CBMS Lectures

Calderón's Inverse Problem

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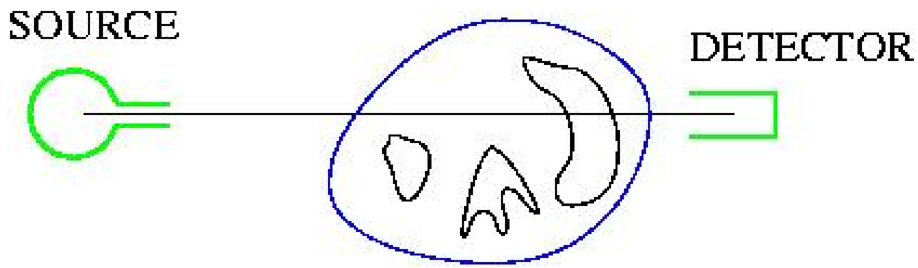
Clemson, June 2024

Inverse Boundary Problems

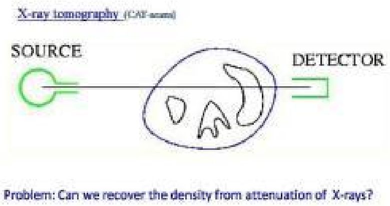
Can one determine the internal properties of a medium by making measurements outside the medium (non-invasive)?

X-ray tomography (CT-scans)



Problem: Can we recover the density from attenuation of X-rays?

X-ray Tomography



Radon solved the problem in 1917

Determine the integral of density over lines

Johann Radon

Nobel Prize in Medicine (1979)

A. Cormack

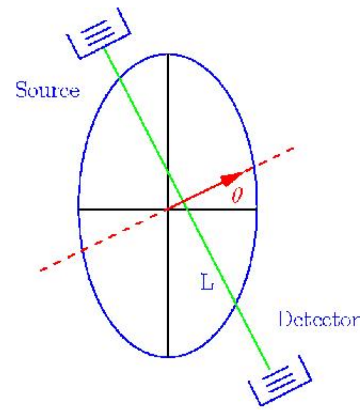
G. Hounfield

CT SCAN

A. Cormack and G. Hounsfield (1979): Nobel prize in medicine for development of CT

Radon Transform

Radon (1917) $n = 2$



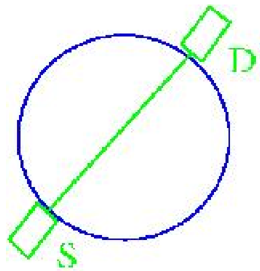
$f(x) =$ Unknown function

$$I_{\text{detector}} = e^{-\int_L f} I_{\text{source}}$$

$$Rf(s,\theta) = g(s,\theta) = \int_{\langle x,\theta\rangle = s} f(x)\,dH = \int_L f$$

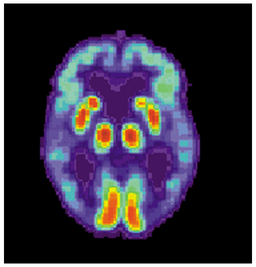
$$f(x) = \frac{1}{4\pi^2}\; p.v. \int_{S^1} d\theta \int \frac{\frac{d}{ds} g(s,\theta)\,ds}{\langle x,\theta\rangle - s}$$

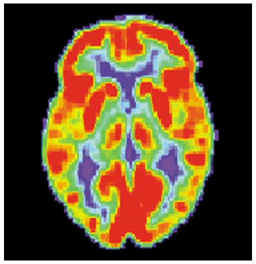
LINEAR (No Scattering)



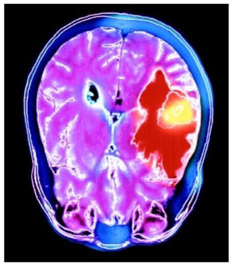
X-ray tomography (CT)

PET

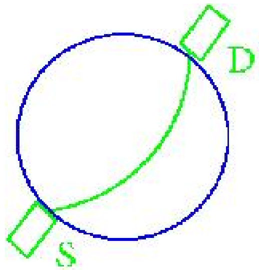




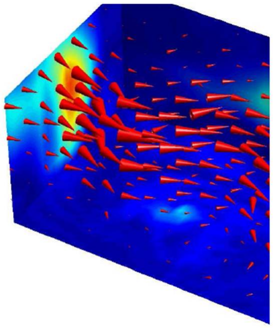
MRI



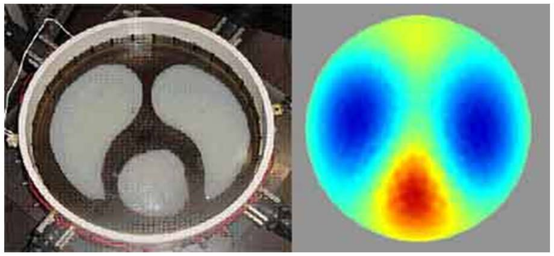
NONLINEAR (Scattering)



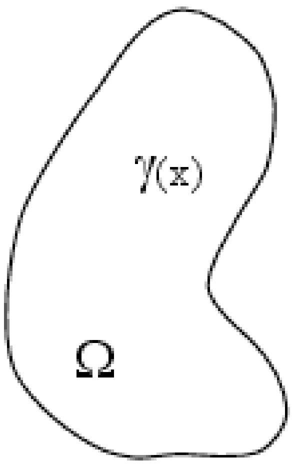
Ultrasound



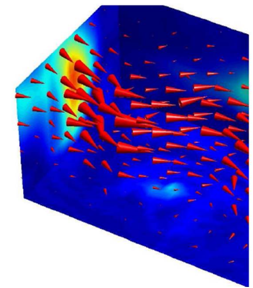
Electrical
Impedance
Tomography
(EIT)

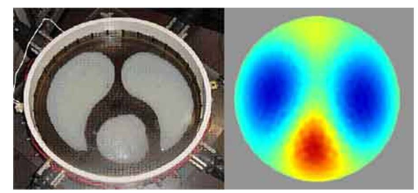


CALDERÓN'S PROBLEM and EIT



$\Omega \subset \mathbb{R}^n$
$(n = 2, 3)$





Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho

Alberto P. Calderón (1920-1997)

REMINISCENCIA DE MI VIDA MATEMATICA

Speech at Universidad Autónoma de Madrid accepting the ‘Doctor Honoris Causa’:

> *My work at “Yacimientos Petroliferos Fiscales” (YPF) was very interesting, but I was not well treated, otherwise I would have stayed there.*

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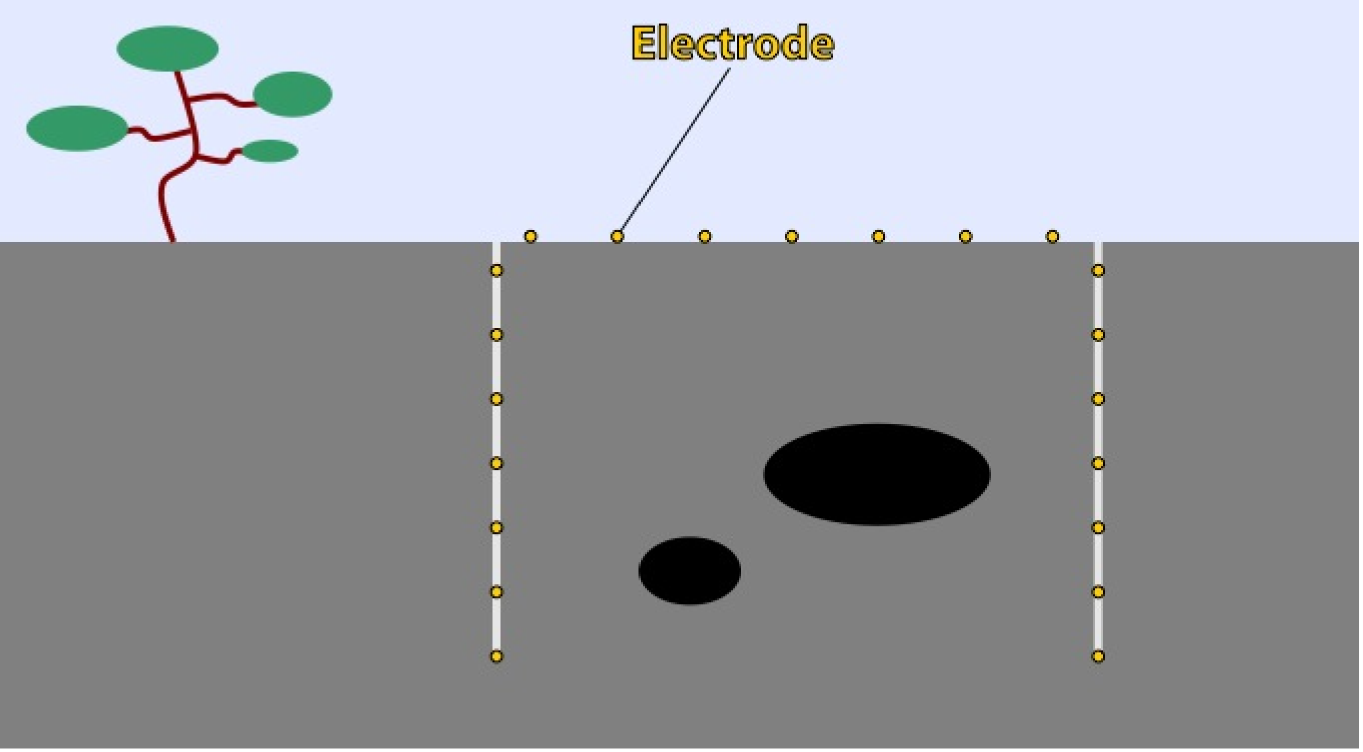
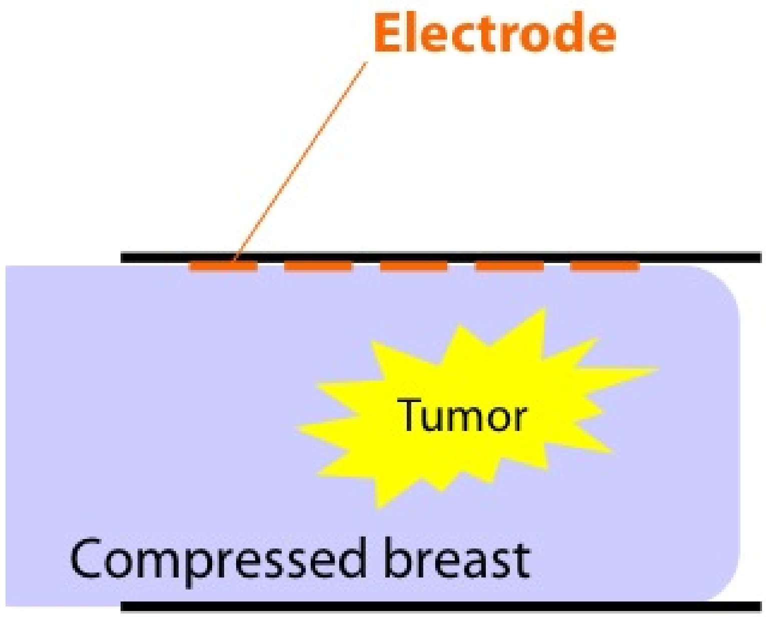
Geological underground probing is the application of EIT considered by Calderón



Early detection of breast cancer is effective using combined X-ray mammography and EIT



Cancerous tissue is up to four times more conductive than healthy tissue. [**Jossinet** -98]

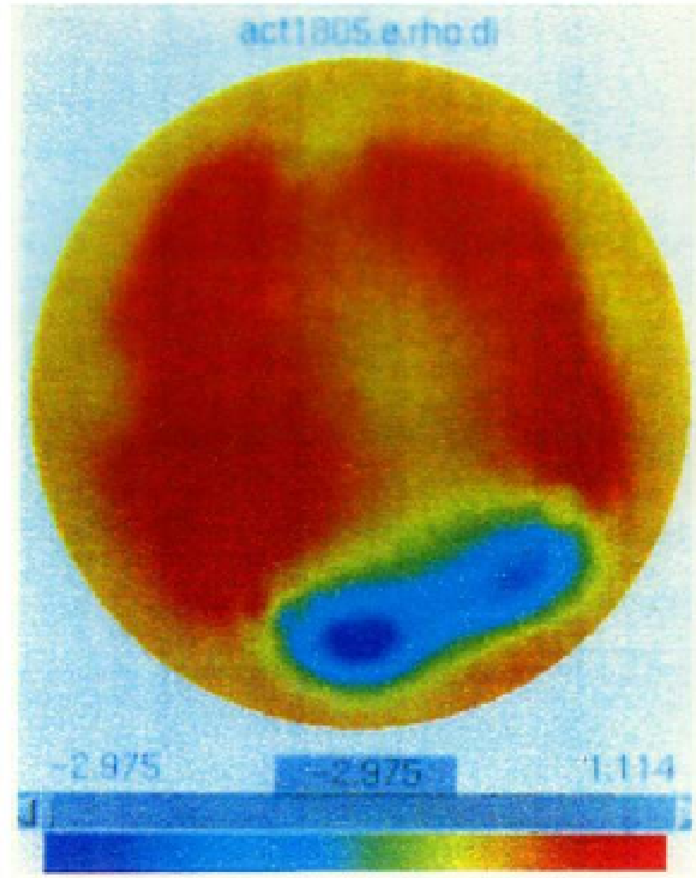
X-ray attenuation is almost the same in cancerous and healthy tissue.

David Isaacson and his team have achieved good results in early detection of breast cancer using EIT.

Other Applications

- Non-destructive testing (corrosion, cracks)
- Seepage of groundwater pollutants
- Medical Imaging (EIT)

Tissue	Conductivity (mho)
Blood	6.7
Liver	2.8
Cardiac muscle	6.3 (longitudinal)
	2.3 (transversal)
Grey matter	3.5
White matter	1.5
Lung	1.0 (expiration)
	0.4 (inspiration)



ACT3 imaging blood as it leaves the heart (blue) and fills the lungs (red) during systole.

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Thanks to D. Issacson

Electrical Impedance Tomography

Advantages

- non-invasive (small chance of infections)

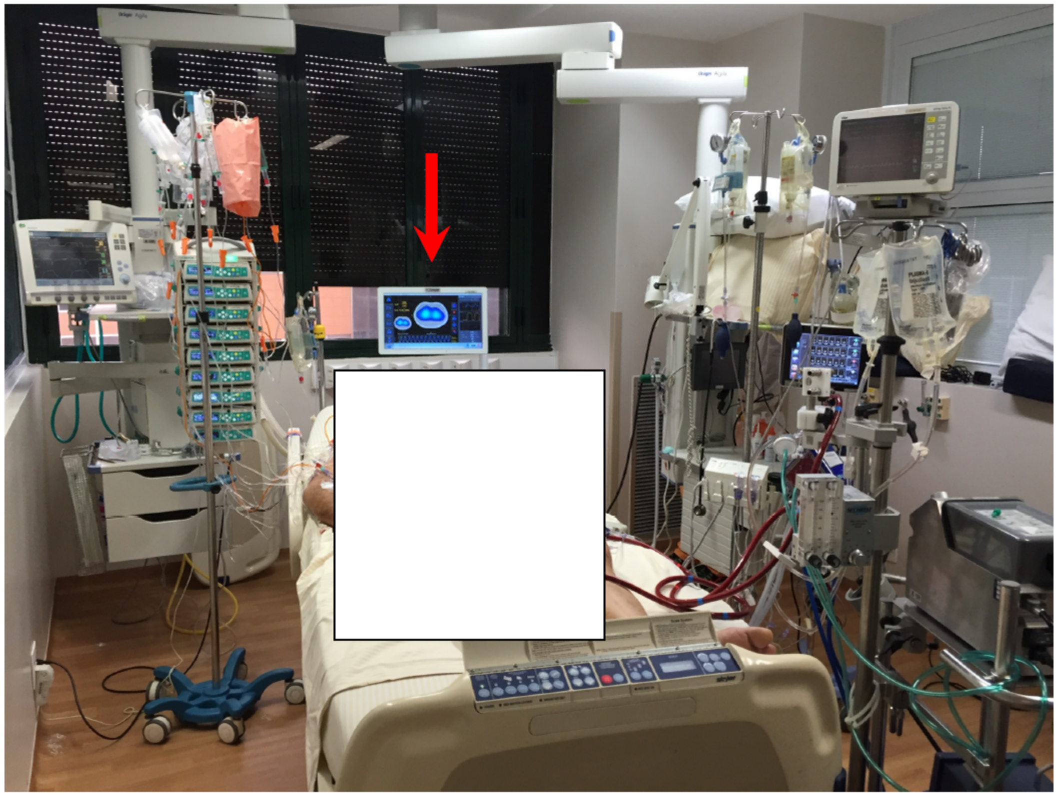
- safe (no ionizing radiation is needed)

- portable (used bedside/easily shared if needed)

- can be used 24/7 (updated information/alerts about events)

- no serious issues if used for long periods (skin allergies)

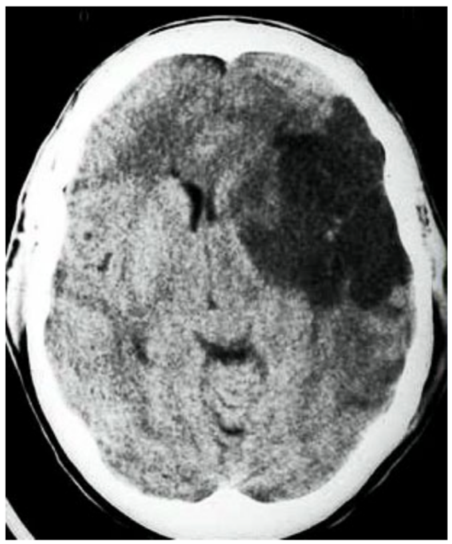
Continuous Monitoring and Portability



Albert Einstein Hospital - São Paulo - Brazil

Imaging stroke with EIT

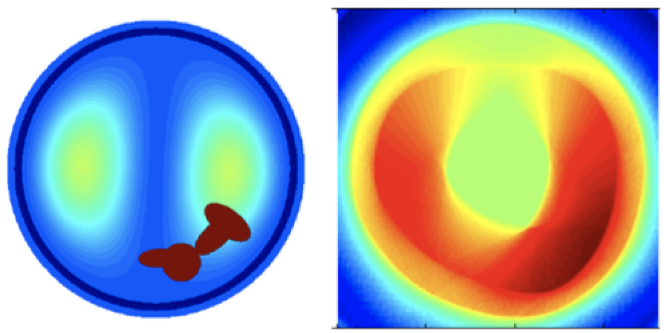
Ischemic stroke:
low conductivity.
CT image from
Jansen 2008



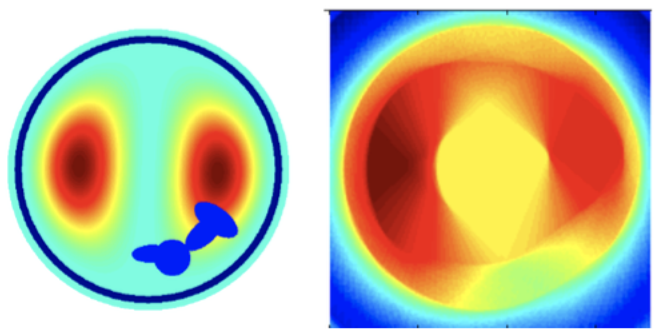
Hemorrhagic stroke:
high conductivity.
CT image from
Nakano *et al.* 2001

Same symptoms in both cases!

Brain Imaging



Simulated hemorrhage in the brain: higher conductivity because of excess blood.
Left: original, right: reconstruction



Simulated ischemic stroke: lower conductivity resulting from a clot blocking the flow of blood.
Left: original, right: reconstruction

Greenleaf, Lassas, Santacesaria, Siltanen and U, 2018

Calderón's Problem (EIT)

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$I(x) = \gamma(x)\nabla u$$

The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

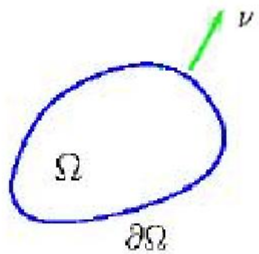
$$\operatorname{div}(\gamma(x)\nabla u) = 0 \text{ in } \Omega$$

Dirichlet-to-Neumann Map

$$\begin{aligned} \text{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}$$

$\gamma(x)$ = conductivity,
f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega = (\nu \cdot \gamma\nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\Lambda_\gamma(f) = \nu \cdot \gamma\nabla u\big|_{\partial\Omega}$$

EIT (Calderón's inverse problem)

Does Λ_γ determine γ?

Λ_γ = Dirichlet-to-Neumann map

Bilinear Form

$$\begin{array}{rl} \text{div}(\gamma\nabla u) &= 0 \\ u|_{\partial\Omega} &= f \end{array}, \quad \begin{array}{rl} \text{div}(\gamma\nabla v) &= 0 \\ v|_{\partial\Omega} &= g \end{array}$$

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

$$Q_\gamma(f,g) = \int_\Omega \gamma \nabla u \cdot \nabla v dx.$$

A. P. Calderón: On an inverse boundary value problem, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, RÍo de Janeiro, 1980.

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u(x)|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f dS.$$

Linearization

$$\lim_{\varepsilon \to 0^+} \frac{Q_{\gamma+\varepsilon h}(f,g) - Q_\gamma(f,g)}{\varepsilon} = \int_\Omega h \nabla u \cdot \nabla v \, dx.$$

Case $\gamma = 1$: $\quad \mathrm{div}(\gamma \nabla u) = \Delta u = 0$

Linearized Problem: Suppose we know

$$\int_\Omega h \nabla u \cdot \nabla v \, dx \qquad \forall \ \Delta u = \Delta v = 0.$$

Can we recover h?

Linearization

Linearized problem at $\gamma = 1$:

$$\boxed{\int_\Omega h \nabla u \cdot \nabla v dx} \qquad \text{data} \quad \forall\ \Delta u = \Delta v = 0.$$

Can we recover h?

$$\begin{matrix} u = e^{x\cdot\rho} \\ v = e^{-x\cdot\overline{\rho}} \end{matrix}\ , \qquad \rho \in \mathbb{C}^n,\ \rho\cdot\rho = 0.$$

$$\rho = \frac{\eta - i\xi}{2}, \qquad \rho\cdot\rho = 0 \ \Leftrightarrow\ |\eta| = |\xi|, \eta\cdot\xi = 0.$$

$$\boxed{|\xi|^2 \int_\Omega h e^{-ix\cdot\xi} dx} \qquad \text{known}$$

we can recover $\boxed{\widehat{\chi_\Omega h}(\xi)}$, therefore h on Ω.

Boundary Determination

Theorem (Kohn-Vogelius, 1984)

Assume $\gamma \in C^\infty(\overline{\Omega})$. From Λ_γ we can determine $\partial^\alpha \gamma\big|_{\partial\Omega}$, $\forall\, \alpha$.

Proof (Sylvester-U, 1988; Lee-U,1989)

Λ_γ is a pseudodifferential operator of order 1 (Calderón).

$\partial\Omega = \{x^n = 0\}$ locally.

Coordinates $x = (x', x^n)$, $x' \in \mathbb{R}^{n-1}$

$$\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'} \lambda_\gamma(x', \xi') \widehat{f}(\xi') d\xi'$$

Boundary Determination

$$\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'} \lambda_\gamma(x',\xi')\widehat{f}(\xi')d\xi'$$

$$\lambda_\gamma(x',\xi') = \gamma(0,x')|\xi'| + a_0(x',\xi') + \cdots + a_j(x',\xi') + \cdots$$

with $a_j(x',\xi')$ pos. homogeneous of degree $-j$ in ξ':

$$a_j(x',\lambda\xi') = \lambda^{-j} a_j(x',\xi'), \quad \lambda > 0.$$

Result From a_j, we can determine $\left.\frac{\partial^j \gamma}{\partial \nu^j}\right|_{x^n=0}$.

Basic Results

Theorem $n \geq 3$ (Sylvester-U, 1987)

$$\gamma \in C^2(\overline{\Omega}), \qquad 0 < C_1 \leq \gamma(x) \leq C_2 \qquad \text{on } \overline{\Omega}$$

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$$

- Extended to $\gamma \in C^{3/2}(\overline{\Omega})$ (Päivarinta-Panchenko-U, Brown-Torres, 2003)
- $\gamma \in C^{1+\epsilon}(\overline{\Omega}), \gamma$ conormal (Greenleaf-Lassas-U, 2003)
- $\gamma \in C^1(\overline{\Omega})$ (Haberman-Tataru, 2013)
- $\gamma \in W^{1,n}(\overline{\Omega})$, $(n = 3, 4)$ (Haberman, 2015)
- $\gamma \in W^{1,\infty}(\overline{\Omega})$ (Caro-Rogers, 2016)

Complex-Geometrical Optics Solutions (CGO)

- Reconstruction A. Nachman (1988)
- Stability G. Alessandrini (1988)
- Numerical Methods (D. Issacson, K. Knudsen, J. Müller, S. Siltanen, $\cdots$)

Schrödinger equation

Reduction to Schrödinger equation

$$\boxed{\text{div}(\gamma \nabla w) = 0}$$

$$u = \sqrt{\gamma} w$$

Then the equation is transformed into:

$$\boxed{(\Delta - q)u = 0, q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}} \qquad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)$$

$$\boxed{\begin{aligned} (\Delta - q)u &= 0 \\ u\Big|_{\partial\Omega} &= f \end{aligned}}$$

Define $\boxed{\Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}}$

ν = unit-outer normal to $\partial\Omega$.

IDENTITY

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} \left((\Lambda_{q_1} - \Lambda_{q_2}) u_1 \Big|_{\partial\Omega} \right) u_2 \Big|_{\partial\Omega} dS$$

$$(\Delta - q_i) u_i = 0$$

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{q_1} = \Lambda_{q_2}$ and

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

GOAL: Find **MANY** solutions of $(\Delta - q_i) u_i = 0$.

CGO Solutions

Calderón: Let $\rho \in \mathbb{C}^n$, $\boxed{\rho \cdot \rho = 0}$

$$\boxed{\rho = \eta + ik} \qquad \eta, k \in \mathbb{R}^n, |\eta| = |k|, \eta \cdot k = 0$$

$$\boxed{u = e^{x \cdot \rho}} = e^{x \cdot \eta} e^{ix \cdot k}$$

$$\Delta u = 0, \qquad u = \begin{cases} \underline{\text{exponentially decreasing}}, & x \cdot \eta < 0 \\ \underline{\text{oscillating}}, & x \cdot \eta = 0 \\ \underline{\text{exponentially increasing}}, & x \cdot \eta > 0 \end{cases}$$

Complex Geometrical Optics

(Sylvester-U) $n \geq 2,\ q \in L^\infty(\Omega)$
Let $\rho \in \mathbb{C}^n$ $(\rho = \eta + ik, \eta, k \in \mathbb{R}^n)$ such that $\boxed{\rho \cdot \rho = 0}$
$(|\eta| = |k|, \eta \cdot k = 0)$.

Then for $|\rho|$ sufficiently large we can find solutions of

$$\boxed{(\Delta - q) w_\rho = 0 \text{ on } \Omega}$$

of the form

$$\boxed{w_\rho = e^{x \cdot \rho}(1 + \Psi_q(x, \rho))}$$

with $\Psi_q \to 0$ in Ω as $|\rho| \to \infty$.

Uniqueness

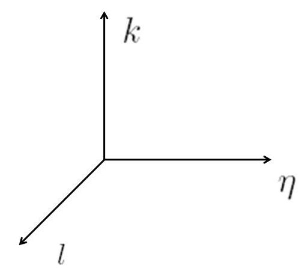
Proof $\boxed{\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2}$

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

$$u_1 = e^{x\cdot\rho_1}(1 + \Psi_{q_1}(x, \rho_1)), \qquad u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x, \rho_2))$$

$\rho_1 \cdot \rho_1 = \rho_2 \cdot \rho_2 = 0, \qquad \rho_1 = \eta + i(k + l)$

$\rho_2 = -\eta + i(k - l)$



$\eta \cdot k = \eta \cdot l = l \cdot k = 0, \quad |\eta|^2 = |k|^2 + |l|^2$

$$\int_\Omega (q_1 - q_2) e^{2ix\cdot k}(1 + \Psi_{q_1} + \Psi_{q_2} + \Psi_{q_1}\Psi_{q_2}) = 0$$

Letting $|l| \to \infty$ $\boxed{\int_\Omega (q_1 - q_2) e^{2ix\cdot k} = 0 \quad \forall k \Longrightarrow q_1 = q_2}$

APPLICATIONS

$n \geq 3 \quad (\Delta - q) = 0$, $\boxed{\Lambda_q}$ determines $\boxed{q}$

- EIT $\boxed{\Lambda_\gamma}$ determines $\boxed{\gamma}$
- Optical Tomography (Diffusion Approximation)

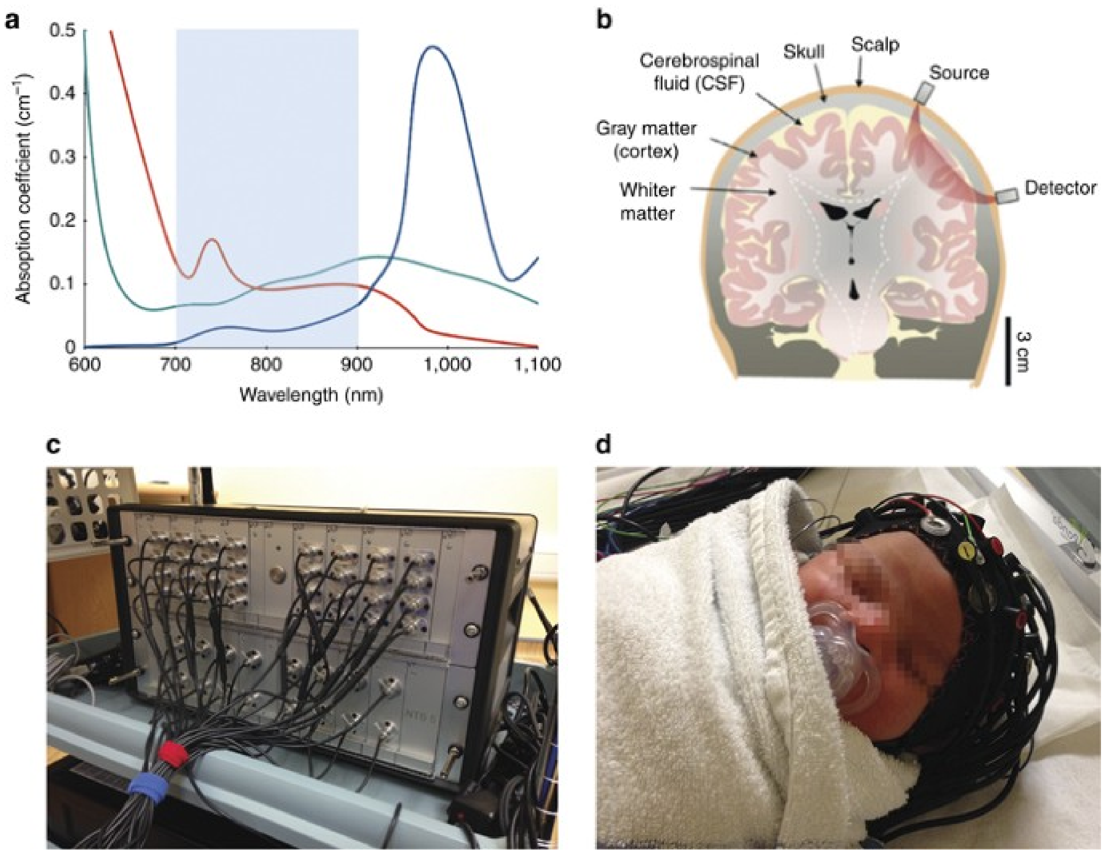
$$\boxed{i\omega U - \nabla \cdot D(x)\nabla U + \sigma_a(x)U = 0} \text{ in } \Omega$$

U= Density of photons, D=Diffusion Coefficient, $\sigma_a(x)$= optical absorption.

RESULT

- If $\omega \neq 0$ we can recover both $D(x)$ and $\sigma_a(x)$.
- If $\omega = 0$ we can recover either $D(x)$ or $\sigma_a(x)$.

Optical Tomography



Wikipedia

Other Applications (Fixed energy)

- Optics $(\Delta - k^2 n(x))u = 0$, $n(x)$ isotropic index of refraction $(q(x) = k^2 n(x))$.
- Acoustic $\text{div}(\frac{1}{\rho(x)}\nabla p) + \omega^2 \kappa(x) p = 0$, ρ density, κ compressibility (need two frequencies ω).
- Inverse quantum scattering at fixed energy $(\Delta - q - \lambda^2)u = 0$, q potential.
- Maxwell's Equation (Isotropic) (Ola-Somersalo): Reduction to $(\Delta - Q)$, Q an 8×8 matrix.
- Quantitative Photoacoustic Tomography (Bal-U, 2010)

Extension of CGO Solutions

$$u = e^{x\cdot\rho}(1 + \Psi_q(x,\rho))$$

$\rho \in \mathbb{C}^n, \rho\cdot\rho = 0$

(Not helpful for localizing)

Kenig-Sjöstrand-U (2007),

$$u = e^{\tau(\varphi(x)+i\psi(x))}(a(x) + R(x,\tau))$$

$\tau \in \mathbb{R}$, φ, ψ real-valued, $R(x,\tau) \to 0$ as $\tau \to \infty$.

φ limiting Carleman weight,

$$\nabla\varphi\cdot\nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

Example: $\varphi(x) = \ln|x - x_0|,\quad x_0 \notin \overline{ch(\Omega)}$

CGO Solutions

$$u = e^{\tau(\varphi(x)+i\psi(x))}(a_0(x) + R(x,\tau))$$
$$R(x,\tau) \xrightarrow{\tau\to\infty} 0 \text{ in } \Omega$$

$$\varphi(x) = \ln|x - x_0|$$

Complex Spherical Waves

Theorem (Kenig-Sjöstrand-U, 2007) Ω strictly convex.

$$\Lambda_{q_1}\Big|_{\Gamma} = \Lambda_{q_2}\Big|_{\Gamma}, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

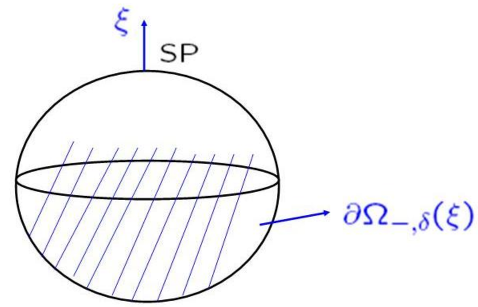
Earlier Result

(Bukhgeim-U, 2002) $n \geq 3$. Let $\xi \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. We define

$$\partial\Omega_\pm = \left\{ x \in \partial\Omega; \langle \nu, \xi \rangle \begin{matrix} > 0 \\ < 0 \end{matrix} \right\}$$

(ν is the unit outer normal). Let $\delta > 0$

$$\partial\Omega_{+,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle > \delta\}$$
$$\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle < \delta\}$$



Theorem (Bukhgeim-U) Suppose we know

$$\boxed{\Lambda_q(f)|_{\partial\Omega_{-,\delta}(\xi)}} \qquad \text{supp } f \subseteq \partial\Omega,$$

Then we can recover $\boxed{q}$.

Carleman Estimate

$\xi \in \mathbb{S}^{n-1}$,

Let $q \in L^\infty(\Omega)$, $u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$. For $\tau \geq \tau_0$

$$\tau^2 \int_\Omega |e^{-\tau\langle x,\xi\rangle}u|^2 \, dx + \underline{\tau} \int_{\underline{\partial\Omega_+}} \langle \xi, \nu(x)\rangle |e^{-\tau\langle x,\xi\rangle}\frac{\partial u}{\partial \nu}|^2 dS(x)$$

$$\leq C\left(\int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2 dx - \underline{\tau} \int_{\underline{\partial\Omega_-}} \underline{\langle \xi, \nu(x)\rangle} |e^{-\tau\langle x,\xi\rangle}\frac{\partial u}{\partial \nu}|^2 dS\right)$$

Carleman Estimate

Remarks $\Delta_\rho u = e^{-x\cdot\rho}\Delta(e^{x\cdot\rho}u)$

- Carleman estimate for domains with boundary for $\Delta_\rho = \Delta + 2\rho\cdot\nabla$

- Weight is linear: $\langle x,\xi\rangle$

Corollary $u = 0$ on $\partial\Omega$, $\frac{\partial u}{\partial\nu}|_{\partial\Omega_-} = 0$

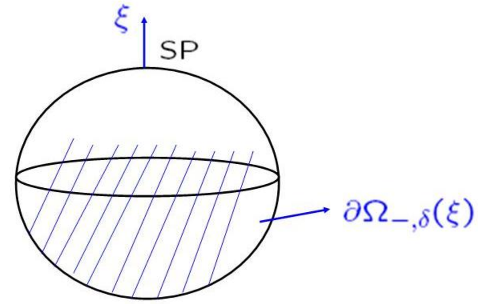
$$\tau\int_{\partial\Omega_+}\langle x,\nu(x)\rangle|e^{-\tau\langle x,\xi\rangle}\frac{\partial u}{\partial\nu}|^2 dS(x) \leq C\int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2 dx$$

We need $\langle\nu(x),\xi\rangle \geq \delta > 0$

Partial Data

<u>Bukhgeim-U</u> $(n \geq 3)$

$$\Lambda_{q_1}(f)|_{\partial\Omega_{-,\delta}(\xi)} = \Lambda_{q_2}(f)|_{\partial\Omega_{-,\delta}(\xi)} \quad \forall f \quad \Longrightarrow \quad q_1 = q_2$$



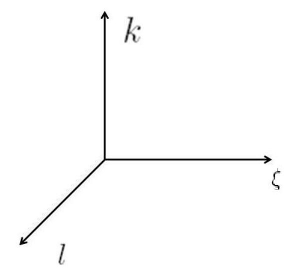
$$\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle < \delta\}$$

<u>Sketch of proof</u>

Choose $u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x, \rho_2))$

solution of $(\Delta - q_2)u_2 = 0$

$$\rho_2 = \tau\xi + i(k + l)$$
$$|k|^2 + |l|^2 = \tau^2$$



Let u_1 be such that $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}, \dfrac{\partial u_1}{\partial \nu}|_{\partial\Omega_{-,\delta}(\xi)} = \dfrac{\partial u_2}{\partial \nu}|_{\partial\Omega_{-,\delta}(\xi)}$

Partial Data

$$u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x, \rho_2))$$

$$(\Delta - q_2)u_2 = 0,\ \rho_2 = \tau\xi + i(k + l)$$

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega},\ \frac{\partial u_1}{\partial\nu_1}|_{\partial\Omega_{-,\delta}(\xi)} = \frac{\partial u_2}{\partial\nu_1}|_{\partial\Omega_{-,\delta}(\xi)}$$

$$u = u_1 - u_2, \quad q = q_1 - q_2$$

$$v_1 = e^{x\cdot\rho_1}(1 + \Psi_{q_1}(x, \rho_1)) \quad \rho_1 = -\tau\xi + i(k - l)$$

solution of $(\Delta - q_1)v_1 = 0$.

$$\int_\Omega q u_2 v_1 dx = \int_{\partial\Omega} \frac{\partial u}{\partial\nu} v_1 dS$$

Note that $u|_{\partial\Omega} = 0$, $\frac{\partial u}{\partial\nu}|_{\partial\Omega_{-,\delta}(\xi)} = 0$

Partial Data

$q = q_1 - q_2$

$$(*) \qquad \int_\Omega q u_2 v_1 dx = \int_{\partial\Omega_{+,\delta}(\xi)} \frac{\partial u}{\partial \nu} v_1 dS$$

$$u_2 = e^{x\cdot\rho_1}(1+\Psi_{q_1})$$
$$v_1 = e^{x\cdot\rho_2}(1+\Psi_{q_2})$$

Fix $k \in \mathbb{R}^n$, $\rho_1 + \rho_2 = 2ik$

Carleman estimate

$$\tau \int_{\partial\Omega_+} \langle \xi, \nu(x)\rangle |e^{-\tau\langle x,\xi\rangle} \frac{\partial u}{\partial \nu}|^2 dS(x) \le C \int_\Omega |(\Delta - q_1) u e^{-\tau\langle x,\xi\rangle}|^2 dS(x)$$

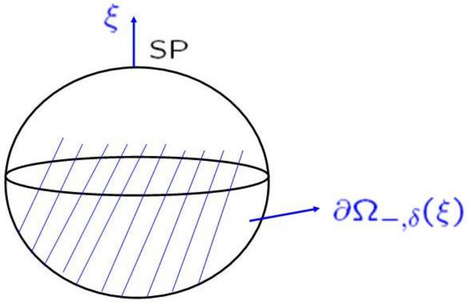
Need $\langle x, \xi\rangle \ge \delta > 0$. $|\text{RHS}| \le C$ as $|\rho| \to \infty$, LHS$\to \int_\Omega q e^{2ix\cdot k} dx$

Partial Data

We get $\int_\Omega e^{2ix\cdot k} q(x)dx = 0$

$k \perp \xi$. But we can move ξ a little bit

$\xi \in \mathbb{S}^{n-1}$



$$\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu(x), \xi\rangle < \delta\}$$

We obtain $\widehat{\chi_\Omega q}(-2k) = 0$ in an open cone $\Longrightarrow q = 0$.

Carleman estimate $\Longrightarrow$ control of $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega_{+,\delta}}$ (with appropriate linear weights) (Stability estimates, Heck-Wang, 2006, 2016)

More general Partial Data

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

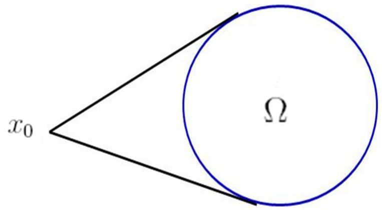
$$\Lambda_{q_1}\Big|_{\Gamma} = \Lambda_{q_2}\Big|_{\Gamma}, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

$u_\tau = e^{\tau(\varphi + i\psi)} a_\tau$ $\qquad$ $\varphi(x) = \ln|x - x_0|, x_0 \notin \overline{ch(\Omega)}$

Eikonal: $\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi| = |\nabla\psi|$

$\psi(x) = d(\frac{x-x_0}{|x-x_0|}, \omega), \omega \in S^{n-1}$: smooth for $x \in \bar{\Omega}$.

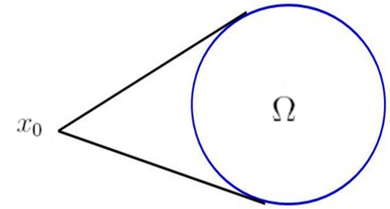


Transport: $(\nabla\varphi + i\nabla\psi) \cdot \nabla a_\tau = 0$

(Cauchy-Riemann equation in plane generated by $\nabla\varphi, \nabla\psi$)

More general Partial Data

$\varphi(x) = \ln|x - x_0|, \quad x_0 \notin \overline{ch(\Omega)}$



Carleman Estimates

$u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega_-} = 0 \qquad\qquad \partial\Omega_\pm = \{x \in \partial\Omega; \nabla\varphi\cdot\nu \gtrless 0\}$

$$\int_{\partial\Omega_+} < \nabla\varphi, \nu > |e^{-\tau\varphi(x)}\frac{\partial u}{\partial \nu}|^2 ds \leq \frac{C}{\tau}\int_\Omega |(\Delta - q)ue^{-\tau\varphi(x)}|^2 ds$$

This gives control of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_{+,\delta}}$,

$$\partial\Omega_{+,\delta} = \{x \in \partial\Omega, \nabla\varphi\cdot\nu \geq \delta\}$$

More general CGO solutions

$$u_\tau = e^{\tau(\varphi + i\psi)} a_\tau,$$

$\tau \gg 0$, $\tau = 1/h$ (semicl.), φ, ψ real-valued

- φ is a limiting Carleman weight

$$e^{\frac{\varphi}{h}} h^2(-\Delta + q) e^{-\frac{\varphi}{h}}$$

has semiclassical principal symbol

$$P_\varphi(x, \xi) = \xi^2 - (\nabla\varphi)^2 + 2i\nabla\varphi \cdot \xi$$

Hörmander's condition:

$$\{\text{Re } P_\varphi, \text{Im } P_\varphi\} \leq 0 \quad \text{on} \quad P_\varphi = 0$$

We need $\varphi, -\varphi$ to be phase of solutions.

$$\text{LCW}: \quad \{\text{Re } P_\varphi, \text{Im } P_\varphi\} = 0$$

$\nabla\varphi \neq 0$ in an open neighborhood of $\bar{\Omega}$.

More general CGO solutions

$u_h = e^{\frac{1}{h}(\varphi + i\psi)} a_h$

• φ LCW, φ real-valued

$$\{\mathrm{Re} P_\varphi, \mathrm{Im} P_\varphi\} = 0 \quad \text{on} \quad P_\varphi = 0$$

$\nabla\varphi \neq 0$ on an open neighborhood of $\bar{\Omega}$.

Examples (Dos Santos Ferreira-Kenig-Salo-U, 2009)

(a) $\varphi(x) = x \cdot \xi$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$

(b) $\varphi(x) = a \ln |x - x_0| + b$, $(a, b$ constants), $x_0 \notin \overline{ch(\Omega)}$

(c) $\varphi(x) = \dfrac{a\langle x - x_0, \xi\rangle}{|x - x_0|^2} + b$, $\xi \in \mathbb{R}^n$

(d) $\varphi(x) = a \arctan \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(e) $\varphi(x) = a \operatorname{arctanh} \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(f) $n = 2$, φ is a harmonic function

More general CGO solutions

Instead of

$$\int_{\Omega} e^{2ix\cdot k} q(x)dx = 0$$

$k \perp \xi$ ($\xi \in \mathbb{S}^{n-1}$) as in Bukhgeim-U argument we get

$$\int_{\Omega} e^{i\lambda f(x)} \boxed{q(x)} a_1 a_2 dx = 0$$

λ any real number, $a_1, a_2 \neq 0$, $f(x)$ real-analytic, a_1, a_2 real analytic

Analytic microlocal analysis $\Longrightarrow q = 0$ (like inversion of real-analytic Radon transforms)

Linearization with Partial Data

(Analog of Calderón)

Theorem (Dos Santos Ferreira, Kenig, Sjöstrand-U, 2009; Sjöstrand-U, 2016)

$$\boxed{\int_\Omega huv = 0}$$

$\Gamma \subseteq \partial\Omega$, Γ open,

$(\Delta - q)u = (\Delta - q)v = 0$, $\quad q$ is analytic, $u, v \in C^\infty(\overline{\Omega})$,

supp $u|_{\partial\Omega}$, supp $v|_{\partial\Omega} \subseteq \Gamma$,

$$\Rightarrow \boxed{h = 0}.$$

Complex Spherical Waves

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$$

$$\varphi(x) = \ln|x - x_0|, \; x_0 \notin \overline{\text{ch}(\Omega)}$$

Also used to determine inclusions, obstacles, etc.

a) Conductivity   Ide-Isozaki-Nakata-Siltanen-U, 2007

b) Helmholtz   Nakamura-Yosida, 2007

c) Elasticity   J.-N. Wang-U, 2007

d) Maxwell   T. Zhou, 2010

Complex Spherical Waves

(Loading reconperfect1.mpg)

The Two Dimensional Case

Theorem ($n = 2$) Let $\gamma_j \in C^2(\overline{\Omega})$, $j = 1, 2$.

Assume $\boxed{\Lambda_{\gamma_1} = \Lambda_{\gamma_2}}$. Then $\boxed{\gamma_1 = \gamma_2}$.

- Nachman (1996)
- Brown-U (1997) Improved to γ_j Lipschitz
- Astala-Päivärinta (2006) Improved to $\gamma_j \in L^\infty(\Omega)$

Recall

$$\begin{gathered}\operatorname{div}(\gamma \nabla u) = 0, \quad \gamma \in L^\infty(\Omega) \\ u|_{\partial\Omega} = f\end{gathered}$$

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u|^2 dx = \langle \Lambda_\gamma f, f \rangle_{L^2(\partial\Omega)}.$$

The Two Dimensional Case

This follows from more general result

Theorem ($n = 2$, Bukhgeim, 2008) Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$.

$$\text{Assume } \boxed{\Lambda_{q_1} = \Lambda_{q_2}}. \text{ Then } \boxed{q_1 = q_2}.$$

Recall

$$\begin{gathered}(\Delta - q)u = 0, \\ u|_{\partial\Omega} = f.\end{gathered} \qquad \Lambda_q(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$$

with ν-unit outer normal.

The Two Dimensional Case

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2}} \Rightarrow \boxed{q_1 = q_2}$$

Sketch of proof New class of CGO solutions

$$u_j(z,\tau) = e^{i\tau z^2}\left(1 + r_j(z,\tau)\right) \quad \tau \gg 1$$

solve $(\Delta - q_j)u_j = 0$ with $r_j(z,\tau) \to 0$ on Ω sufficiently fast.

Notation $z = x_1 + ix_2$

Remark $z^2 = x_1^2 - x_2^2 + 2ix_1x_2 = \varphi + i\psi$

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

φ harmonic, ψ conjugate harmonic.

The Two Dimensional Case

$$\Lambda_{q_1} = \Lambda_{q_2} \ \Rightarrow \ \boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

$$(\Delta - q_j) u_j = 0$$

$$u_j = e^{i\tau z^2 (1 + r_j(z,\tau))}$$

Substituting

$$\int_\Omega (q_1 - q_2) e^{2i\tau(x_1^2 - x_2^2)} (1 + r_1 + r_2 + r_1 r_2) dx = 0.$$

Letting $\tau \to \infty$ and using stationary phase

$$(q_1 - q_2)(0) = 0.$$

Changing z to $z - z_0$ we get

$$(q_1 - q_2)(z_0) = 0.$$

The Two Dimensional Case

This follows from more general result

Theorem ($n = 2$, Bukhgeim, 2008) Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$.

Assume $\boxed{\Lambda_{q_1} = \Lambda_{q_2}}$. Then $\boxed{q_1 = q_2}$.

Recall

$$\begin{gathered}(\Delta - q)u = 0, \\ u|_{\partial\Omega} = f.\end{gathered} \qquad \Lambda_q(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$$

with ν-unit outer normal.

The Two Dimensional Case

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2}} \Rightarrow \boxed{q_1 = q_2}$$

Sketch of proof New class of CGO solutions

$$\boxed{\begin{array}{c} u_1(z,\tau) = e^{\tau z^2}(1 + r_1(z,\tau)) \\ u_2(z,\tau) = e^{-\tau \bar{z}^2}(1 + r_2(z,\tau)) \end{array}} \qquad \tau \gg 1$$

solve $(\Delta - q_j)u_j = 0$ with $r_j(z,\tau) \to 0$ on Ω sufficiently fast.

Notation $z = x_1 + ix_2$

Remark $z^2 = x_1^2 - x_2^2 + 2ix_1x_2 = \varphi + i\psi$

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

φ harmonic, ψ conjugate harmonic.

The Two Dimensional Case

$$\Lambda_{q_1} = \Lambda_{q_2} \;\Rightarrow\; \boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

$$(\Delta - q_j) u_j = 0$$

$$u_1 = e^{\tau z^2} \left(1 + r_1(z, \tau)\right), \quad u_2 = e^{-\tau \bar{z}^2} \left(1 + r_2(z, \tau)\right)$$

Substituting

$$\int_\Omega (q_1 - q_2) e^{4i\tau x_1 x_2} (1 + r_1 + r_2 + r_1 r_2) dx = 0.$$

Letting $\tau \to \infty$ and using stationary phase

$$(q_1 - q_2)(0) = 0.$$

Changing z to $z - z_0$ we get

$$(q_1 - q_2)(z_0) = 0.$$

Partial data

Let $\Gamma \subseteq \partial\Omega$, Γ open.
Let $q_j \in C^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$, $j = 1, 2$.

Theorem (Imanuvilov-U-Yamamoto 2010) n=2. Assume

$$\boxed{\Lambda_{q_1}(f)\Big|_\Gamma = \Lambda_{q_2}(f)\Big|_\Gamma}$$

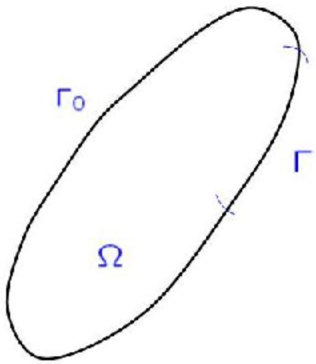
$\forall\, f$, supp$f \subseteq \Gamma$. Then

$$\boxed{q_1 = q_2}.$$

- Riemann Surfaces: Guillarmou-Tzou (2011)

Partial Data

$\Gamma_0 = \partial\Omega - \Gamma$



Construct CGO solutions

$$\begin{array}{c} \Delta u_j - q_j u_j = 0 \quad \text{in } \Omega \\ u_j|_{\Gamma_0} = 0 \end{array}$$

In this case

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$$

if $\Lambda_{q_1}(f)|_\Gamma = \Lambda_{q_2}(f)|_\Gamma$, $\text{supp} f \subseteq \Gamma$.

Partial Data

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

$$u_j|_{\Gamma_0} = u_j|_{\partial\Omega - \Gamma} = 0$$

$$u_1(x) = e^{\tau\Phi(z)}\left(a(z) + \frac{a_0(z)}{\tau}\right) + \overline{e^{\tau\Phi(z)}\left(a(z) + \frac{a_1(z)}{\tau}\right)} + e^{\tau\varphi} R_\tau^{(1)}$$

$$u_2(x) = e^{-\tau\bar{\Phi}(z)}\left(\bar{a}(z) + \frac{b_0(z)}{\tau}\right) + \overline{e^{-\tau\bar{\Phi}(z)}\left(\bar{a}(z) + \frac{b_1(z)}{\tau}\right)} + e^{\tau\varphi} R_\tau^{(2)}$$

$\Phi = \varphi + i\psi$ holomorphic

Partial Data

$$u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots), \quad u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots)$$

$$\boxed{\Phi(z) = \varphi + i\psi} \quad \text{holomorphic}$$

$$\boxed{u_j|_{\partial\Omega - \Gamma} = 0}$$

$p \in \Omega$, Φ has non-degenerate critical point at p (Morse function)

$$\bar{\partial} a = 0 \quad \text{Re } a|_{\partial\Omega-\Gamma} = 0$$

$a = 0$ at other critical points

Stationary phase in

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

Partial Data

$$u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots)$$
$$u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots)$$
$$u_j|_{\partial\Omega-\Gamma} = 0$$

$\Phi(z)$ Morse function with non-degenerate critical point at p.

$$\int_\Omega (q_1 - q_2)u_1u_2 = 0$$

Stationary phase

$$\Longrightarrow \boxed{(q_1 - q_2)(p) = 0}$$

Carleman Estimate With Degenerate Weights

Lemma 1

Let $\partial\Omega - \Gamma = \{x \in \partial\Omega;\ \nu \cdot \nabla\varphi = 0\}$. Then for τ sufficiently large, $\exists$ solution of

$$\boxed{\begin{array}{c}\Delta u - qu = f \quad \text{in} \quad \Omega \\ u|_{\partial\Omega - \Gamma} = g\end{array}}$$

such that

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C\left(|\tau|^{-1/2}\|fe^{-\tau\varphi}\|_{L^2(\Omega)} + \|ge^{-\tau\varphi}\|_{L^2(\Omega-\Gamma)}\right)$$

Phase Function

Lemma 2 (Vekua) Given points $x_1, ...x_N$ in Ω and constants $b_i, i = 1, 2$, $C_{0,j}, C_{1,j}, C_{2,j}, j = 1, ..., N$, there exists an open and dense set

$$\Theta \subseteq \overline{C^2(\partial\Omega - \Gamma)} \times \overline{C^2(\partial\Omega - \Gamma)} \times \mathbb{C}^{3N}$$

solution of

$$\phi + i\psi \text{ holomorphic in } \Omega,$$

$$\boxed{\begin{array}{c} (\phi, \psi)|_{\partial\Omega - \Gamma} = (b_1, b_2) \\ (\phi + i\psi)(x_j) = C_{0,j} \\ \frac{\partial}{\partial z}(\phi + i\psi)(x_j) = C_{1,j} \\ \frac{\partial^2}{\partial z^2}(\phi + i\psi)(x_j) = C_{2,j} \end{array}}$$

CGO Solutions ($n = 2$)

$$\partial_{\bar{z}}^{-1} g(z) := -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\xi_1 + i\xi_2 - z} \, d\xi_1 d\xi_2, \qquad \partial_{z}^{-1} g := \overline{\partial_{\bar{z}}^{-1} \bar{g}}$$

$$R_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\bar{z}}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})$$

$$\widetilde{R}_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{z}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})$$

$\mathcal{H}$ = non-degenerate critical points of Φ.

CGO Solutions ($n = 2$)

$$\Delta u_1 - q_1 u_1 = 0 \quad \text{in } \Omega$$

$$u_1\big|_{\partial\Omega\setminus\Gamma} = 0.$$

Let Φ be a holomorphic Morse function, such that $\mathrm{Im}\Phi = 0$ on $\partial\Omega\setminus\Gamma$. Let $\Phi = \varphi + i\psi$.

$$\begin{aligned} u_1(x) &= e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi z}}\overline{(a(z) + a_0(z)/\tau)} \\ &\quad + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}. \end{aligned}$$

CGO Solutions (n=2)

$$u_1(x) = e^{\tau\Phi(z)}(a(z)+a_0(z)/\tau) + e^{\tau\overline{\Phi z}}\overline{(a(z)+a_0(z)/\tau)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}.$$

Choice of a, a_0, a_1:

$$a, a_0, a_1 \in C^2(\bar{\Omega}), \quad \partial_{\bar{z}}a = \partial_{\bar{z}}a_0 = \partial_{\bar{z}}a_1 \equiv 0$$

$$\mathrm{Re}\, a\Big|_{\partial\Omega\setminus\Gamma} = 0, \quad a = \partial_z a = 0 \quad \text{on } \mathcal{H}\cap\partial\Omega.$$

$$(a_0(z)+\overline{a_1(z)})\Big|_{\partial\Omega\setminus\Gamma} = \frac{\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)}{4\partial_z\Phi} + \frac{\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})}{4\overline{\partial_z\Phi}}$$

CGO Solutions (n=2)

The polynomials $M_1(z)$ and $M_3(z)$ satisfy

$$\partial_z^j \Big(\partial_{\bar z}^{-1}(aq_1) - M_3(z)\Big) = 0, \quad z \in \mathcal{H}, j = 0, 1, 2$$

$$\partial_{\bar z}^j \Big(\partial_z^{-1}(\bar a q_1) - M_3(\bar z)\Big) = 0, \quad z \in \mathcal{H}, j = 0, 1, 2.$$

Let $e_i, i = 1, 2$ be smooth, $e_1 + e_2 = 1$ on $\overline{\Omega}$ with $e_1 = 0$ in a neighborhood of $\mathcal{H} - \partial\Omega$ and $e_2 = 1$ in a neighborhood of $\partial\Omega$.

Remainder Term

Choice of u_{11}:

$$
\begin{aligned}
u_{11} \;=\; & -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}\Big(e_1\Big(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)\Big)\Big) \\
& -\frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}\Big(e_1\Big(\partial_z^{-1}(\overline{a(z)})q_1 - M_3(\bar{z})\Big)\Big) \\
& -\frac{e^{i\tau\psi}}{\tau}\,\frac{e_2\Big(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)\Big)}{4\partial_z\Phi} \\
& -\frac{e^{i\tau\psi}}{\tau}\,\frac{e_2\Big(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})\Big)}{4\overline{\partial_z\Phi}}
\end{aligned}
$$

Other Remainder Term

Find u_{12} such that

$$\Delta(u_{12}e^{\tau\varphi}) - q_1 u_{12}e^{\tau\varphi} - q_1 u_{11}e^{\tau\varphi} + h_1 e^{\tau\varphi} \quad \text{in } \Omega,$$

$$\begin{aligned} u_{12}\Big|_{\partial\Omega\setminus\Gamma} &= \frac{1}{4}\widetilde{R}_{\Phi,\tau}\Big(e_1\Big(\partial_{\bar z}^{-1}(a(z)q_1) - M_1(z)\Big)\Big) \\ &+\frac{1}{4}R_{\Phi,-\tau}\Big(e_1\Big(\partial_{z}^{-1}(\overline{a(z)})q_1 - M_3(\bar z)\Big)\Big). \end{aligned}$$

$$\|u_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty.$$

Other Remainder Term

Here

$$
\begin{aligned}
h_1 &= e^{i\tau\psi}\Delta\left(\frac{e_2\Big(\partial_{\bar z}^{-1}(a(z)q_1) - M_1(z)\Big)}{4\tau\partial_z\Phi}\right) \\
&\quad + e^{-i\tau\psi}\Delta\left(\frac{e_2\Big(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar z)\Big)}{4\tau\overline{\partial_z\Phi}}\right) \\
&\quad - \frac{a_0 q_1}{\tau}e^{i\tau\psi} - \frac{\overline{a_1}q_1}{\tau}e^{-i\tau\psi}.
\end{aligned}
$$

Other Remainder Term

Similarly:

$$\Delta v - q_2 v = 0 \quad \text{in } \Omega, \qquad v\Big|_{\partial\Omega\setminus\Gamma} = 0.$$

Construct solution v of the form

$$\begin{aligned} v(x) &= e^{-\tau\Phi(z)}\Big(a(z) + b_0(z)/\tau\Big) + e^{-\tau\overline{\Phi(z)}}\Big(\overline{a(z) + b_0(z)/\tau}\Big) \\ &\quad + e^{-\tau\varphi} v_{11} + e^{-\tau\varphi} v_{12}. \end{aligned}$$

Main Term

$$R = \int_\Omega (q_1 - q_2)(a(a_0 + b_0) + \bar{a}(\overline{a_1} + \overline{b_1}))dx$$

$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\,\frac{\partial_{\bar{z}}^{-1}(aq_2) - M_2(z)}{\partial_z\Phi} + \bar{a}\,\frac{\partial_{\bar{z}}^{-1}(\bar{a}q_2) - M_4(\bar{z})}{\overline{\partial_z\Phi}}\right)dx$$

$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\,\frac{\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)}{\partial_z\Phi} + \bar{a}\,\frac{\partial_{\bar{z}}^{-1}(\bar{a}q_1) - M_3(\bar{z})}{\overline{\partial_z\Phi}}\right)dx$$

Proof of Uniqueness for Partial Data

- Take geometiric optics solution u_1 to

$$\Delta u_1 - q_1 u_1 = 0, \quad u_1\Big|_{\partial\Omega\backslash\Gamma} = 0.$$

- u_2: $$\Delta u_2 - q_2 u_2 = 0, \quad u_2\Big|_{\partial\Omega} = u_1\Big|_{\partial\Omega}.$$

  DN maps are equal $\Rightarrow \nabla u_2 = \nabla u_1$ on Γ.

$$u = u_1 - u_2 \;\Rightarrow\; \Delta u - q_2 u = (q_1 - q_2) u_2$$

$$u\Big|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma} = 0.$$

- Take complex geometric optics solution v to

$$\Delta v - q_2 v = 0, \quad v\Big|_{\partial\Omega\backslash\Gamma} = 0.$$

Uniqueness for Partial Data

$$0 = \int_\Omega v(\Delta u - q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx :$$

Stationary phase + estimates for $u_{12} \Rightarrow$

$$2\sum_{k=1}^{l} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x_k})\mathrm{Re}\, e^{i2\tau \mathrm{Im}\,\Phi(\widetilde{x_k})}}{|(\det \mathrm{Im}\,\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}} + R = o(1), \quad \text{as } \tau \to \infty.$$

[left side] $=$ almost perodic function in τ.

Bohr's theorem inplies [left side] $= 0$ for all τ.

Phase function

We can choose Φ such that

$$\operatorname{Im}\Phi(\widetilde{x_k}) \neq \operatorname{Im}\Phi(\widetilde{x_j}), \quad j \neq k.$$

Let $a(\widetilde{x_k}) \neq 0$. Then stationary phase implies

$$q_1(\widetilde{x_k}) = q_2(\widetilde{x_k}).$$

Partial Data for Second Order Elliptic Equations ($n = 2$)

(Imanuvilov–U–Yamamoto, 2011)

$$\boxed{\Delta_g + A(z)\frac{\partial}{\partial z} + B(z)\frac{\partial}{\partial \overline{z}} + q} \qquad z = x_1 + ix_2$$

$g = (g_{ij})$ positive definite symmetric matrix;

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \qquad g^{ij} = (g_{ij})^{-1}$$

Includes:

- Anisotropic Calderón's Problem
- Magnetic Schrödinger Equation
- Convection terms

Anisotropic case

Cardiac muscle	6.3 mho	(longitudinal)
	2.3 mho	(transversal)

$\gamma = (\gamma^{ij})$ positive-definite, symmetric matrix

conductivity

$\Omega \subseteq \mathbb{R}^n, \Omega$ bounded. Under assumptions of no sources or sinks of current the potential u satisfies

$$\text{div}(\gamma \nabla u) = 0$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_i}\right) = 0 \text{ in } \Omega \qquad (*)$$

$$u\big|_{\partial\Omega} = f$$

f = voltage potential at boundary

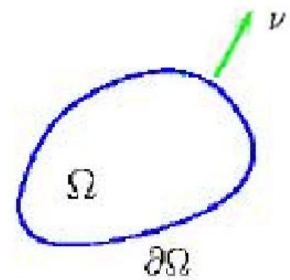
Isotropic

$$\gamma^{ij}(x) = \alpha(x)\delta^{ij}; \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Anisotropic Case

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ in } \Omega$$
$$u\Big|_{\partial\Omega} = f \qquad (*)$$

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}$$



$\nu = \left(\nu^1, \cdots, \nu^n\right)$ is the unit outer normal to $\partial\Omega$

$\Lambda_\gamma(f)$ is the induced current flux at $\partial\Omega$.

Λ_γ is the voltage to current map or Dirichlet - to - Neumann map

Anisotropic Case

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ in } \Omega$$
$$u\Big|_{\partial\Omega} = f \qquad (*)$$

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}$$

EIT: Can we recover γ in Ω from Λ_γ ?

Invariance

$$\text{div}(\gamma \nabla u) = 0 \qquad u\big|_{\partial\Omega} = f$$

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \gamma^{ij} \nu^i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}$$

$$\Lambda_\gamma \Rightarrow \gamma\ ?$$

Answer: No

$$\Lambda_{\psi_*\gamma} = \Lambda_\gamma$$

where $\psi : \Omega \to \Omega$ change of variables

$$\psi|_{\partial\Omega} = \text{Identity}$$

$$\psi_*\gamma = \left(\frac{(D\psi)^T \circ \gamma \circ D\psi}{|\det D\psi|} \right) \circ \psi^{-1}$$

$$v = u \circ \psi^{-1}$$

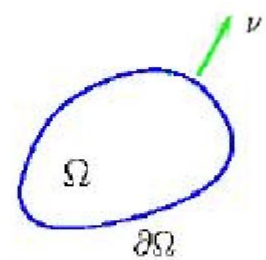
Geometric inverse problems (Lee-U, 1989)

(M,g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\Delta_g u = 0 \text{ on } M \qquad u\big|_{\partial M} = f$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

Geometric inverse problems

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

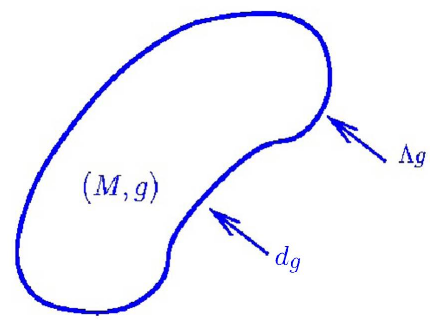
Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

Another Motivation (String Theory)

HOLOGRAPHY



Inverse problem: Can we recover (M, g) (bulk) from Dirichlet-to-Neumann map ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 01 (2004) 034

B. Czech, L. Lamprou, S. McCandlish and J. Sully, Integral geometry and holography, JHEP 10 (2015) 175

Anisotropic Case

Theorem ($n \geq 3$) (Lassas-U 2001, Lassas-Taylor-U 2003) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f, \ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma = \text{Identity}$, so that

$$\boxed{g_1 = \psi^* g_2}$$

In fact one can determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Anisotropic Case

Theorem (Guillarmou-Sa Barreto, 2009) $(M, g_i), i = 1, 2$, are compact Riemannian manifolds with boundary that are Einstein. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi|_{\partial M} = Identity$ such that

$$\boxed{g_1 = \psi^* g_2}$$

Note: Einstein manifolds with boundary are real analytic in the interior.

Anisotropic Case

Theorem ($n = 2$)(Lassas-U, 2001)

(M, g_i), $i = 1, 2$, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f,\ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma =$ Identity, and $\beta > 0, \beta\Big|_\Gamma = 1$ so that

$$\boxed{g_1 = \beta \psi^* g_2}$$

In fact, one can determine topology of M as well.

Moding Out the Diffeomorphism Group

Some conformal class $\Lambda_{\beta g} = \Lambda_g$, $\beta \in C^\infty(M)$

$$\Longrightarrow \beta = 1?$$

More general problem

$$\begin{array}{c}(\Delta_g - q)u = 0,\ q \in C^\infty(M)\\ u|_{\partial M} = f,\\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}.\end{array}$$

Inverse Problem: Does Λ_g determines q?

Moding Out the Diffeomorphism Group

$(\Delta_g - q)u = 0, \qquad \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}, \qquad \boxed{\Lambda_g \to q?}$

Theorem (n=2) (Guillarmou-Tzou, 2009) **YES**

Earlier results:

- $\mathbb{R}^2$, q small (Sylvester-U, 1986)
- $\mathbb{R}^2$, q generic (Sun-U, 2001)
- $\mathbb{R}^2$, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$ (Nachmann 1996)
- Riemannian surfaces, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$, (Henkin-Michel, 2008)
- $q \in L^\infty$, (Bukhgeim, 2008)

Moding Out the Diffeomorphism Group

$(n \geq 3)$

$$\begin{aligned}
(\Delta_g - q)u = 0,\ q \in C^\infty(M) \\
u|_{\partial M} = f, \\
\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}.
\end{aligned}$$

$$(*) \quad g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad c > 0.$$

Theorem (Dos Santos-Kenig-Salo-U) Assume that there is a global coordinate system so that (*) is true. In addition g_0 is simple. Then Λ_g determines uniquely q.

Simple: No conjugate points and strictly convex.

Moding Out the Diffeomorphism Group

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad x' \in \mathbb{R}^{n-1}$$

Examples

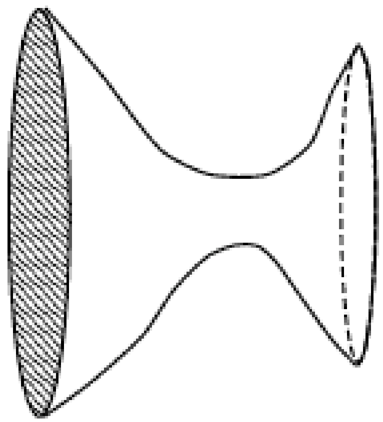
(a) $g(x)$ conformal to Euclidean metric (Sylvester-U, 1987)

(b) $g(x)$ conformal to hyperbolic metric (Isozaki, 2004)

(c) $g(x)$ conformal to metric on sphere (minus a point)

Non-uniqueness for EIT (Invisibility)

Motivation (Greenleaf-Lassas-U, MRL, 2003)



When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.