, respectively) liberation set of A.
2. $\Psi[\alpha \cup \{e\}, :]$ has full row-rank for all $e \in \beta$, where $\alpha = E(\overline{G}) \setminus \beta$.
3. There exists a nonzero vector $\mathbf{x} \in \mathrm{Col}(\Psi)$ such that $\mathrm{supp}(\mathbf{x}) = \beta$ and A has the SSP (the SAP, respectively) with respect to $G + \beta$.

For $M \in M_{m,n}$ and $\alpha \subsetneq [m]$, the following conditions are equivalent.

1. $M[\alpha \cup \{k\}, :]$ has full row-rank for all $k \in [m] \setminus \alpha$.
2. There exists a nonzero vector $\mathbf{v} \in \mathrm{Col}(M)$ such that $\mathrm{supp}(\mathbf{v}) = [m] \setminus \alpha$ and $M[\alpha, :]$ has full row-rank.

Example

Let us look at all the equivalent conditions applied to Ψ from the previous example. Recall, the first two rows are labeled by the elements of α.
The column reduced echelon form of Ψ is

$$\Psi' = \left(\begin{array}{cc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} I_2 & O \\ ? & B \end{pmatrix}.$$

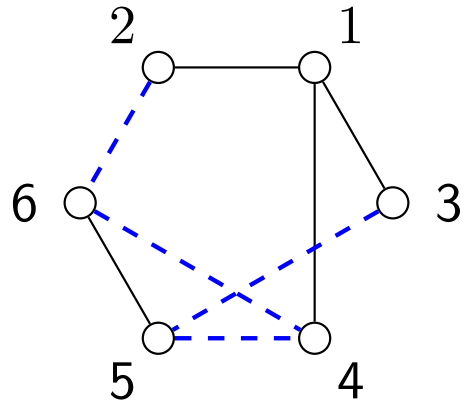
- (2) holds for $\mathbf{x} = (0, 0, 1, -1)^\top \in \mathrm{Col}(\Psi) = \mathrm{Col}(\Psi')$, by noting that $\mathrm{supp}(\mathbf{x}) = \beta$ and A has the SSP with respect to $G + \beta$.
- Since column operations do not change the dependency of rows, we can easily deduce from Ψ' that both $\Psi[\alpha \cup \{\{3, 5\}\}, :]$ and $\Psi[\alpha \cup \{\{4, 5\}\}, :]$ have full row-rank, hence conditions (1) holds.

Example

$$A = \begin{pmatrix} b & t & t & t & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a & a \end{pmatrix} + c\, I_6 \in \mathcal{S}(K_{1,3} \dot{\cup} K_2),\ a, b, c, t \in \mathbb{R} \setminus \{0\}.$$

- This family of matrices realizes the ordered multiplicity lists $(1, 3, 2)$, $(2, 3, 1)$, $(1, 1, 3, 1)$ and $(1, 3, 1, 1)$ spectrally arbitrary.
- $\beta = \{\{3, 5\}, \{4, 5\}, \{2, 6\}, \{4, 6\}\}$ is a liberation set of all matrices in this family.

The multiplicity lists listed above can be realized spectrally arbitrarily for $\mathsf{G}_{151} = (K_{1,3} \dot{\cup} K_2) + \beta$:



Direct sum of matrices with the SSP

Let G and H be graphs on m and n vertices, respectively, β a set of edges between G and H, and Y an $m \times n$ matrix.

We say that Y *vanishes on* β, if under the labeling of the rows of Y by $V(G)$ and the columns of Y by $V(H)$ we have $Y_{u,v} = 0$ for all $\{u, v\} \in \beta$. We denote this by $Y|_\beta = 0$.

Assume that A and B have the SSP, and that $Y = O$ is the only matrix in $\mathbb{R}^{m \times n}$ with $Y|_\beta = 0$ that satisfies $AY - YB = O$. Then $A \oplus B \in \mathcal{S}(G \dot{\cup} H)$ has the SSP with respect to $(G \dot{\cup} H) + \beta$.

Homogeneous Sylvester equation

Let $A \in \mathcal{S}_m$, $B \in \mathcal{S}_n$, with k distinct common eigenvalues $\lambda_1, \ldots, \lambda_k$.

$$\mathcal{R}(A, B) = \{Y \in M_{m,n}(\mathbb{R}); AY - YB = O\}$$
$$= \operatorname{span}\{\mathbf{u}\mathbf{v}^\top : A\mathbf{u} = \lambda_i \mathbf{u},\ B\mathbf{v} = \lambda_i \mathbf{v} \text{ for some } i \in [k]\}.$$

We are looking for the sets β so that $Y \in \mathcal{R}(A, B)$ and $Y|_\beta = 0$ implies $Y = 0$. Eigenvectors of A and B will play a role.

Generic Subspace

Let $\mathcal{W}$ be a d-dimensional subspace of $\mathbb{R}^n$ and W be an $n \times d$ matrix whose columns form a basis of $\mathcal{W}$. Then $\mathcal{W}$ is said to be *generic* if and only if every $d \times d$ submatrix of W is invertible.

- The definition above does not depend on the choice of W.
- A one-dimensional subspace $\mathcal{W}$ is generic if and only if it is spanned by a nowhere zero vector.

Liberation sets for $A \oplus B$ when A and B have one eigenvalue in common

(Lin, Oblak, Š, 2023)

Let $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ have the SSP,

- $\operatorname{spec}(A) \cap \operatorname{spec}(B) = \{\lambda\}$ with $\operatorname{mult}_A(\lambda) = k$ and $\operatorname{mult}_B(\lambda) = \ell$,
- $\ker(A - \lambda I)$ and $\ker(B - \lambda I)$ generic.

Then for any $V_G \subseteq V(G)$ and $V_H \subseteq V(H)$ with

- $|V_G| = k$ and $|V_H| = \ell + 1$ or
- $|V_G| = k + 1$ and $|V_H| = \ell$,

the set

$$\beta = \{\{u, v\} : u \in V_G \text{ and } v \in V_H\}$$

is an SSP liberation set for $A \oplus B$. In particular, there exists a matrix $C \in S(G \dot\cup H + \beta)$ with the SSP such that $\operatorname{spec}(C) = \operatorname{spec}(A) \cup \operatorname{spec}(B)$.

Proof

Let U_A and U_B be $|V(G)| \times k$ and $|V(H)| \times \ell$ matrices whose columns form bases of $\ker(A - \lambda I)$ and $\ker(B - \lambda I)$, respectively.

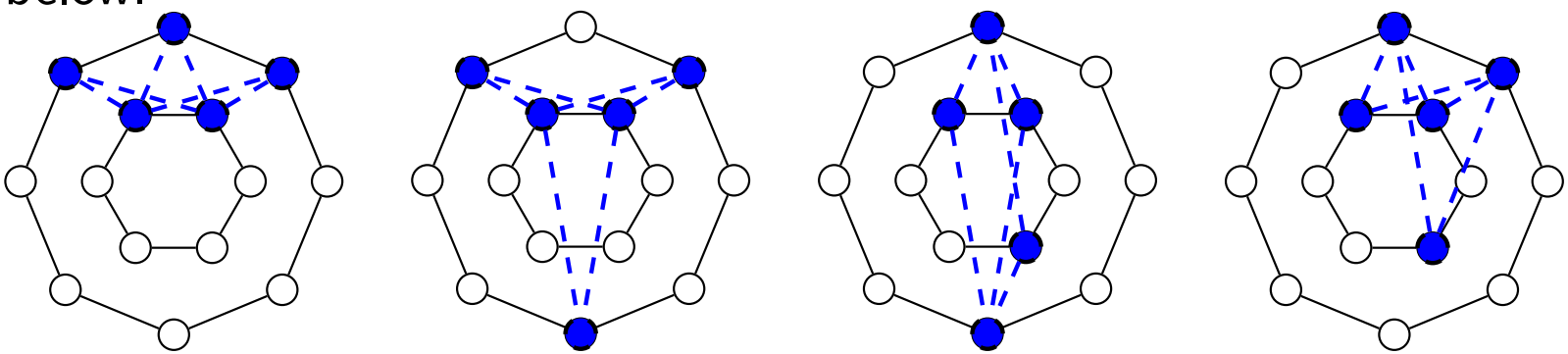
$$\mathcal{R}(A, B) = \{U_A S U_B^\top : S \in M_{k,\ell}(\mathbb{R})\}$$

- Generic eigenspaces: If $Y \in \mathcal{R}(A, B)$ and $Y|_{\mathcal{I} \times \mathcal{J}} = 0$ for some $k \times \ell$ grid $\mathcal{I} \times \mathcal{J}$, then $Y = 0$,
- any $\beta' \subset \beta$ with $|\beta'| = |\beta| - 1$ contains a $k \times \ell$ grid.

Example

With a little effort one can find $A \in \mathcal{S}(C_6)$ and $B \in \mathcal{S}(C_8)$, both having all eigenvalues with multiplicity 2, that satisfy the assumption of the previous result.

The Liberation Lemma gives us the existence of $M \in \mathcal{S}(H)$ with the SSP and the multiplicity list $\{4, 2, 2, 2, 2, 2\}$ for any of the graphs H listed below.



The assumption on genericity of eigenspaces can be hard to prove, but is straightforward to check for specific matrices of small dimension.

$\mathrm{rank}(Y) = 1$

Assume $\ell = 1$, $Y \in \mathcal{R}(A, B)$, and $\beta' \subset V(G) \times V(H)$ so that $|\beta' \cap (\{u\} \times V(H))| = 1$ for k distinct $u \in V(G)$. Then $Y|_{\beta'} = 0$ implies $Y = 0$.

This observation can be used to construct a liberation set β for $A \oplus B$ when $\ell = 1$.

Genericity of Eigenspaces

- Let T be a tree and $A \in \mathcal{S}(G)$. Then the smallest and the largest eigenvalues of A must be simple by the Perron–Frobenius Theorem, and their eigenspaces are generic.
- Let G be a connected graph on n vertices and $\lambda_1 < \cdots < \lambda_n$. Then there is a matrix $A \in \mathcal{S}(G)$ with $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$ such that the eigenspace of each eigenvalue is generic. This will be discussed in Lecture 16.
- Let $A \in \mathcal{S}(K_n)$. By choosing appropriate orthogonal matrices Q, we may replace A by $Q^\top AQ$ and assume the eigenspace of each eigenvalue is generic.

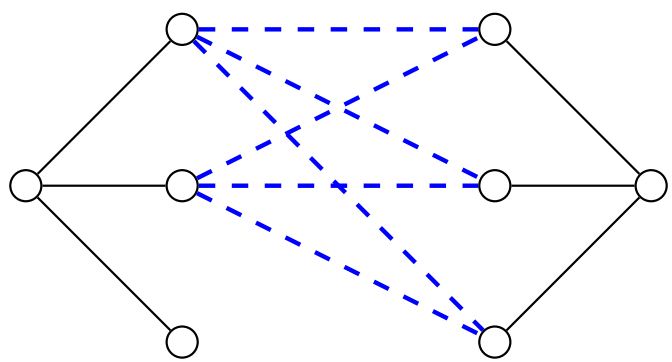
The cases listed above demonstrate situations where the genericity of eigenspaces can be proven. However, the existence of a liberation set does not need this property. In fact, such genericity is impossible for multiple eigenvalues of matrices whose pattern is a tree.

Example

$$A = \begin{pmatrix} a & \mathbf{a}^\top \\ \mathbf{a} & \theta I_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} b & \mathbf{b}^\top \\ \mathbf{b} & \theta I_3 \end{pmatrix} \text{ in } \mathcal{S}(K_{1,3}).$$

- A, B have an eigenvalue θ of multiplicity 2.
- the eigenspaces of θ are not generic (eigenvectors share a 0 entry).
- the eigenspaces are "locally" generic on $\{2, 3, 4\}$.

The set $\beta = V_G \times V_H$ is a liberation set of $A \oplus B$ for any $V_G \subseteq \{2, 3, 4\}$ with $|V_G| = 2$ and $V_H \subseteq \{6, 7, 8\}$ with $|V_H| = 3$.
Therefore, there is a matrix in $\mathcal{S}(K_{1,3} \dot{\cup} K_{1,3} + \beta)$ with the SSP and the ordered multiplicity list $(1, 1, 4, 1, 1)$



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