Strong Matrix Properties and the Inverse Eigenvalue Problem Outline of Lectures

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1. **Overview of inverse eigenvalue problems.**
   This lecture discusses questions like the following: What's an inverse problem? How do inverse problems give rise to matrix theoretic problems? How are the analytic, geometric and algebraic properties of a matrix impacted by the combinatorial arrangement of its zero and nonzero entries? What does the topic this conference have to do with skyscrapers, archery, power lines, and F1-racecars?

2. **Foundations of the Inverse Eigenvalue Problem of a Graph.**
   This lecture will describe the early work that led to the Inverse Eigenvalue Problem for a Graph, the historical roadblocks encountered that frustrated many a mathematician, and the breakthrough idea of using the Inverse Function Theorem to implicitly solve many inverse eigenvalue problems.

3 & 4. **Theoretical Underpinnings and Consequences of Strong Properties**
   Historically, the study of inverse eigenvalue problems focused on explicit methods. A major turning point was the introduction of implicit methods for inferring the solvability of families of inverse problems from an "especially nice" solution to a specific inverse problem. Here, the meaning of "especially nice" for the solution is generic in a sense that can expressed as a certain linear algebraic condition. This genericity condition leads to the notion of strong properties.

   In this lecture we motivate and establish the theoretical underpinnings of strong properties. These underpinnings and their consequences will be a common thread in the remaining lectures.

   The key theorems are:

   - The Supergraph Theorem:
     If $A \in \mathcal{S}(G)$ has the SAP and H is a supergraph of G, then there is matrix in $\mathcal{S}(G)$ with the same nullity as A.
   - The Proximity Theorem: if A has the SAP and graph G, and B is sufficiently close to A, then there is a matrix with graph G and the same nullity as B.
   - Minor Monotonicity Theorem: If G is a graph minor of H, and $\mathcal{S}(G)$ contains a matrix with the SAP and nullity k, then so does $\mathcal{S}(H)$.

5. **SAP, ξ and graph minors.**
   The maximum nullity of a graph G, $M(G)$ is an ill-behaved graph parameter. A nicer parameter, $\xi(G)$ is the largest nullity of a matrix in $\mathcal{S}(G)$ that has the SAP. We show that $\xi(G)$ is minor monotone; that is, if G is a graph minor of H, then $\xi(G) \leq \xi(H)$. Then we survey results about graphs with extreme values of $\xi(G)$.

6 & 8. **The Strong Spectral Property and Applications**
   The Strong Arnol'd Property allows one to study the maximum multiplicity of one eigenvalue at a time. Yet, inverse eigenvalue problems often concern the entire spectrum of a matrix. In Lecture 6, we introduce and develop a property, known as the Strong Spectral Property (SSP), that provides a tool for implicitly inferring the existence to a family of inverse eigenvalue problems from an "especially nice" solution to one inverse problem. Applications to previously discussed problems are presented in Lecture 8.

7. **Jacobian Methods**

   We explore a number of different Jacobian methods for providing information about eigenvalues allowed by a matrix sign pattern or zero-nonzero pattern. The methods are applied to a number of inverse eigenvalue problems, such as determining if a sign pattern or zero-nonzero matrix pattern is inertially or spectrally arbitrary. In the context of a nonderogatory matrix, one of the Jacobian methods is shown to be equivalent to a strong property.

9. **Liberation Sets**

   We revisit the SAP and the SSP with respect to a graph, and recall how the verification matrix is used to determine when a given matrix has the SAP or the SSP. The main question considered is what can still be said when we find a matrix does not have a given property. Liberation sets encompass the idea that a desired property may be attainable for matrices for some, but not all, supergraphs of the graph of the original matrix. These concepts are demonstrated through various examples.

10. **Inverse problems for not necessarily symmetric matrices.**

    We describe a strong spectral property and a strong multiplicity property for not necessarily symmetric matrices. Considering matrix sign patterns or zero-nonzero patterns, we describe some classes of patterns that allow or require these strong properties. We also explore how the methods provide information about the number of distinct eigenvalues allowed by a pattern.

11. **The ν and μ parameters related to the Strong Arnol'd Property**
    $\xi(G)$ is related to <u>all</u> symmetric matrices graph G.

    For many applications, the set matrices of interest is more constrained. $\nu(G)$ is the maximum nullity among the positive semidefinite matrices with graph G. $\mu(G)$ is the maximum nullity among the so-called Colin de Verdiere (CDV)-matrices; that is, the matrices have the SAP, graph G, all off-diagonal entries non-positive, and exactly one negative eigenvalue.

    We survey results about $\mu(G)$ and $\nu(G)$; many of these are related to planarity or embedability properties of G.

12. **SAP Connections.**
    This talk will discuss various combinatorial and geometric concepts and invariants related to maximum

nullity, including orthogonal representations of graphs, and zero forcing (which can be used to provide an upper bound on the maximum multiplicity of a graph). Different (but equivalent) formulations of the SAP are considered and their motivations from problems other than maximum nullity are presented. This includes SAP for subspaces of a vector space and positive semidefinite connections.

13. **Strong Nullity Interlacing Property.**

    Given an $n \times n$ symmetric matrix A and an index r, the *nullity pair* of A with respect to r is

    $$(\mathrm{nul}(A), \mathrm{nul}(A(r))).$$

    In this lecture we introduce and discuss a strong property for nullity pairs, and then use it to give characterizations of which graphs allow certain nullity pairs.

14. **Minimum number of distinct eigenvalues of a graph.**

    We list various parameters associated with the multiplicity lists of a given graph. To contrast behavior with cases where a multiplicity list can be achieved by a matrix with the SMP or the SSP, we show that multiplicities and associated eigenvalues, can be extremely rigid. In particular, we given an example when five high multiplicities determine all other multiplicities, and force a specific construction of the realizing matrix. Focusing on the minimum number of distinct eigenvalues of a graph G, denoted by $q(G)$, we showcase some constructive techniques that produce matrices with small number of distict eigenvalues. We determine $q(G)$ for some families of graphs G, and investigate graphs G with $q(G) = 2$.

15. **On the number of distinct eigenvalues: nonsymmetric patterns.**

    We explore implications of strong properties, and Jacobian methods, for the minimum number of distinct eigenvalues among matrices with a given sign or zero-nonzero pattern. We describe bounds on $q(\mathcal{A})$, the minimum number of eigenvalues allowed by a pattern, in terms of digraph structure. The allow sequence for the number of distinct eigenvalues of a pattern is described, with results about eigenvalue multiplicities. Some general results about allow sequences are provided and characterizations with respect to certain digraph structures are given, along with some open problems for consideration.

16. **Eigenvectors and Strong Properties.**

    In the study of inverse eigenvalue problems, constructions of matrices with prescribed spectra often depend on methods that build larger matrices from smaller components. These constructions frequently depend on the specific eigenvectors of the matrices that are used as building blocks. While eigenvectors are even harder to study than eigenvalues, we show how the SSP allows us to approach them, at least in the case of distinct eigenvalues. As an application of these results, we demonstrate a method for constructing symmetric orthogonal matrices with prescribed patterns.

17 & 19. **Inverse Singular Value Problems.**

    In Lecture 17 we discuss the inverse singular value problem for patterns and the Strong Singular Value Property: Given an $m \times n$ pattern matrix P with $m \leq n$, which multi-sets of m nonnegative real numbers are the multiset of singular values of some matrix with pattern P? In Lecture 19, we ask which patterns P have the property that no matrix in $\mathcal{Z}(P)$ has a multiple singular value?

18. **Nonnegative Inverse Eigenvalue Problem.**

The Nonnegative Inverse Eigenvalue Problem (NIEP), seeks to characterize all the multi-sets of eigenvalues that can be realized as the spectrum of an entry-wise nonnegative matrix. After reviewing selected highlights from the current state-of-the-art, we recast the problem to characterize the characteristic polynomials of nonnegative matrices. We retrace the steps taken in the previous lectures, and investigate how strong properties can help understand characteristic polynomials of nonnegative Hessenberg matrices.

20. **Inverse Symplectic Eigenvalue Problems.**

The appropriate setting for Newtonian physics is Euclidean space (that is, $\mathbb{R}^n$ endowed with the standard inner product). Mathematically, this gives rise to symmetric matrices, their eigenvalues, and orthogonal similarities that preserve the above. Hamiltonian mechanices is set in symplectic space (that is, $\mathbb{R}^{2p}$ endowed with a nondegenerate alternating bilinear form). Mathematically, this gives rise to positive definite matrices, their so-called symplectic eigenvalues, and symplectic similarity which preserves the above. In this lecture, we introduce the needed background, and then study the inverse symplectic eigenvalue problem.