CBMS Lectures

Inverse Problems for Nonlinear Elliptic Equations

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Inverse Problems for Nonlinear Conductivity Equations

Sun, 1996 (linearization)
Sun–U, 1997: $\gamma(x, u)$ (anisotropic, 2nd-order linearization)
Hervas–Sun, 2002: $\gamma(x, \nabla u)$,
Muñoz–Uhlmann, 2020, Shankar, 2019: $\gamma(u, \nabla u)$.

Consider the boundary value problem

$$\begin{cases} \nabla \cdot (\gamma(x, u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We define the associated Dirichlet-to-Neumann map by

$$\Lambda_\gamma(f) = (\gamma(x, u)\partial_\nu u)|_{\partial\Omega}$$

where ν is the unit outer normal to $\partial\Omega$.

Theorem (Sun, 1996)

Let $n \geq 2$*. Assume* $\gamma_i, \in C^{1,1}(\bar{\Omega} \times [-T, T])\ \forall T > 0, i = 1, 2,$ *and* $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$*. Then* $\gamma_1(x,t) = \gamma_2(x,t)$ *on* $\bar{\Omega} \times \mathbb{R}$*.*

The following linearization proposition is the key for the proof.

Proposition

Let $\gamma(x,t)$ *be as in the theorem. Let* $1 < p < \infty, 0 < \alpha < 1$*.*
Denote $\gamma^t(x) = \gamma(x,t)$*.*
Then for any $f \in C^{2,\alpha}(\partial\Omega), 0 < \alpha < 1, t \in \mathbb{R}$

$$\lim_{s \to 0} \left\| \frac{1}{s} \Lambda_\gamma(t + sf) - \Lambda_{\gamma^t}(f) \right\|_{H^{1/2}(\partial\Omega)} = 0.$$

Non-linearity Helps!

Let us consider the Dirichlet problem for the following semilinear elliptic equation,

$$\begin{cases} -\Delta u + q(x)u^m = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega, \end{cases}$$

with $q \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, and $m \geq 2$. Let $\Gamma \subset \partial\Omega$ be arbitrary open non-empty. When $f \in C^{2,\alpha}(\partial\Omega)$ with $\|f\|_{C^{2,\alpha}(\partial\Omega)}$ sufficiently small, the problem has a unique small solution $u \in C^{2,\alpha}(\overline{\Omega})$. Define the partial Dirichlet–to–Neumann map $N_q^\Gamma f = \partial_\nu u|_\Gamma$, where supp $(f) \subset \Gamma$.

Theorem (Krupchyk–U; Lassas–Liimatainen–Lin–Salo, 2019)

Let $N_{q_1}^\Gamma f = N_{q_2}^\Gamma f$ for all $f \in C^{2,\alpha}(\partial\Omega)$ small with supp $(f) \subset \Gamma$. Then $q_1 = q_2$ in Ω.

Idea of the Proof

Assume that $m = 2$. A second order linearization of the partial Dirichlet–to–Neumann map leads to the following integral identity,

$$\int_\Omega (q_1 - q_2) v^{(1)} v^{(2)} v^{(3)} dx = 0$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma$, $l = 1, 2, 3$.

A result on the linearized partial data inverse problem by Dos Santos Ferreira–Kenig–Sjöstrand–U, 2009: the set of products of two harmonic functions in $C^\infty(\overline{\Omega})$ which vanish on $\partial\Omega \setminus \Gamma$ is dense in $L^1(\Omega)$. (the proof is highly non-trivial and is based on the FBI transform approach to analytic microlocal analysis).

We conclude that

$$(q_1 - q_2) v^{(3)} = 0 \quad \text{in} \quad \Omega.$$

Now taking $v^{(3)} \not\equiv 0$ harmonic, and using that the set $(v^{(3)})^{-1}(0)$ is of measure zero, we get $q_1 = q_2$ in Ω.

The Calderón problem with partial data

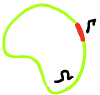
In practice impedance tomography measurements cannot be taken on the entire boundary due to limitations in resources or obstructions from natural obstacles.

This leads us to consider the Calderón problem with partial data.
Consider

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. The partial Dirichlet–to–Neumann map,

$$\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f) = (\gamma \partial_\nu u)|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



The Calderón problem with partial data: Does $\Lambda_\gamma^{\Gamma_1,\Gamma_2}$ determine γ in Ω? Open in general.

Known results in linear case:

- Bukhgeim–U, 2002:

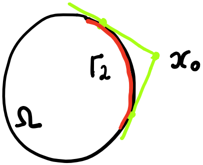
$$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \xi \cdot \nu(x) < \varepsilon\}, \quad \xi \in \mathbb{S}^{n-1}, \quad \varepsilon > 0.$$

  Note: Γ_2 is slightly more than a half of the boundary

- Kenig–Sjöstrand–U, 2007:

$$\Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|} \cdot \nu(x) < \varepsilon\}, \quad x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0,$$

Γ_1 = small neighborhood of complement of Γ_2.



Note: when Ω is strictly convex, Γ_2 could be arbitrarily small

Partial data inverse problems for semilinear conductivities

Consider the Dirichlet problem for the following isotropic semilinear conductivity equation,

$$\begin{cases} \nabla \cdot (\gamma(x,u)\nabla u) = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

Assume that the function $\gamma : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions,

(i) the map $\mathbb{C} \ni \tau \mapsto \gamma(\cdot, \tau)$ is holomorphic with values in the Hölder space $C^{1,\alpha}(\overline{\Omega})$ with some $0 < \alpha < 1$,

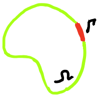
(ii) $\gamma(x, 0) = 1$, for all $x \in \Omega$.

There exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}$, the problem (1) has a unique solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|u\|_{C^{2,\alpha}(\overline{\Omega})} < C\delta$.

Let $\Gamma \subset \partial\Omega$ be an arbitrary open non-empty subset of the boundary $\partial\Omega$. We define the partial Dirichlet–to–Neumann map

$$\Lambda_\gamma^\Gamma(f) = (\gamma(x,u)\partial_\nu u)|_\Gamma,$$

where $f \in B_\delta(\partial\Omega)$ with supp $(f) \subset \Gamma$.



Theorem (Kian–Krupchyk–U, 2023)
If $\Lambda_{\gamma_1}^\Gamma = \Lambda_{\gamma_2}^\Gamma$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega} \times \mathbb{C}$.

Remark To the best of our knowledge, these results are the first partial data results for nonlinear conductivity equations.

Remark An analog of these partial data results is still not known in the case of the linear conductivity equation in dimensions $n \geq 3$. This is a major open problem in the field! Non-linearity helps!

Idea of the proof

First it follows from (i) and (ii) that γ can be expanded into the following power series,

$$\gamma(x,\lambda) = 1 + \sum_{k=1}^{\infty} \partial_\lambda^k \gamma(x,0) \frac{\lambda^k}{k!}, \quad \partial_\lambda^k \gamma(x,0) \in C^{1,\alpha}(\overline{\Omega}), \quad \lambda \in \mathbb{C},$$

converging in the $C^{1,\alpha}(\overline{\Omega})$ topology.
Using the mth order linearization, $m \geq 2$, we reduce the proof of

$$\partial_\lambda^{m-1} \gamma_1(x,0) = \partial_\lambda^{m-1} \gamma_2(x,0)$$

to the following density result:

Theorem (Kian–Krupchyk–U, 2023)

Let $m = 2, 3, \dots,$ be fixed and let $f \in L^\infty(\Omega)$ be such that

$$\int_\Omega f \Big(\sum_{k=1}^{m} \prod_{r=1, r\neq k}^{m} u_r \nabla u_k \Big) \cdot \nabla u_{m+1} dx = 0,$$

for all functions $u_l \in C^\infty(\overline{\Omega})$ harmonic in Ω with supp $(u_l|_{\partial\Omega}) \subset \Gamma$, $l = 1, \dots, m+1$. Then $f = 0$ in Ω.

The proof uses Analytic Microlocal Analysis, FBI transform techniques.

Partial data for semilinear elliptic PDE

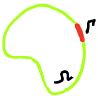
Consider next the following Dirichlet problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here $q \in C^{\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, $(\nabla u)^2 = \nabla u \cdot \nabla u$.

For any $f \in C^{2,\alpha}(\partial\Omega)$ small, there exists a unique small solution $u \in C^{2,\alpha}(\overline{\Omega})$. Define the partial Dirichlet–to–Neumann map,

$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



Theorem (Krupchyk–U, 2020)
$\Lambda_{q_1}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{\Gamma_1,\Gamma_2} \Longrightarrow q_1 = q_2$ *in* Ω.

Remark. Slightly more general nonlinearities can also be treated.

Idea of the proof

Performing a second order linearization, we get

$$\int_\Omega (q_1 - q_2)(\nabla v^{(1)} \cdot \nabla v^{(2)}) v^{(3)} dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, 2, 3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Our inverse theorem follows therefore from the following density result.

Theorem (Krupchyk–U, 2020)
Span$\{\nabla v^{(1)} \cdot \nabla v^{(2)} : v^{(l)} \in C^\infty(\overline{\Omega})$ *harmonic*, $v^{(l)}|_{\partial\Omega\setminus\Gamma_1} = 0, l = 1, 2\}$ *is dense in* $L^1(\Omega)$.

The proof uses Analytic Microlocal Analysis, FBI transform techniques.

Partial Data Quasilinear Elliptic Equation

Theorem (Kian–Krupchyk–U, 2020) Let γ_1, γ_2 satisfy the conditions above.

$$\Lambda^{\Gamma}_{\gamma_1}(\lambda + f) = \Lambda^{\Gamma}_{\gamma_2}(\lambda + f)$$

for all f sufficiently small with $supp\, f \subset \Gamma$ and all $\lambda \in \Sigma \subset \mathbb{C}$ that has a limit point in $\mathbb{C}$. Then $\gamma_1 = \gamma_2$.

Remark. We also established similar partial data results for the semi-linear conductivity equation $\nabla \cdot (\gamma(x, u)\nabla u) = 0$.

Remark. An analog of these partial data results is still not known in the case of the linear conductivity equation in dimensions $n \geq 3$. This is a major open problem in the field! Non-linearity helps!

Completeness

The proof of the theorem uses the ideas of the following lemma:

Lemma (Krupchyk–U, 2019) Let $h \in L^1(\Omega)$ and

$$\int_\Omega h \nabla u \cdot \nabla v = 0$$

for all u, v satisfying $\Delta u = \Delta v = 0$ on Ω and $supp\, u.v|_{\partial\Omega} \subset \Gamma$. Then $h = 0$.

Quasilinear Conductivities

$$\begin{cases} div(\gamma(x,u,\nabla u)\nabla u) = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases}$$

$0 < \gamma(x,0,0) \in C^\infty(\bar{\Omega})$ and the map $(\tau, z) \to \gamma(\cdot, \tau, z)$ is holomorphic with values in $C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$.

$$\Lambda_\gamma(f) = \gamma(x,u,\nabla u)\frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

where ν is the unit outer normal.
Theorem (Carstea–Feizmohammadi–Kian–Krupchyk–U, 2021) Let $n \geq 3$, under the assumptions above, if

$$\Lambda_{\gamma_1} f = \Lambda_{\gamma_2} f$$

for all f sufficiently small. Then $\gamma_1 = \gamma_2$.

Idea of the proof

Letting $\lambda = (\rho, \mu) \in \mathbb{C} \times \mathbb{C}^n$, we write by Taylor's formula,

$$\gamma_j(x, \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \gamma_j^{(k)}(x, 0; \underbrace{\lambda, \dots, \lambda}_{k \text{ times}}), \quad x \in \Omega, \quad j = 1, 2.$$

Here $\gamma_j^{(k)}(x, 0)$ is the kth differential of the holomorphic function $\lambda \mapsto \gamma_j(x, \lambda)$ at $\lambda = 0$, which is a symmetric tensor of rank k, given by

$$\gamma_j^{(k)}(x, 0; \lambda, \dots, \lambda) = \sum_{j_1, \dots, j_k = 0}^{n} (\partial_{\lambda_{j_1}} \dots \partial_{\lambda_{j_k}} \gamma_j)(x, 0) \lambda_{j_1} \dots \lambda_{j_k}, \quad x \in \Omega.$$

It suffices to prove that

$$\gamma_1^{(k)}(\cdot, 0) = \gamma_2^{(k)}(\cdot, 0) \quad \text{in} \quad \Omega.$$

Sketch of Proof

When $m = 1$, the result follows from a polarization trick and the fact that

$$\text{span}\{\gamma_0 \nabla v_1 \cdot \nabla v_2 : v_j \in C^\infty(\overline{\Omega}), \nabla \cdot (\gamma_0 \nabla v_j) = 0, j = 1, 2\}$$

is dense in $L^2(\Omega)$. (Sylvester–U, 1987, Krupchyk–Lassas–Siltanen, 2011)

When $m \geq 2$, we observe that there are at least four solutions in our integral identity and we use crucially this observation.

To explain the idea, let $m = 2$. We need to show that if

$$\sum_{(l_1, l_2, l_3) \in \pi(3)} \sum_{j,k=1}^{n} \int_\Omega T^{jk}(x) \partial_{x_j} u_{l_1} \partial_{x_k} u_{l_2} \nabla u_{l_3} \cdot \nabla u_4 \, dx = 0,$$

for all $u_l \in C^\infty(\overline{\Omega})$, $l = 1, \dots, 4$ solving the conductivity equation

$$\nabla \cdot (\gamma_0 \nabla u_l) = 0 \quad \text{in} \quad \Omega,$$

then $T = 0$ in Ω.

Quasilinear anisotropic conductivities in dimension 2

Let $A(x, \rho, \mu) = (a_{jl}(x, \rho, \mu))_{2\times 2}$ be an anisotropic quasilinear conductivitiy. Consider the boundary value problem

$$\begin{cases} \nabla \cdot (A(x, u, \nabla u)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Theorem (Liimatainen-Wu, 2024)

Let $n = 2$, A_1 and A_2 be quasilinear anisotropic conductivities such that $\Lambda_{A_1}(f) = \Lambda_{A_2}(f)$, for all f in $C^{2,\alpha}(\partial\Omega)$ small, then there exists a $W^{1,2}$ diffeomorphism Φ which is the identity map on the boundary such that $A_2 = H_\Phi(A_1)$ where

$$(H_\Phi A)(x, t) = \frac{(D\Phi(x))^T A(x, t)(D\Phi(x))}{|D\Phi|} \circ \Phi^{-1}(x)$$

.

Idea of the proof

First reduce the problem to isotropic case using isothermal coordinates, and consider isotropic quasilinear conductivity $\gamma(x, u, \nabla u)$;

then by higher order linearization, reduce the proof of

$$\partial_\lambda^{m-1}\gamma_1(x,0) = \partial_\lambda^{m-1}\gamma_2(x,0), \quad \lambda = (\rho,\mu) \in \mathbb{C}\times\mathbb{C}^n$$

to the following density result, which is the same as in the isotropic problem in dimension $n \geq 3$:

Proposition

Let $m \in \mathbb{N}$, if T is a continuous function on $\bar{\Omega}$ with values in the space of symmetric tensors of rank m such that

$$\sum_{(l_1,\ldots,l_{m+1})\in\pi(m+1)} \sum_{j_1,\ldots,j_m=0}^{2} \int_\Omega T^{j_1\cdots j_m}(x)\,(u_{l_1},\nabla u_{l_1})_{j_1} \ldots (u_{l_m},\nabla u_{l_m})_{j_m} \nabla u_{l_{m+1}} \cdot \nabla u_{m+2}\, dx = 0$$

for all $u_l \in C^\infty(\bar{\Omega})$ solving $\nabla\cdot(\gamma_0\nabla u_l) = 0$ in Ω, $l = 1,\ldots,m+2$. Then $T = 0$ in Ω. Here $(u_l,\nabla u_l)_j$, $j = 0,1,2$ stands for the jth component of the vector $(u_l, \partial_{x_1} u_l, \ldots, \partial_{x_n} u_l)$, and $\gamma_0 = \gamma(x,0)$.

Remark. Let $T^{j_1\cdots j_m} = \partial_{\lambda_1}\ldots\partial_{\lambda_m}\gamma$, the theorem follows from the proposition.

Idea of the proof

To prove the main proposition, we first do boundary determination to show T vanish up to infinite order on the boundary, using singular solutions with prescribed singularity close to the boundary outside Ω:

$$u(x) = \log|x - z_\sigma| + w(x - z_\sigma)$$

where $z_\sigma = x_0 + \sigma\nu$ for some small $\sigma > 0$.

Next, we use **Bukhgeim's CGO solutions** (phases having nondegenerate critical points), as well as **limiting Carleman weights** (phases having no critical point) in the integral identity. Then we apply the stationary phase method.

An example of Bukhgeim's CGO solutions:

$$u = \frac{1}{\sqrt{\gamma}} e^{\pm z^2/h}(1 + r_h), \quad ||r_h||_{L^2} = O(h^{\frac{1}{2}+\epsilon})$$

An example of limiting Carleman weights defined by Kenig-Sjöstrand-U:

$$\tilde{u} = \frac{1}{\sqrt{\gamma}} e^{\pm(z^2+z)/h}(1 + \tilde{r}_h), \quad ||\tilde{r}_h||_{L^2} = O(h)$$

Idea of the proof

When $m = 1$, The integral identity in the proposition becomes

$$0 = \sum_{(l_1,l_2)\in\pi(2)} \sum_{j=0}^{n} \int_\Omega T^j(x) \left(u_{l_1}, \nabla u_{l_1}\right)_j \nabla u_{l_2} \cdot \nabla u_3 dx$$

Let $u_2 = 1$, we get

$$\int_\Omega T^0(x) \nabla u_1 \cdot \nabla u_2 dx = 0$$

for any u_1, u_2 solving $\nabla \cdot (\gamma_0 \nabla u) = 0$ in Ω. Integrate by parts, we could obtain

$$\int uv \nabla \cdot \left(\gamma_0 \nabla \left(\frac{T^0}{\gamma_0}\right)\right) = 0$$

Denote $A := \nabla \cdot \left(\gamma_0 \nabla \left(\frac{T^0}{\gamma_0}\right)\right)$. Substituting Bukhgeim's CGO solutions:

$$u = \frac{1}{\sqrt{\gamma}} e^{z^2/h}(1 + r_h), \quad v = \frac{1}{\sqrt{\gamma}} e^{-\bar{z}^2/h}(1 + \tilde{r}_h)$$

We have

$$\int e^{(z^2 - \bar{z}^2)/h} \frac{A}{\gamma_0}(1 + r_h + \tilde{r}_h + r_h \tilde{r}_h) = 0.$$

Idea of the proof

Recall the **stationary phase** method: for any $u(x) \in C_0^\infty(\mathbb{R}^2)$, we have the following asymptotic expansion for the oscillatory integral

$$\frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{i(z^2-\bar{z}^2)/4h}u(z)dz = \sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\left(\frac{1}{i}(\bar{\partial}^2-\partial^2)\right)^k u\right)(0,0)+S_N(u,h)$$

where
$$|S_N(u,\tau)| \leq \frac{Ch^N}{N!}\sum_{|\alpha+\beta|\leq 3}\left\|\partial_x^\alpha\partial_y^\beta\left(\partial_x\partial_y\right)^N u\right\|_{L^1}$$

Therefore, for the integral identity

$$\int e^{(z^2-\bar{z}^2)/h}\frac{A}{\gamma_0}(1+r_h+\tilde{r}_h+r_h\tilde{r}_h)=0,$$

we have that the principal term is of order h, while the other terms can be shown to have higher order in h using the estimates we have for $r_h, \tilde{r}_h$. Since the coefficient in the principal term is A/γ_0, by letting $h\to 0$, we conclude that $A=0$, which implies $T^0=0$.
For the other terms, in a similar way we can show that $T^1=T^2=0$.
The proof for $m\geq 2$ follows a similar way.