CBMS Lectures

Inverse Problems for Nonlinear Hyperbolic Equations

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University of Washington

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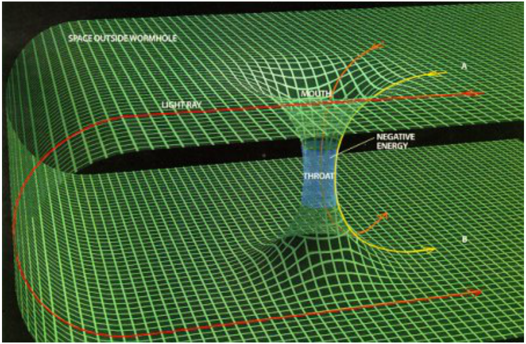
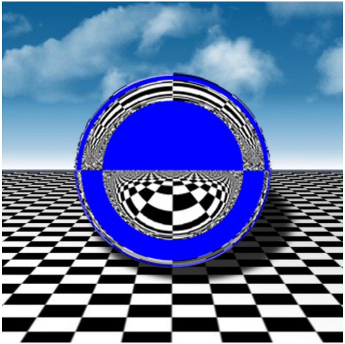
Goal: To Determine the Topology and Metric of Space-Time





How can we determine the topology and metric of complicated structures in space-time with a radar-like device?

Figures: Anderson institute and Greenleaf-Kurylev-Lassas-U.

Non-linearity Helps

We will consider inverse problems for non-linear wave equations, e.g.

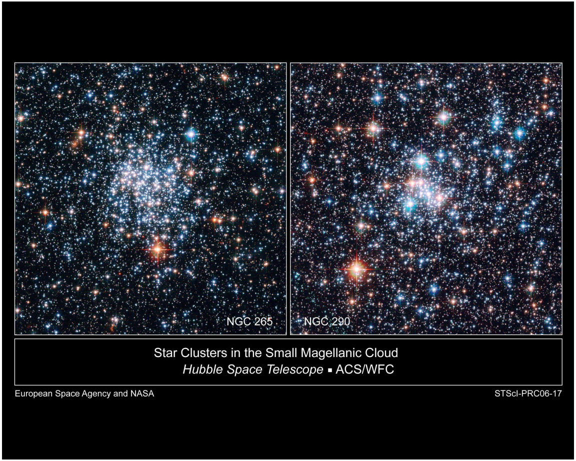
$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^2 = f(t,y).$$

We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

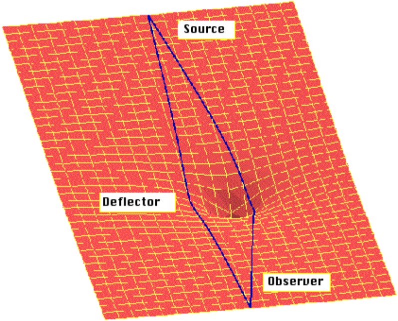
Inverse Problems in Space-Time: Passive Measurements

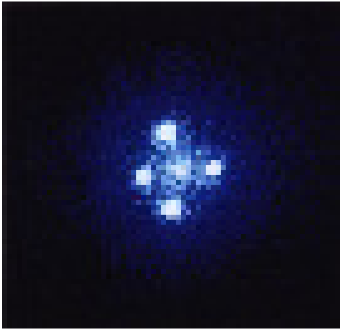


Can we determine the structure of space-time when we see light coming from many point sources varying in time? We can also observe gravitational waves.

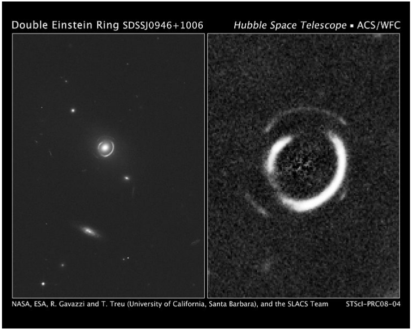
Gravitational Lensing

We consider e.g. light or X-ray observations or measurements of gravitational waves.

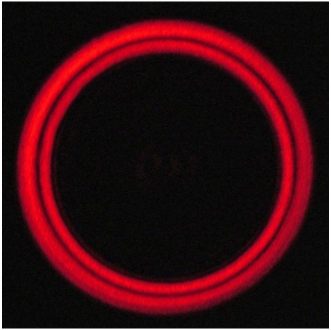




Gravitational Lensing



Double Einstein Ring

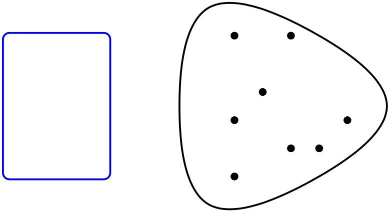


Conical Refraction

Passive Measurements: Gravitational Waves

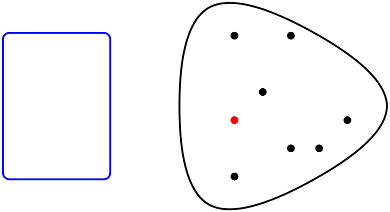
NSF Announcement, Feb 11, 2015

Inverse Problem for Passive Measurements



Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



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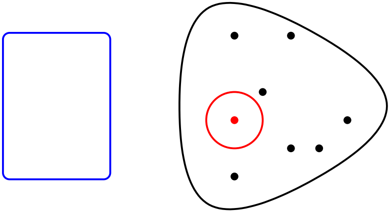

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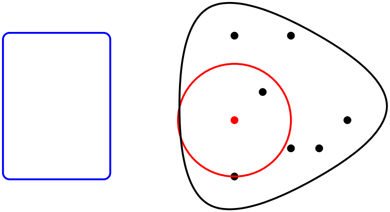

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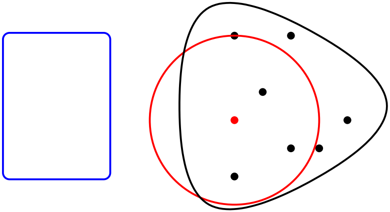

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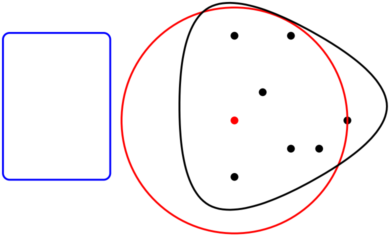

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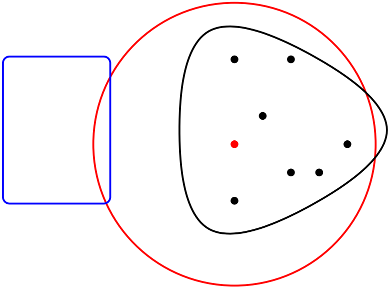

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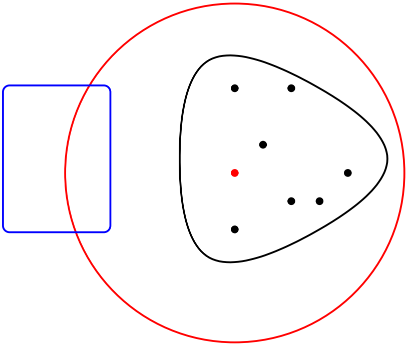

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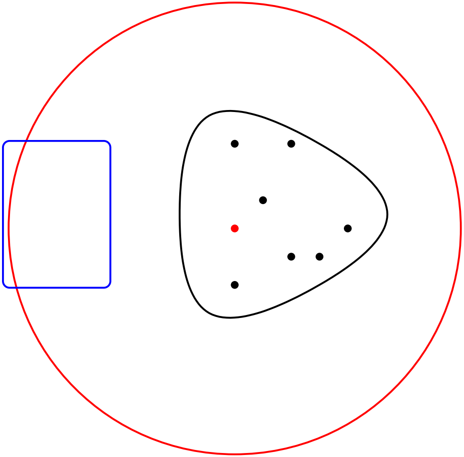

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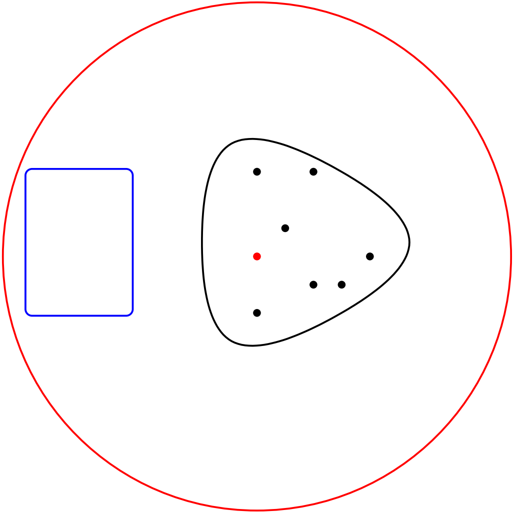

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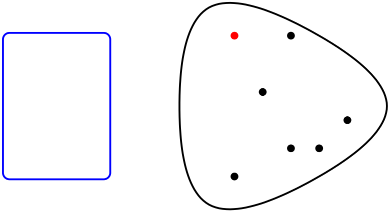

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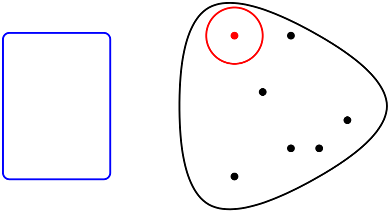

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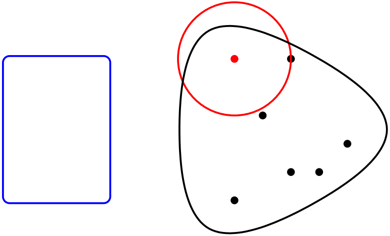

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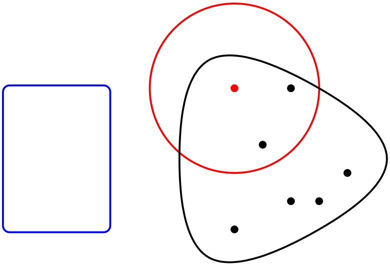

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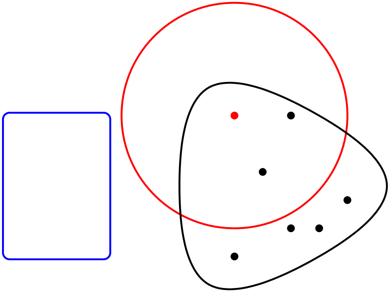

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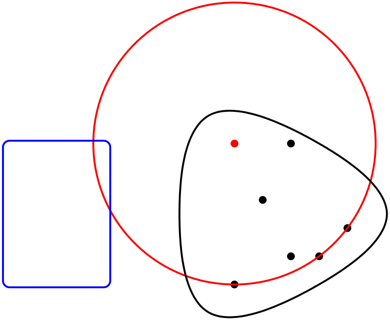

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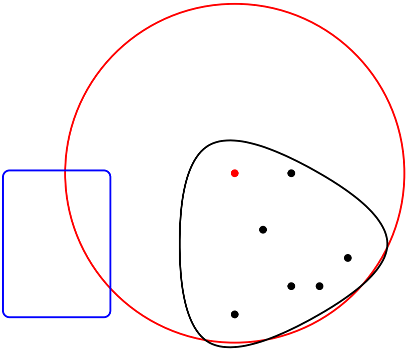

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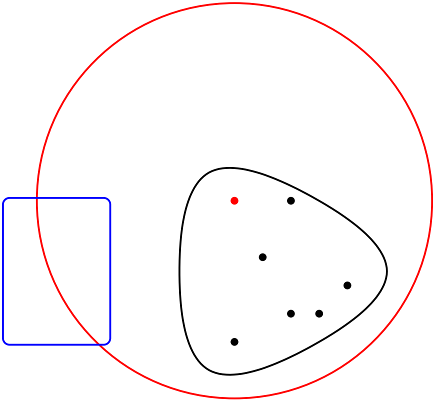


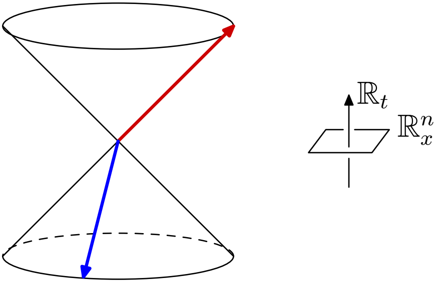
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Lorentzian Geometry

$(n+1)$-dimensional Minkowski space: (M, g)

$$M = \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x, \quad \text{metric: } g = -dt^2 + dx^2.$$

Null/lightlike vectors: $V \in T_q M$ with $g(V, V) = 0$.



$L^{\pm}_q M$: future/past null vectors

Lorentzian Geometry

In general:

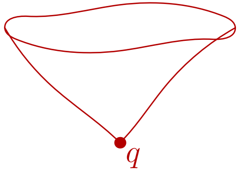
$M = (n+1)$-dimensional manifold, g Lorentzian $(-,+,\dots,+)$.

Assume: existence of time orientation.

$$T_q M \cong (\mathbb{R}^{1+n}, \text{Minkowski metric}).$$

Null-geodesics: $\gamma(s) = \exp_q(sV)$, $V \in T_q M$ null.

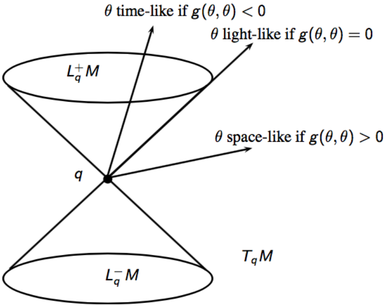
Future light cone: $\mathcal{L}_q^+ = \{\exp_q(V) \colon V \text{ future null}\}$



Lorentzian Manifolds

Let (M, g) be a $1+3$ dimensional time oriented Lorentzian manifold.
The signature of g is $(-,+,+,+)$.
Example: Minkowski space-time $(\mathbb{R}^4, g_m)$, $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.

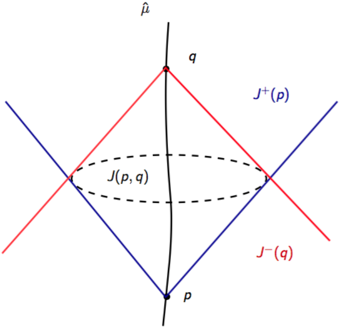
- $L_q^{\pm} M$ is the set of future (past) pointing light like vectors at q.
- Casual vectors are the collection of time-like and light-like vectors.
- A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).



Causal Relations

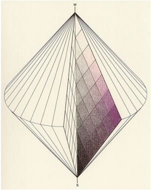
Let $\widehat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.

- The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : q < p\}$.
- $J(p,q) = J^+(p) \cap J^-(q)$, $I(p,q) = I^+(p) \cap I^-(q)$.



Global Hyperbolicity

A Lorentzian manifold (M, g) is globally hyperbolic if



- there is no closed causal paths in M;
- for any $p, q \in M$ and $p < q$, the set $J(p, q)$ is compact.

Then hyperbolic equations are well-posed on (M, g)
Also, (M, g) is isometric to the product manifold

$$\mathbb{R} \times N \text{ with } g = -\beta(t, y)dt^2 + \kappa(t, y).$$

Here $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t.
We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M.

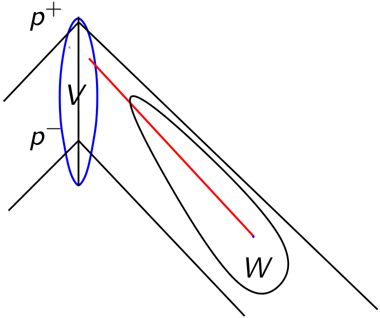
Light Observation Set

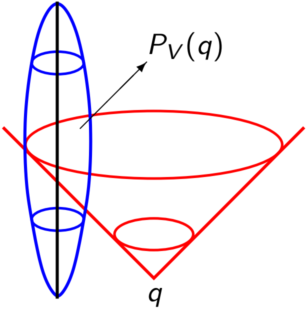
Let $\mu = \mu([-1,1]) \subset M$ be time-like geodesics containing p^- and p^+. We consider observations in a neighborhood $V \subset M$ of μ.

Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

The light observation set for $q \in W$ is

$$P_V(q) := \{\gamma_{q,\xi}(r) \in V;\ r \geq 0,\ \xi \in L_q^+ M\}.$$





Inverse Problems with Passive Measurements

The **earliest light observation set** of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there is no } y \in \mathcal{P}_V(q) \text{ and future pointing time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts (Engelhardt-Horowitz, arXiv 2016)

Theorem (Kurylev-Lassas-U 2018, arXiv 2014)

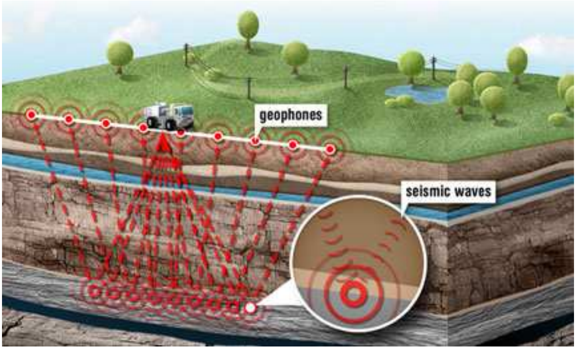
Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\widehat{\mu}([-1,1]) \subset M, p^\pm = \widehat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\widehat{\mu}([-1,1])$ and $W \subset M$ be a relatively compact set. Assume that we know

$$\mathcal{E}_V(W).$$

Then we can determine the topological structure, the differential structure, and the conformal structure of W, up to diffeomorphism.

Inverse Problems for Linear Hyperbolic Equations

- Rakesh-Symes 1987: Inverse problem for $\partial_t^2 - \Delta + q$.
- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with time-independent metric.
- Unique continuation needed for Belishev-Kurylev-Tataru results fail for time-depending wave speed.



Active Measurements

Wave equation: Let $g = [g_{jk}(y)]_{j,k=1}^n$ and $u = u^f(y,t)$ be the solution of

$$(\partial_t^2 u - \Delta_g)u = 0 \quad \text{on } N \times \mathbb{R}_+,$$
$$u|_{\partial N \times \mathbb{R}_+} = f,$$
$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0.$$

Here N is a Riemannian manifold, ν is the unit normal of ∂N,

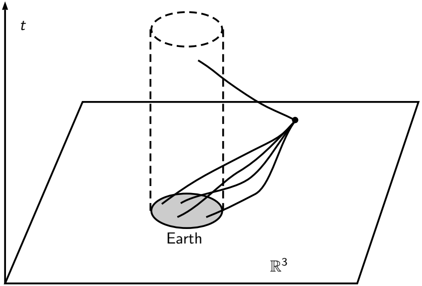
$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial y^j}(|g|^{1/2} g^{jk} \frac{\partial}{\partial y^k} u),$$

where $|g| = \det(g_{ij})$ and $[g_{ij}] = [g^{jk}]^{-1}$. Let

$$\Lambda f = \partial_\nu u^f|_{\partial N \times \mathbb{R}_+}.$$

We are given boundary data $(\partial N, \Lambda)$.

Interaction of Nonlinear Waves



Inverse Problem for a Non-linear Wave Equation

Consider the non-linear wave equation

$$\square_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M^0 = (-\infty, T) \times N,$$
$$\text{supp }(u) \subset J_g^+(\text{supp }(f)),$$

where $\text{supp}(f) \subset V$, $V \subset M$ is open,

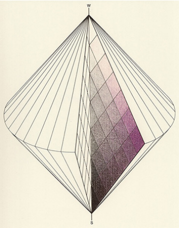
$$\square_g u = -\sum_{p,q=1}^{4} (-\det(g(x)))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g(x)))^{1/2} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

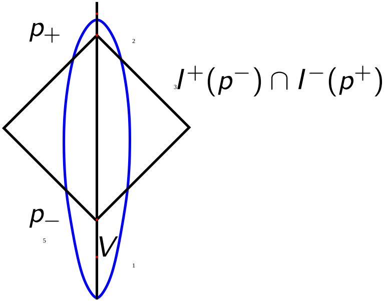
$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(V)$ is a controllable source, and $a(x)$ is a non-vanishing C^∞-smooth function.
In a neighborhood $\mathcal{W} \subset C_0^2(V)$ of the zero-function, define the measurement operator by

$$L_V : f \mapsto u|_V, \quad f \in C_0^6(V).$$

Theorem (Kurylev-Lassas-U, 2018)

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1+3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a : M \to \mathbb{R}$ be a non-vanishing function. Then $(V, g|_V)$ and the measurement operator L_V determines the set $I^+(p^-) \cap I^-(p^+) \subset M$ and the conformal class of the metric g, up to a change of coordinates, in $I^+(p^-) \cap I^-(p^+)$.

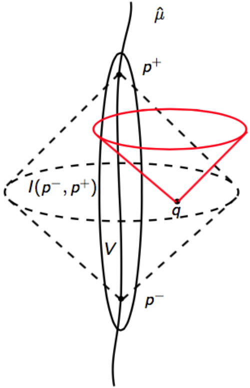




Idea of the Proof in the Case of Quadratic Nonlinearity: Interaction of Singularities

We construct the earliest light observation set by producing artificial point sources in $I(p_-, p_+)$. The key is the singularities generated from nonlinear interaction of linear waves.

- We construct sources f so that the solution u has new singularities.
- We characterize the type of the singularities.
- We determine the order of the singularities and find the principal symbols.



Non-linear Geometrical Optics

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\begin{aligned} \square_g u + au^2 &= f, \quad \text{in } M^0 = (-\infty, T) \times N, \\ u\big|_{(-\infty,0)\times N} &= 0 \end{aligned}$$

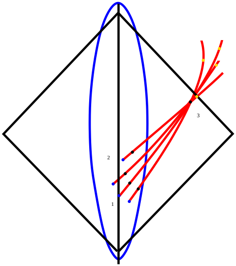
with $f = \varepsilon f_1$. When $Q = \square_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\ \|E_\varepsilon\| &\leq C\varepsilon^5. \end{aligned}$$

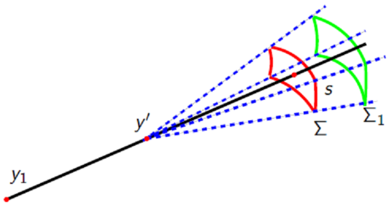
Non-linear Geometrical Optics

The product has, in a suitable microlocal sense, a principal symbol.

There is a lot of technology availale for the interaction analysis of conormal waves: intersecting pairs of conormal distributions (Melrose-U, 1979, Guillemin-U, 1981, Greenleaf-U, 1991).



Pieces of spherical waves



Consider solutions of $\square_g u_1 = f_1$, where f_1 is a conormal distribution that is singular on $\{t_0\} \times \Sigma$. The solution u_1 is a distribution associated to two intersecting Lagrangian manifolds. We can control the width s of the waves.

Lagrangian Manifolds Intersecting

From $\Box_g u_1 = f_1$ we have

$$u_1 = \Box_g^{-1} f_1.$$

Thus,

$$\mathrm{WF} u_1 \subset \mathrm{WF} f_1 \cup \Lambda_p(\mathrm{WF} f_1)$$

where

$$\Lambda_p(\mathrm{WF} f_1) = \text{forward flow out by } H_p \text{ starting at WF} f_1 \text{ intersected with } \{p = 0\}.$$

Here $p = \tau^2 - \sum g^{ij}(y)\xi_i\xi_j$.

H_p is the Hamiltonian vector field.

Notice that $\{p = 0\}$ is the light cone.

Interaction of Waves in Minkowski Space $\mathbb{R}^4$

Let x^j, $j = 1, 2, 3, 4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

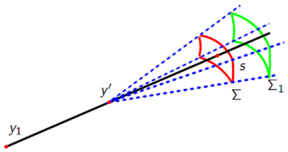
$$\begin{aligned} u_j(x) &= v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R}, j = 1, 2, 3, 4. \\ x^j &= t - x \cdot \omega_j, \quad |\omega_j| = 1 \end{aligned}$$

Waves u_j are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j = \{x^j = 0\}, \quad j = 1, 2, 3, 4.$$

The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of Two Waves (Second order linearization)

If we consider sources $f_{\vec\varepsilon}(x) = \varepsilon_1 f_{(1)}(x) + \varepsilon_2 f_{(2)}(x)$, $\vec\varepsilon = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec\varepsilon}$, we have

$$\begin{aligned} W_2(x) &= \frac{\partial}{\partial\varepsilon_1}\frac{\partial}{\partial\varepsilon_2} u_{\vec\varepsilon}(x)|_{\vec\varepsilon=0} \\ &= Q(a\, u_{(1)} \cdot u_{(2)}), \end{aligned}$$

where $Q = \Box_g^{-1}$ and

$$u_{(j)} = Qf_{(j)}.$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since the normal bundle N^*K_{12} contain only light-like directions $N^*K_1 \cup N^*K_2$,

$$\mathrm{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no new interesting singularities are produced by the interaction of two waves (Greenleaf-U, 1991).

Three plane waves interact and produce a conic wave. (Bony, 1986, Melrose-Ritter, 1987, Rauch-Reed, 1982)

Interaction of Three Waves (Third order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \\ &= \quad 4Q(a\, u_{(1)}\ Q(a\, u_{(2)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(2)}\ Q(a\, u_{(1)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(3)}\ Q(a\, u_{(1)}\ u_{(2)})), \end{aligned}$$

where $Q = \square_g^{-1}$. The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_3$.
The normal bundle N^*K_{123} contains light-like directions that are not in $N^*K_1 \cup N^*K_2 \cup N^*K_3$ and hence new singularities are produced.

Interaction of Four Waves (Fourth order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have following. Consider

$$W_4 = \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$ we have $N^*K_{1234} = T_q^*M$. Hence new singularities are produced and

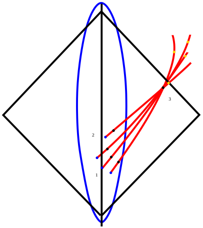
$$\mathrm{singsupp}(W_4) \subset (\cup_{j=1}^{4} K_j) \cup \Sigma \cup \mathcal{L}_q^+ M,$$

where Σ is the union of conic waves produced by sources on K_{123}, K_{134}, K_{124}, and K_{234}. Moreover, $\mathcal{L}_q^+ M$ is the union of future going light-like geodesics starting from the point q.

Interaction of Four Waves

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



Active and Passive Measurements

(M, g) $(2+1)$-dimensional, $\Box_g u = u^3 + f$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \Box_g^{-1} f_i.$$

Take $f_i =$ conormal distribution, e.g.

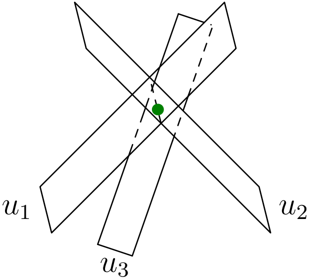
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

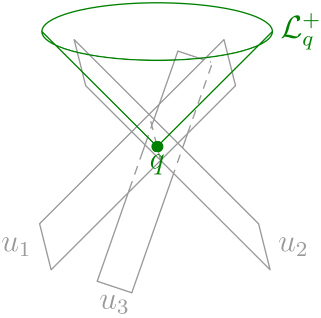
Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \Box_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$





$$q = \bigcap_{i=1}^{3} \mathit{sing\ supp}\ u_i, \quad \mathcal{L}_q^+ = \mathit{sing\ supp}\ \Box_g^{-1}(u_1 u_2 u_3)$$

$\Rightarrow$ singularities of $\partial^3_{\epsilon_1\epsilon_2\epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

Further Developments

1. Einstein's equations coupled with scalar fields (Kurylev-Lassas-U, 2013; Kurylev-Lassas-Oksanen-U, 2022)

2. Einstein-Maxwell's equations in vacuum (Lassas-U-Wang, 2017)

3. Einstein's equations (U-Wang, 2020)

4. Non-linear elasticity (de Hoop-U-Wang, 2020; U-Zhai, 2021)

5. Yang-Mills (Chen-Lassas-Oksanen-Paternain, 2021, 2022)

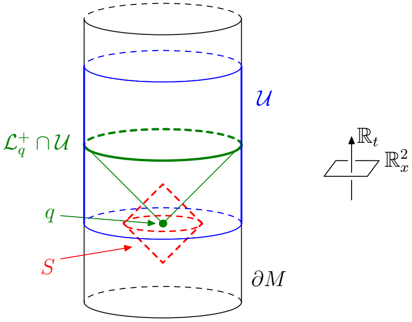
6. Inverse Scattering (Sa Barreto-U-Wang, 2022)

7. Semilinear equations (Kurylev-Lassas-U, 2018; Wang-U, 2018; Wang-Zhou, 2019; Hintz-U-Zhai, 2022; Stefanov-Sa Barreto, 2021; U-Zhang 2021; Hintz-U-Zhai, 2022)

8. Non-linear Acoustics (Acosta-U-Zhai, 2023; U-Zhang, 2023)

Boundary Light Observation Set

$M = \{(t, x) \colon |x| < 1\} \subset \mathbb{R}^{1+2}.$



Set of sources $S \subset M^\circ$.

Observations in $\mathcal{U} \subset \partial M$.

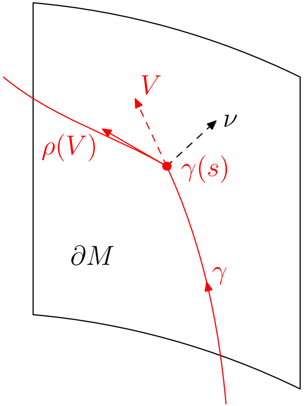
Data: $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} \colon q \in S\}$

Theorem

The collection $\mathscr{S}$ determines the topological, differentiable, and conformal structure $[g|_S] = \{fg|_S \colon f > 0\}$ of S.

Reflection at the Boundary

γ null-geodesic until $\gamma(s) \in \partial M$.



$\rho(V)$ = reflection of V across ∂M. (Snell's law.)

$\rightarrow$ continuation of γ as broken null-geodesic

Null-convexity

Simplest case:

$$\text{All null-geodesics starting in } M^\circ \text{ hit } \partial M \text{ transversally.} \qquad (1)$$

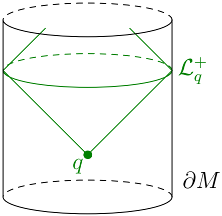
Proposition

(1) *is equivalent to null-convexity of* ∂M*:*

$$II(W, W) = g(\nabla_W \nu, W) \geq 0, \quad W \in T\partial M \text{ null.}$$

Stronger notion: strict null-convexity. ($II(W, W) > 0$, $W \neq 0$.)

Define light cones $\mathcal{L}_q^+$ using broken null-geodesics.



Main Result

Setup:

- (M,g) Lorentzian, dim ≥ 2, strictly null-convex boundary
- existence of $t\colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

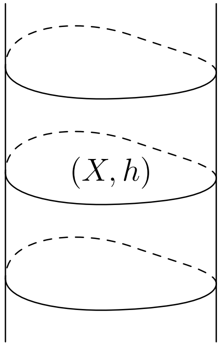
1. $\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ *and the unlabelled collection* $\mathscr{S} = \{\mathcal{L}^+_q \cap \mathcal{U} \colon q \in S\} \subset 2^{\mathcal{U}}$ *uniquely determine* $(S, [g|_S])$ *(topologically, differentiably, and conformally).*

Example for (M, g)

(X, h) compact Riemannian manifold with boundary.

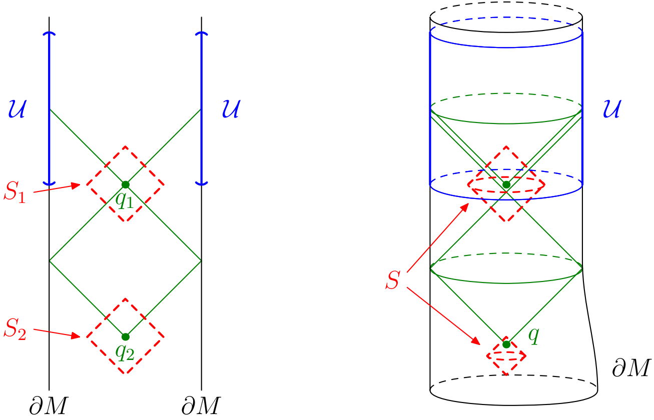


$$M = \mathbb{R}_t \times X, \quad g = -dt^2 + h.$$

(Strict) null-convexity of $\partial M \iff$ (strict) convexity of ∂X

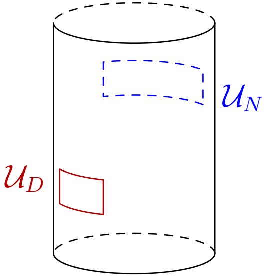
'Counterexamples'

Necessity of assumption 1. ($\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$)



S_1 and $S_1 \cup S_2$ are indistinguishable from $\mathcal{U}$.

Active Measurements for Boundary Value Problems



(Special case: $\mathcal{U}_N = \mathcal{U}_D$.)

Propagation of singularities: (strict) null-convexity assumption simplifies structure of null-geodesic flow. (Melrose 1975, Taylor 1975, Melrose–Sjöstrand 1978.)

Inverse Boundary Value Problem

Assume $M = \mathbb{R} \times N$ is a Lorentzian manifold of dimension $(1+3)$ with time-like boundary.

$$\begin{aligned} \Box_g u(x) + a(x)u(x)^4 &= 0, && \text{on } M, \\ u(x) &= f(x), && \text{on } \partial M, \\ u(t,y) &= 0, && t < 0, \end{aligned}$$

Inverse Problem: determine the metric g and the coefficient a from the Dirichlet-to-Neumann map.

The Main Result

Theorem (Hintz-U-Zhai, 2022)

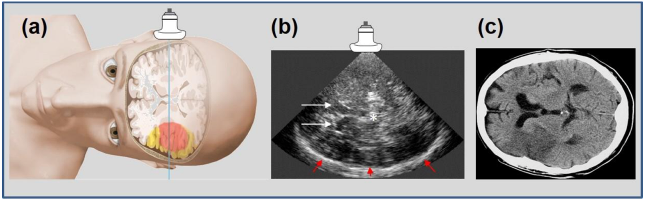
Consider the semilinear wave equations

$$\square_{g^{(j)}} u(x) + a^{(j)} u(x)^4 = 0, \qquad j = 1, 2,$$

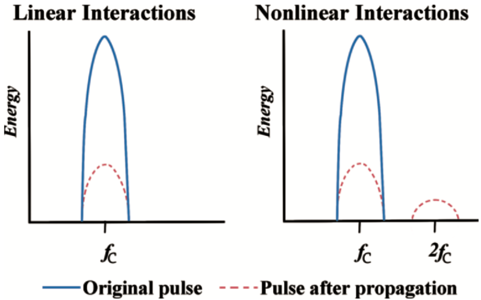
on Lorentzian manifold $M^{(j)}$ *with the same boundary* $\mathbb{R} \times \partial N$. *If the Dirichlet-to-Neumann maps* $\Lambda^{(j)}$ *acting on* $\mathcal{C}^5([0, T] \times \partial N)$ *are equal,* $\Lambda^{(1)} = \Lambda^{(2)}$, *then there exist a diffeomorphism* $\Psi : U_{g^{(1)}} \to U_{g^{(2)}}$ *with* $\Psi|_{(0,T)\times\partial N} = \mathcal{I}d$ *and a smooth function* $\beta \in C^\infty(M^{(1)})$, $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$, *so that, in* $U_{g^{(1)}}$,

$$\Psi^* g^{(2)} = e^{-2\beta} g^{(1)}, \quad \Psi^* a^{(2)} = e^{-\beta} a^{(1)}, \quad \square_g e^{-\beta} = 0.$$

Ultrasound Imaging



Nonlinear interaction: waves at frequency f_C generate waves at frequency $2f_C$:



Inverse Boundary Value Problem

The acoustic waves are modeled by the Westervelt-type equation

$$\begin{aligned} \frac{1}{c^2(x)}\partial_t^2 p(t,x) - \beta(x)\partial_t^2 p^2(t,x) &= \Delta p(t,x), \quad \text{in } (0,T)\times\Omega, \\ p(t,x) &= f, \quad \text{on } (0,T)\times\partial\Omega, \\ p = \frac{\partial p}{\partial t} &= 0, \quad \text{on } \{t=0\}, \end{aligned}$$

- c: wavespeed
- β: nonlinear parameter

Inverse problem: recover β from the Dirichlet-to-Neumann map Λ.

Second Order Linearization

Second order linearization and the resulted integral identity:

$$\int_0^T \int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Lambda(\epsilon_1 f_1 + \epsilon_2 f_2)\Big|_{\epsilon_1=\epsilon_2=0} f_0 \, dS \, dt$$
$$= 2\int_0^T \int_{\Omega} \beta(x)\partial_t(u_1 u_2)\partial_t u_0 \, dx \, dt.$$

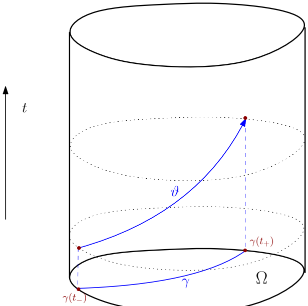
where u_j, $j = 1, 2$ are solutions to the linear wave equation

$$\frac{1}{c^2}\partial_t^2 u_i(t,x) - \Delta u_j(t,x) = 0$$

with $u_j|_{(0,T)\times\partial\Omega} = f_j$, and u_0 is the solution to the backward wave equation with $u_0|_{(0,T)\times\partial\Omega} = f_0$

Reduction to a Weighted Ray Transform

Construct Gaussian beam solutions u_0, u_1, u_2 traveling along the same null-geodesic $\vartheta(t) = (t, \gamma(t))$, where $\gamma(t), t \in (t_-, t_+)$ is the geodesic in (Ω, g) joining two boundary points $\gamma(t_-), \gamma(t_+) \in \partial\Omega$.



Insert into the integral identity, one can extract the Jacobi-weighted ray transform of $f = \beta c^{3/2} \Rightarrow$ invert this weighted ray transform (Paternain-Salo-U-Zhou, 2019; Feizmohammadi-Oksanen, 2020)

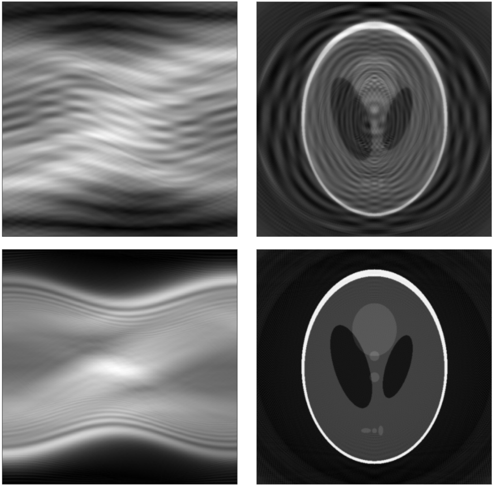


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

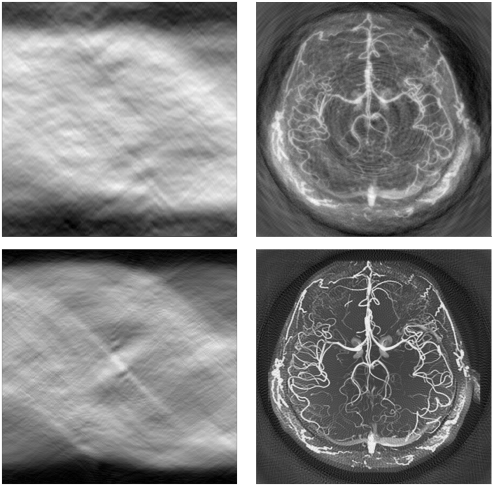


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

Einstein's Equations

The Einstein equation for the $(-,+,+,+)$-type Lorentzian metric g_{jk} of the space time is

$$\mathrm{Ein}_{jk}(g) = T_{jk},$$

where

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

In vacuum, $T = 0$. In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x)\frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$
$$\nabla_p(g^{pj}\,T_{jk}) = 0.$$

Einstein's Equations Coupled with Matter Fields

$$\begin{aligned}
&\mathrm{Ein}(g) = T, \quad T = \mathrm{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, T) \times N, \\
&\square_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L, \\
&g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}.
\end{aligned}$$

Here, $\widehat{g}$ and $\widehat{\phi}$ are C^∞-smooth and satisfy the equations above with zero sources and

$$\mathrm{T}_{jk}(g, \phi) = \sum_{\ell=1}^{L} \partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

$$\nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4.$$

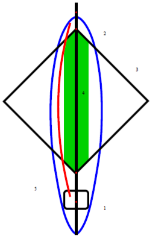
Let $V_{\widehat{g}} \subset M$ be a neighborhood of the geodesic μ and $p^-, p^+ \in \mu$.

Theorem (U-Wang, 2020; Kurylev-Lassas-U, 2013; Kurylev-Lassas-Oksanen-U, 2022)

Let

$$\mathcal{D} = \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g});\ g \text{ and } \phi \text{ satisfy Einstein equations with a source } \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2),\ \text{supp}\,(\mathcal{F}) \subset V_g, \text{ and } \nabla_j(\mathrm{T}^{jk}(g,\phi) + \mathcal{F}_1^{jk}) = 0\}.$$

The data set $\mathcal{D}$ determines uniquely the metric on the double cone $(J^+(p^-) \cap J^-(p^+), \widehat{g})$.



Inverse Problems in Cosmology

The existence of gravitational waves was predicted by Einstein and confirmed by the LIGO project in 2015. The gravitational waves generated in the early Universe, called primordial gravitational waves, are of great interest in cosmology. The detection of these gravitational waves is quite challenging:

".. will involve waves today whose wave lengths will extend all the way up to our present cosmological horizon (the distance out to which we can currently observe in principle) and that are likely to be well beyond the reach of any direct detectors for the foreseeable future."

quoted from Krauss, Dodelson and Meyer in Science, 2010.

Inverse Problems in Cosmology

Cosmic Microwave Background (CMB) is the thermal radiation remnant from the Big Bang (discovered by Penzias and Wilson 1964). It is considered as a primary source of information regarding the early universe. For example, the EGS (Ehlers-Geren-Sachs, 1968) theorem roughly states that the isotropy of observed $\mathcal{CMB}$ implies the isotropy of the universe.

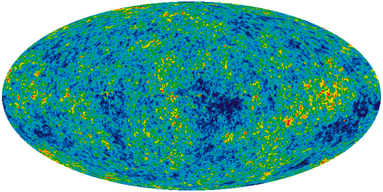
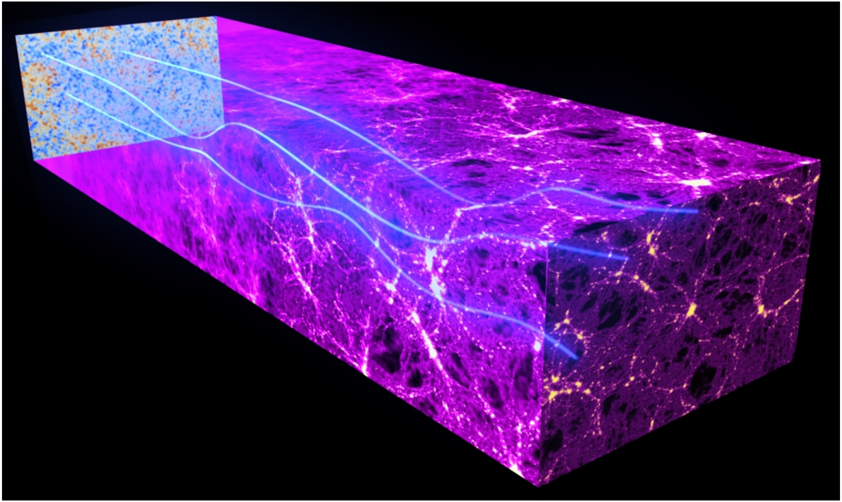


Figure: All-sky picture of the infant universe created from Wilkinson Microwave Anisotropy Probe (WMAP) data. Picture courtesy to NASA.

The inverse problem we study is the determination of early gravitational perturbations from CMB measurements.

Gravitational Lensing



CMB photons are deflected by the gravitational lensing effect of massive cosmic structures as they travel across the Universe.

Cosmological X-ray Tomography

Let $(\mathscr{M}, g_0)$ be a Friedman-Lemaître-Robertson-Walker (FLRW) cosmological model, where

$$\mathscr{M} = (0, \infty) \times \mathbb{R}^3, \quad g_0(x) = -dt^2 + a^2(t)dy^2$$

and $x = (t, y), t \in \mathbb{R}_+, y \in \mathbb{R}^3$, and $a(t) > 0$ is smooth. In this model, the Universe starts from a Big Bang at $t = 0$ and inflates. The factor R reflects the rate of expansion.

- when the Universe was very young and dominated by radiation, the factor $a(t) \approx t^{\frac{1}{2}}$.
- At later times, when matter became to dominate, $a(t) \approx t^{2/3}$.
- Based on more recent observations, the Universe is expanding with a rate $a(t) = e^{\Lambda t}$ with Λ a positive cosmological constant.

Cosmological X-ray Tomography

Consider a perturbation $(\mathscr{M}, g)$ of $(\mathscr{M}, g_0)$.

- let $\mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3$ be the "surface of last scattering" which is the moment that photons start to travel freely in space-time.
- let $\mathscr{S} = \{t_1\} \times \mathbb{R}^3$ be the surface where we observe the $\mathcal{CMB}$.
- let $\gamma(\tau), \tau \geq 0$ be a light-like geodesic on $(\mathscr{M}, g)$ with $\gamma(0) \in \mathscr{S}_0$. We think of $\gamma(\tau)$ as the trajectory of photons emitted from $\mathscr{S}_0$.

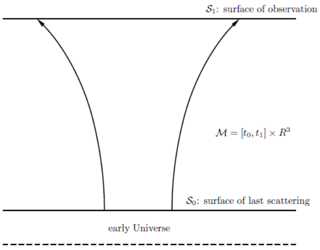


Figure: The FLRW cosmology model.

The Inverse Sachs-Wolfe Problem

Let's assume that the actual cosmos is a metric perturbation $g = g_0 + \delta g$ on $\mathscr{M}$ where δg is a small perturbation compared to g_0.
We use the conformal time s such that $ds = a^{-1}dt$. Then we get $g_0 = a^2(s)\left(ds^2 - \delta_{ij}dx^i dx^j\right) = a^2(s)g_M$ where g_M is the Minkowski metric on $M = (0, \infty)$. In the longitudinal gauge, also called the conformal Newtonian gauge, we consider the metric g of the form

$$g = a^2(s)\left[(1 + 2\Phi)ds^2 - (1 - 2\Psi)dx^2\right]$$

Here, Φ, Ψ are scalar functions on M and this type of perturbation is called scalar perturbations.

The Inverse Sachs-Wolfe Problem

Let T be the temperature observed at $\mathscr{S}$ in the isotropic background g_0. Let δT be the temperature fluctuation from the isotropic background. Then one component of $\delta T/T$ is the integrated Sachs-Wolfe (ISW) effects

$$\left(\frac{\delta T}{T}\right)^{ISW} = \int_0^{s_1 - s_0} (\partial_s \Phi(\gamma(\tau)) + \partial_s \Psi(\gamma(\tau)) d\tau$$

see e.g. Durrer 2008 . The integrated Sachs-Wolfe effect can be extracted from the CMB and other astrophysical measurements, see for example Manzotti, Dodelson 2014.
The integral is the light ray transform of $\partial_s(\Phi + \Psi)$. So the inverse problem is a tomography problem in the cosmological setting, see Guillemin, 1989; Lassas- Oksanen-Stefanov-U, 2018.

A Simple Mathematical Model for CMB Measurements

- $M = (0, \infty) \times \mathbb{R}^3$ and the metric tensor g is close to

$$g_0(t, y) = -dt^2 + a^2(t)dy^2, \quad (t, y) \in (0, \infty) \times \mathbb{R}^3,$$

  where the warping factor $a(t)$ is strictly positive. For example, $a(t) = t^{2/3}$ gives the Einstein-de Sitter cosmological model.

- The CMB photons are emitted with a fixed energy E_0 uniformly in all future pointing lightlike directions on

$$\Sigma = \{t_0\} \times \mathbb{R}^3.$$

  The physical meaning of $t_0 > 0$ is "380,000 years after the Big Bang".

- The CMB photons are observed by (p, ∂_t), $p \in \mathcal{U}$, where

$$\mathcal{U} = \{t_1\} \times \mathcal{U}_1.$$

  Here $t_1 > t_0$ and $\mathcal{U}_1 \subset \mathbb{R}^3$ is open.

Photons in the Theory of General Relativity

Let (M, g) be a $1+3$ dimensional Lorentzian manifold. On each tangent plane T_pM, $p \in M$, there is a basis $V_0, \ldots, V_3$ such that in this basis

$$g(p) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We suppose that (M, g) is time-oriented, that is, there is a vector field X on M satisfying $(X, X)_g < 0$ everywhere. We say that X is timelike and gives the direction of the future.

A geodesic $\beta : [0, \ell] \to M$ models a *photon* if it is lightlike and future pointing, that is,

$$(\dot\beta(\tau), \dot\beta(\tau))_g = 0 \quad \text{and} \quad (\dot\beta(\tau), X)_g < 0$$

for one and hence for all $\tau \in [0, \ell]$.

Observers and Energy Measurements

A point $(p, Z) \in TM$ is called an *observer* if Z is future pointing and $(Z, Z)_g = -1$.

If $\beta : [0, \ell] \to M$ is a photon and $\beta(\ell) = p$. Then the *energy*[1] E and *Newtonian velocity* V of β as measured by (p, Z) are

$$E = -(\dot{\beta}(\ell), Z)_g, \qquad V = \frac{\dot{\beta}(\ell)}{E} - Z.$$

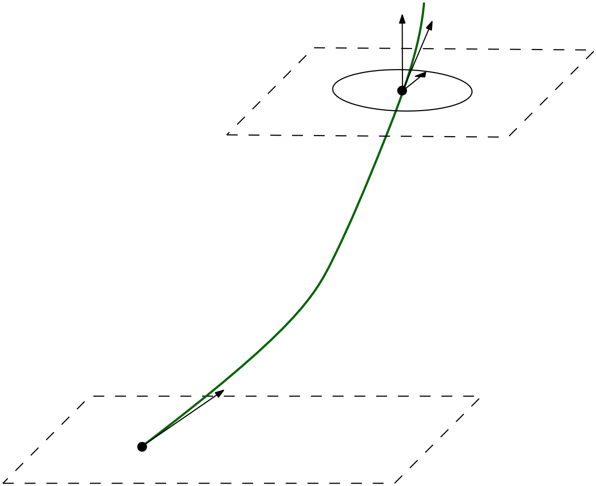
The energies of CMB photons as measured by (Z, p) can be parametrized by the velocities V. The velocity V satisfies

$$(V, Z)_g = 0, \quad (V, V)_g = 1. \tag{2}$$

The equations (2) define the *celestial sphere* of (p, Z). The physical meaning of the celestial sphere is "all the directions in the sky of (p, Z)".

[1] The energy of a photon is directly proportional to its frequency.

Parametrization of the CMB Measurements



The observer (p, ∂_t) measures the energy $E_g(p, V)$ of the CMB photon β coming from the direction V in the celestial sphere. Here $p \in \mathcal{U}$.

Linearization of the CMB Measurements

Let g_ϵ, $\epsilon \in [0,1]$, be a one parameter family of the Lorentzian metric tensor on $(0,\infty)\times\mathbb{R}^3$, and suppose that

$$g_0(t,y) = -dt^2 + a^2(t)dy^2, \quad (t,y)\in(0,\infty)\times\mathbb{R}^3.$$

We define the *redshift* R_ϵ, $\epsilon \in [0,1]$, by

$$R_\epsilon(p,V) = \frac{E_0}{E_{g_\epsilon}(p,V)} - 1,$$

where $E_{g_\epsilon}(p,V)$ is the energy of the CMB photon with respect to g_ϵ, coming from the direction V, as measured by (p,∂_t).

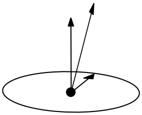
Linearized CMB inverse problem. Given $\partial_\epsilon R_\epsilon|_{\epsilon=0} = E_0\partial_\epsilon E_{g_\epsilon}^{-1}|_{\epsilon=0}$ determine $\partial_\epsilon g_\epsilon|_{\epsilon=0}$ up to natural invariances.

The Light Ray Transform of Tensors

We define the *light ray transform* Lf of a 2-tensor f as the restriction of the geodesic ray transform on lightlike geodesics. That is

$$Lf(p,V) = \int_{\mathbb{R}} f_{jk}(\gamma(\tau))\dot{\gamma}^j(\tau)\dot{\gamma}^k(\tau)d\tau,$$

where V is a vector in the celestial sphere S_p^2 of (p, ∂_t), and γ is the geodesic on (M, g_0) with the initial data $\gamma(0) = p$, $\dot{\gamma}(0) = V + \partial_t$.



Reduction to the Light Ray Transform

We recall that the CMB photons are emitted on Σ and observed $\mathcal{U}$ where

$$\Sigma = \{t_0\} \times \mathbb{R}^3 \quad \text{and} \quad \mathcal{U} = \{t_1\} \times \mathcal{U}_1.$$

Here $0 < t_0 < t_1$ and $\mathcal{U}_1 \subset \mathbb{R}^3$ is open. We define the "slab"

$$M_1 = (t_0, t_1) \times \mathbb{R}^3.$$

Theorem [Lassas-Oksanen-Stefanov-U 2018, Sachs-Wolfe 1967].
Let g_ϵ, $\epsilon \in [0,1]$, be a one parameter family of Lorentzian metric tensors on $(0,\infty) \times \mathbb{R}^3$. Suppose that $g_0(t,y) = -dt^2 + a^2(t)dy^2$, and that $g_\epsilon = g_0$ in $\Sigma \cup \mathcal{U}$. Then

$$\partial_\epsilon R_\epsilon|_{\epsilon=0}(p,V) = Lf(p,V), \quad p \in \mathcal{U}, \; V \in S_p^2,$$

where $f = (2a(t_0))^{-1} a^2 \mathcal{L}_{a\partial_t} a^{-2} \partial_\epsilon g_\epsilon|_{\epsilon=0}$ on M_1 and $f = 0$ elsewhere.

The Light Ray Transform of Functions

for $n \geq 2$, consider the Minkowski space $(\mathbb{R}^{1+n}, g)$, where

$$(t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad g = -dt^2 + dx_1^2 + \ldots + dx_n^2$$

lightlike vector: $g(v, v) = 0 \Rightarrow v = r(1, \theta)$, where $r > 0$, $\theta \in S^n$

light ray: lines with tangent vectors are lightlike vectors

$$\gamma_{(x,\theta)}(s) = (s, x + s\theta), \quad \theta \in S^{n-1},$$

if we fix a parameterization by using the initial condition at $t = 0$

Light Ray Transform

Integral of a function (or distribution) over light rays

$$Lf(x,\theta) = \int f(s, x+s\theta)\,ds, \quad (x,\theta) \in \mathbb{R}^n \times S^{n-1}$$

Fourier Slice Theorem: for any $f \in \mathcal{S}(\mathbb{R}^{n+1})$,

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} Lf(x,\theta)\,dx, \quad \text{when } (1,\theta) \perp \zeta,\ \theta \in S^{n-1},$$

where $\zeta = (\tau, \xi) \in T^*_{(t,x)}\mathbb{R}^{1+n}$.

- knowing $Lf(\cdot,\theta)$ for some $\theta \in S^{n-1}$, then we know all $\widehat{f}(\zeta)$ for ζ on the plane $\tau + \xi \cdot \theta = 0$
- if $Lf = 0$, then $\widehat{f}(\zeta) = 0$ for all ζ satisfying $|\tau| \leq |\xi|$

Injectivity of the Light Ray Transform

The light ray transform L is injective on $C_0^\infty(\mathbb{R}^{1+n})$.

- $f \in C_0^\infty(\mathbb{R}^{1+n}) \Rightarrow \widehat{f}$ is real analytic
- if $Lf = 0$, then $\widehat{f} = 0$ in the cone $|\tau| \leq |\xi|$
- real analytic functions vanishing in an open set $\Rightarrow f = 0$

The light ray transform L is not injective on $\mathcal{S}(\mathbb{R}^{1+n})$.

- let $\psi \in C_0^\infty(\mathbb{R}^{1+n})$ supported in the cone $|\tau| > |\xi|$
- set $f = \check{\psi}$, then $\widehat{f}(-\theta \cdot \xi, \xi) = 0$, for all $\theta \in S^{n-1}$ and all ξ
  $\Rightarrow \quad Lf = 0$
- thus, we can construct $f \neq 0$ but $Lf = 0$

Theorem (stronger result for injectivity)

Let $f \in \mathcal{S}(\mathbb{R}^{1+n})$. Suppose $\exists R > 0$ such that $f(t,x) = 0$, for $|x| > R$. If for θ in some open set one has $Lf(\cdot,\theta) = 0$, then $f = 0$.

- idea: use the analyticity of the partial FT of f w.r.t. x
- for fixed τ, consider

  $$\widehat{f}(\tau,\xi) = \int e^{-i(t\tau + x\cdot\xi)} f(t,x)\,dt\,dx = \int e^{-ix\cdot\xi}\widehat{f}^t(\tau,x)\,dx,$$

  where $\widehat{f}^t(\tau,x) = \int e^{-it\tau} f(t,x)\,dt$
- $\widehat{f}(\cdot,\xi)$ extends to an analytic function, for any fixed τ
- simple case: if $Lf = 0$ for all (x,θ), then
  - $\widehat{f}(\tau,\xi) = 0$ for in the cone $|\tau| \leq |\xi|$
  - by analyticity, $\widehat{f} = 0$ for all ξ, for all fixed τ
- general case: $\widehat{f} = 0$ in an certain open set, which has a nonempty interior in any slice $\{\tau = \tau_0\}$

Injectivity of the Light Ray Transform of Distributions

Extend L to a larger class of functions or distributions

- extend L to $\mathcal{E}'(\mathbb{R}^{1+n})$ by duality
  - does not preserve compactness of the support
  - too restrictive: excludes time-independent distributions
- extend $L\chi$ to $\mathcal{S}'(\mathbb{R}^{1+n})$ distributions vanishing for $|x| > R$ with some $R > 0$, where $\chi \in C^\infty(\mathbb{R}^{1+n})$ is properly supported
  - preserve compactness of the support: $(L\chi)^* = \chi L^*$
  - allows tempered distributions with compact support for x only

Theorem (injectivity for termpered distribution)

Let $f \in \mathcal{S}'(\mathbb{R}^{1+n})$. Suppose $\exists R > 0$ such that $f(t, x) = 0$, for $|x| > R$. If for θ in some open set one has $Lf(\cdot, \theta) = 0$, then $f = 0$.

- idea: generalize the argument before: fix τ and exploit the analyticity w.r.t. ξ

Normal Operator

$$L^*Lf(t,x) = \int_{S^{n-1}}\int_{\mathbb{R}} f(s, x - t\theta + s\theta)\, ds\, d\theta$$
$$= \int_{\mathbb{R}^n} \frac{f(t - |x-x'|, x') + f(t + |x-x'|, x')}{|x-x'|^{n-1}}\, dx',$$

which has the Schwartz kernel

$$N(t,x;t',x') = \frac{\delta(t - t' - |x-x'|, x') + \delta(t - t' + |x-x'|, x')}{|x-x'|^{n-1}}.$$

Here we define $\delta(t \mp |x|)/|x|^{n-1}$ as the linear functional

$$\phi(t,x) \mapsto \int \frac{\phi(\pm|x|, x)}{|x|^{n-1}}\, dx,$$

for $\phi \in C_0^\infty(\mathbb{R}^{1+n})$.

Normal Operator

L^*L is a convolution:

$$L^*L = \mathcal{N} * f, \quad \mathcal{N}(t,x) = \frac{\delta(t-|x|)+\delta(t+|x|)}{|x|^{n-1}}$$

- $\mathcal{N}$ is a tempered distribution homogeneous of order $-n$

If we denote by $\mathcal{F}f$ ($\mathcal{F}^{-1}f$) the FT (inverse FT) of f, then

$$L^*Lf = 2\pi|S^{n-2}|\mathcal{F}^{-1}\frac{(|\xi|^2-\tau^2)_+^{\frac{3}{2}}}{|\xi|^{n-2}}\mathcal{F}f, \quad f \in \mathcal{S}(\mathbb{R}^{1+n})$$

- the FT $\mathcal{F}f$ can be constructed stably in the timelike cone
- the estimate deteriorates at the light cone
- no stable inversion can be done in the space-like cone

Linearized Gauge Invariances

We recall that $\Sigma = \{t_0\} \times \mathbb{R}^3$, $\mathcal{U} \subset \{t_1\} \times \mathbb{R}^3$ and $M_1 = (t_0, t_1) \times \mathbb{R}^3$.

The energy $E_g(p, V)$, $p \in \mathcal{U}$, is invariant under

1. diffeomorphisms $g \mapsto \Phi^* g$ fixing $\Sigma \cup \mathcal{U}$.
2. conformal scalings $g \mapsto cg$ with $c = 1$ on $\Sigma \cup \mathcal{U}$.

These correspond to subspaces in the null space of the light ray transform

1. $L(d^s \omega) = 0$ for 1-forms ω supported on M_1.
2. $L(cg_0) = 0$ for functions c supported on M_1.

Here d^s is the symmetric differential defined in coordinates as follows

$$(d^s \omega)_{ij} = \frac{(\nabla_i \omega)_j + (\nabla_j \omega)_i}{2},$$

and $\nabla_i = \nabla_{\partial_{x^i}}$, $x = (t, y)$, is the covariant derivative with respect to g_0.

Microlocal Inversion of the Light Ray Transform

We write $g = g_0$ where $g_0(t, y) = -dt^2 + a^2(t)dy^2$ as before. We recall that $\Sigma = \{t_0\} \times \mathbb{R}^3$, $\mathcal{U} \subset \{t_1\} \times \mathbb{R}^3$ and $M_1 = (t_0, t_1) \times \mathbb{R}^3$.

Theorem [Lassas-Oksanen-Stefanov-U 2018].

Let $(x, \xi) \in T^*M_1$ be spacelike, that is, $(\xi, \xi)_g > 0$. Suppose that there is a lightlike geodesic γ of (M, g) and $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$\gamma(\tau_1) = x, \quad \xi(\dot{\gamma}(\tau_1)) = 0, \quad \gamma(\tau_2) \in \mathcal{U}. \tag{3}$$

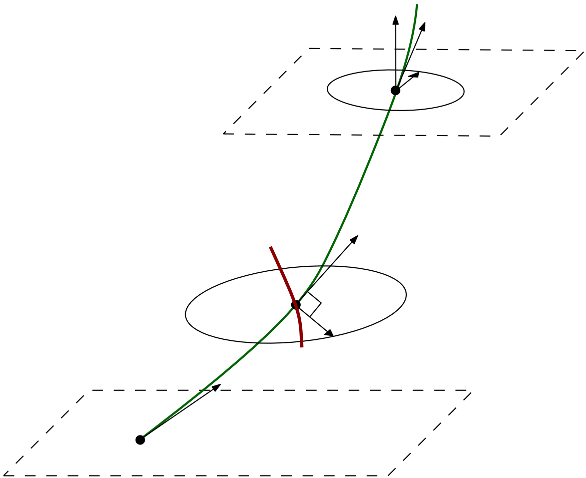
Then there is a microlocal cutoff χ **vanishing outside** $\mathcal{U}$ such that for all 2-tensors f the following are equivalent

(i) $(x, \xi) \in \mathrm{WF}(L^*\chi L f)$,

(ii) $(x, \xi) \in \mathrm{WF}(f + h)$ for all h of the form $h = d^s\omega + cg$.

Moreover, L is smoothing on timelike covectors in T^*M_1, and also on the spacelike covectors that do not satisfy the visibility condition (3).

The Visibility Condition



The visibility condition for spacelike (x, ξ): there is a lightlike geodesic γ and $\tau_1, \tau_2 \in \mathbb{R}$ such that $\gamma(\tau_1) = x$, $\xi(\dot{\gamma}(\tau_1)) = 0$ and $\gamma(\tau_2) \in \mathcal{U}$.

Note the analogy with the limited angle X-ray tomography.

Microlocal Inversion of the Light Ray Transform

Theorem [Lassas-Oksanen-Stefanov-U 2018].
Let $(x,\xi) \in T^*M_1$ be spacelike and suppose that it satisfies the visibility condition. Then there is a microlocal cutoff χ vanishing outside $\mathcal{U}$ such that for all 2-tensors f the following are equivalent

(i) $(x,\xi) \in \mathrm{WF}(L^*\chi L f)$,

(ii) $(x,\xi) \in \mathrm{WF}(f+h)$ for all h of the form $h = d^s\omega + cg$.

Moreover, L is smoothing on timelike covectors in T^*M_1, and also on the spacelike covectors that do not satisfy the visibility condition.

- The theorem is sharp except that it doesn't cover the case of lightlike $(x,\xi) \in T^*M_1$, that is, $(\xi,\xi)_g = 0$. This is an open question.
- The cutoff χ can be chosen so that $L^*\chi L$ is a pseudodifferential operator of order -1 and its principal symbol can be given explicitly.
- Theorem is invariant under diffeomorphisms and conformal scalings.

The Principal Symbol of the Normal Operator

After a change of coordinates and a conformal transformation we have $g = -dt^2 + dy^2$. We write $x = (x^0, x') \in \mathbb{R}^{1+3}$ and $\xi = (\xi_0, \xi') \in \mathbb{R}^{1+3}$. The cutoff χ can be chosen so that $L^* \chi L$ has the principal symbol

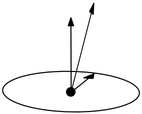
$$\sigma(x,\xi) = \frac{2\pi\chi_1(|\xi_0|/|\xi'|)}{\sqrt{|\xi'|^2 - |\xi_0|^2}} N^{jklm}, \quad N^{jklm} = \int_{S^1_\xi} \chi_2(x' - x^0 v)\theta^j\theta^k\theta^l\theta^m dv,$$

where $\chi_1 \in C_0^\infty(-1,1)$, $\chi_2 \in C_0^\infty(\mathcal{U}_1)$, $\mathcal{U} = \{0\} \times \mathcal{U}_1$, $\theta = (1, v) \in \mathbb{R}^{1+3}$, $S^1_\xi = \{v \in S^2;\ \xi_0 + \xi' v = 0\}$, and $j,k,l,m \in \{0,1,2,3\}$.

- N^{jklm} is homogeneous of degree zero in ξ.
- χ_1 localizes on spacelike covectors, and χ_2 localizes on the set where we have data. We choose χ_1 and χ_2 to be non-negative.
- $v \in S^1_\xi$ and $\chi_2(x' - x^0 v)$ correspond to the visibility condition.
- If the visibility condition holds then the null space of the linear map $N : f_{lm} \mapsto N^{jklm} f_{lm}$ on 2-tensors f is

$$\{c g_{lm} + \xi_l \omega_m + \xi_m \omega_l;\ c \in \mathbb{R},\ \omega \in \mathbb{R}^4\}.$$

On the Microlocal Structure of the Light Ray Transform



- The manifold $\mathcal{C}$ of lightlike geodesics can be parametrized by $p \in \{t_1\} \times \mathbb{R}^3$ and the vectors V in the celestial spheres S^2_p.
- L is a Fourier integral operator of order $-3/4$ whose canonical relation is the twisted conormal bundle N^*Z' of the point-line relation

$$Z = \{(x, \gamma) \in M \times \mathcal{C};\ x = \gamma(\tau) \text{ for some } \tau \in \mathbb{R}\}.$$

  Here $M = (0, \infty) \times \mathbb{R}^3$.
- We have $\dim M = 4$ and $\dim \mathcal{C} = 5$.

On the Microlocal Structure of the Light Ray Transform

Let us consider the projections

$$T^*M \xleftarrow{\ \pi\ } N^*Z \xrightarrow{\ \rho\ } T^*\mathcal{C}$$

- $d\rho$ is not injective on lightlike $(x, \xi) \in T^*M$.
- ρ is an injective immersion away from the lightlike vectors (that is, the Bolker condition holds there).
- The clean intersection calculus implies that $L^*\chi L$ is a pseudodifferential operator when χ cuts off the lightlike covectors.
- When the lightlike covectors are **not** cut off, the canonical relation is a $(1, 2)$-fibered folding canonical relation in the sense of [Greenleaf-U,1991], and $L^*\chi L$ is an IPL class operator.

Backprojection in the "Full Angle" Case

Let us consider the case $g = -dt^2 + dy^2$, and suppose (unrealistically) that we have data the whole slice $\{0\} \times \mathbb{R}^3$. We parametrize the light ray transform

$$Lf(y, v) = \int_{\mathbb{R}} f_{lm}(s, y + sv)\theta^l \theta^m ds, \quad y \in \mathbb{R}^3, \; v \in S^2,$$

where $\theta = (1, v) \in \mathbb{R}^{1+3}$. Then the normal operator L^*L is the convolution $K^{jklm} * f_{lm}$, with the kernel

$$(K^{jklm}, \phi)_{\mathcal{D}' \times \mathcal{D}(\mathbb{R}^4)} = \int_{S^2} \int_{\mathbb{R}} \theta^j \theta^k \theta^l \theta^m \phi(\rho\theta) d\rho dv.$$

The kernel is supported on the light cone $\{(x^0, x') \in \mathbb{R}^{1+3}; \; |x^0| = |x'|\}$.

Fourier Transform of the Backprojection

We write $a(v) = \theta^j\theta^k\theta^l\theta^m$, $\theta = (1, v)$, and have

$$(\widehat{K}^{jklm}, \phi) = \int_{S^2}\int_{\mathbb{R}} a(v) \int_{\mathbb{R}^4} e^{-i\xi(\rho\theta)}\phi(\xi)d\xi d\rho dv$$
$$= 2\pi \int_{\mathbb{R}^4}\int_{S^2} \delta(\xi\theta)a(v)dv\phi(\xi)d\xi.$$

The equation $\xi\theta = 0$ for v defines the affine plane $\xi^0 + \xi' v = 0$.

- If ξ is timelike (that is, $|\xi_0| > |\xi'|$) then the affine plane does not intersect with the unit sphere $S^2 \subset \mathbb{R}^3$. Hence $\widehat{K}^{jklm}(\xi) = 0$.
- If ξ is spacelike (that is, $|\xi_0| < |\xi'|$) then

$$\widehat{K}^{jklm}(\xi) = 2\pi \int_{S^2} \delta(\xi\theta)a(v)dv = \frac{2\pi}{\sqrt{|\xi'|^2 - |\xi_0|^2}} \int_{S^1_\xi} a(v)dv.$$

Here $S^1_\xi = \{v \in S^2;\ \xi\theta = 0\}$ is a circle of radius $|\xi'|^{-1}\sqrt{|\xi'|^2 - |\xi_0|^2}$.