An introduction to the fractional Calderón problem

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Clemson, SC
Jun 3, 2024

Notations

- $n \geq 2$: space dimension
- $0 < s < 1$: fractional power
- Ω : a bounded domain with smooth boundary $\partial\Omega$
- $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$
- $\langle \cdot, \cdot \rangle$: standard L^2-distributional pairing
- $H^s(U)$: the Sobolev space $W^{s,2}(U)$

Fractional Calderón Problem

Definition of $(-\Delta)^s$ $(0 < s < 1)$ in $\mathbb{R}^n$:

- (Fourier transform definition)

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi))(x)$$

  where $\mathcal{F}$ denotes the Fourier transform.

- (Principal value definition)

$$(-\Delta)^s u(x) := c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

  where $B_\epsilon(x)$ denotes the open ball centered at x with radius ϵ.

Fractional Calderón Problem

- (Caffarelli-Silvestre extension definition)

$$((-\Delta)^s f)(x) := c_s \lim_{y \to 0^+} y^{1-2s} \partial_y u(x, y)$$

where u is the solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u(x, 0) = f(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

- Unique continuation property of $(-\Delta)^s$ has been proven based on CS definition and Carleman estimates (Rüland, 15).

(Ghosh–Salo–Uhlmann, 16)

Let $0 < s < 1$ and $u \in H^s(\mathbb{R}^n)$. Let W be nonempty and open. If

$$(-\Delta)^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ in $\mathbb{R}^n$.

Fractional Calderón Problem

Formulation of the fractional Calderón problem:

- We consider the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad u|_{\Omega_e} = g$$

  where $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$ and define the Dirichlet-to-Neumann map

$$\Lambda_q : g \to (-\Delta)^s u|_{\Omega_e}.$$

- Inverse problem: Can we determine q from Λ_q?

Fractional Calderón Problem

Fundamental uniqueness theorem:

(Ghosh–Salo–Uhlmann, 16)

Suppose $n \geq 2$. Let $0 \leq q_1, q_2 \in L^\infty(\Omega)$ and let $W_1, W_2 \subset \Omega_e$ be nonempty and open. If

$$\Lambda_{q_1} g|_{W_2} = \Lambda_{q_2} g|_{W_2}, \qquad g \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω.

Main ingredients of the proof:

- Integral identity for Dirichlet-to-Neumann maps
- Runge approximation property

The following RAP was proved based on UCP.

Runge approximation property (Ghosh–Salo–Uhlmann, 16)

Let $0 \leq q \in L^\infty(\Omega)$ and let $W \subset \Omega_e$ be open. Then

$$S := \{u_g|_\Omega : g \in C_c^\infty(W)\}$$

is dense in $L^2(\Omega)$. Here u_g is the solution of the exterior problem corresponding to the exterior data g.

Fractional Calderón Problem

Proof of the fundamental theorem:

- The assumption on DN maps can be interpreted as

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

  for any $g_j \in C_c^\infty(W_j)$ $(j = 1, 2)$ where u_j is the solution of

$$((-\Delta)^s + q_j)u = 0 \text{ in } \Omega, \qquad u|_{\Omega_e} = g_j.$$

- Given $f \in L^2(\Omega)$, by RAP we can choose $g_j \in C_c^\infty(W_j)$ $(j = 1, 2)$ s.t. $u_1 \to f$, $u_2 \to 1$ in $L^2(\Omega)$.
- Conclude that

$$\int_\Omega (q_1 - q_2) f = 0,$$

  $q_1 = q_2$ since f is arbitrary.

Fractional Calderón Problem

Variants of the fractional Calderón problem: Inverse problems for

- Variable coefficients fractional elliptic operators (Ghosh-Lin-Xiao, 17)
- Local perturbation of fractional Laplacian (Cekić-Lin-Rüland, 18; Covi-Mönkkönen-Railo-Uhlmann, 20)
- Nonlocal perturbation of fractional Laplacian (Bhattacharyya-Ghosh-Uhlmann, 20; Covi, 21)
- Coupled space-time fractional parabolic operator (Lai-Lin-Rüland, 20)
- Fractional magnetic operators (Covi, 19; L, 20; Lai-Zhou, 21)

Fractional Calderón Problem

Inverse problems for

- Fractional elasticity (L, 21; Covi-de Hoop-Salo, 22)
- Fractional porous medium equation (L, 21)
- Operators involving fractional gradients (Covi, 18; Lai-Ohm, 20; Railo-Zimmermann, 22)
- Fractional operators on closed manifolds (Feizmohammadi-Ghosh-Krupchyk-Uhlmann, 21; Quan-Uhlmann, 22; Chien, 22)