Introduction to Inverse Problems for Transport Equations

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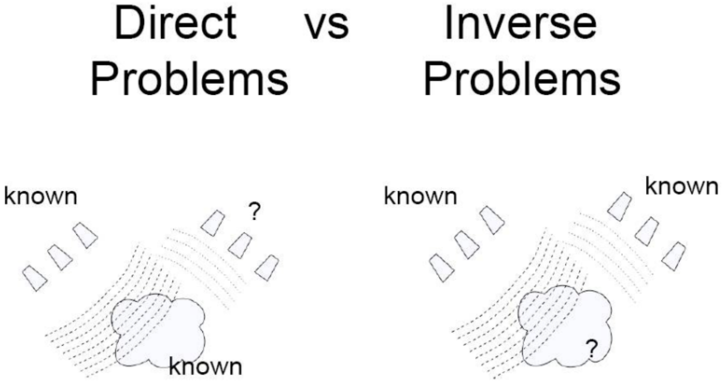
Outline

What is an inverse problem?

In general, an inverse problem is to extract knowledge from data.



X-ray tomography

- **Forward problem:** determine the X-ray projection data from a body whose internal structure we know precisely.
- **Inverse problem:** reconstruct the inner structure of an unknown body from the knowledge of X-ray data from different directions and positions. [1]

2D example:

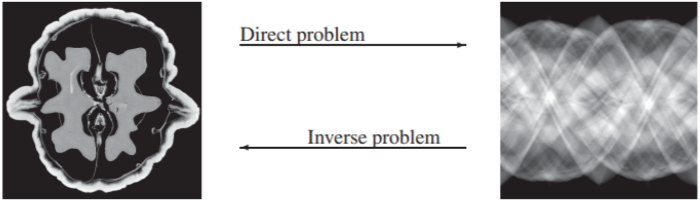
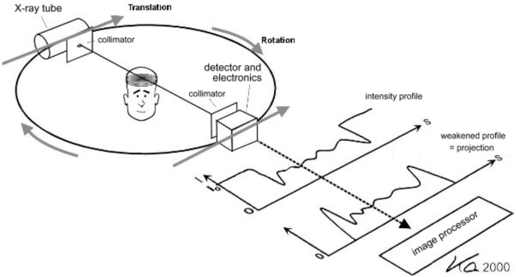
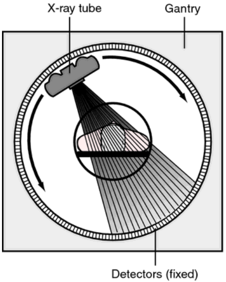


Figure: Cause: the slice of a walnut (left); Effect: the X-ray data (right).

[1] Figure 1. J. L. Mueller-S. Siltanen, *Linear and nonlinear inverse problems with practical applications*, 2012

CT = X-ray tomography



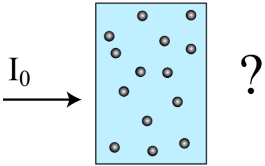


A

cross-section (2D) of the human body is scanned by X-ray beams. The X-ray beam is attenuated when passing through the body.

X-ray

I_0 density of incoming X-ray.



How does the intensity change during the propagation?

Beer's law

The relative intensity loss of X-ray is proportional to the distance it travels.

$I(x)$ – intensity of X-ray along the beam line L at point x.

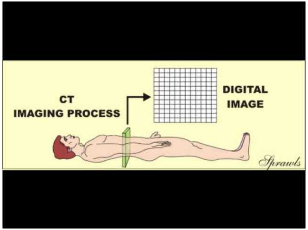
Beer's law: $$\frac{dI}{I} = -\underbrace{\sigma(x)}_{\text{attenuation coefficient / density at } x} dx$$

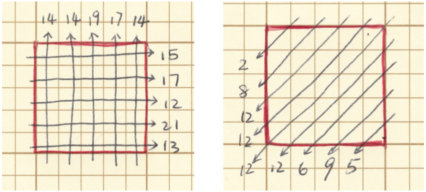
$$-\int_L \sigma(x)\,dx = \ln I(x)\Big|_{\text{source}:x_0}^{\text{receiver}:x_1} = \underbrace{\ln(I_1/I_0)}_{\text{known!}}$$

$$\boxed{\int_L \sigma(x)\,dx} \overset{\text{Processor}}{\Longrightarrow} \boxed{\sigma(x)} \overset{\text{Plot}}{\Longrightarrow} \textbf{tomogram}$$

for "all" L

CT (square body)





Line integrals of σ

reconstruction of σ

Inverse Problems in Imaging (inverse v.s. direct)

Direct problem:

$$\boxed{\text{Input signal}} \Longrightarrow \boxed{A} \Longrightarrow \boxed{\text{Output signal } =?}$$

Inverse problem:

$$\boxed{\text{Input signal}} \Longrightarrow \boxed{A \ =?} \Longrightarrow \boxed{\text{Output signal}}$$

e.g.

$$\boxed{\text{X-ray beam } L} \Longrightarrow \boxed{\sigma \ =?} \Longrightarrow \boxed{\int_L \sigma(x)\, dx}$$

Outline

Transport of photons/particles (A general model)

We consider the density distribution of particles

$$f(\mathrm{x}, \mathrm{v})$$

position velocity

at a position x with velocity v satisfying

Transport Equation

$$v \cdot \nabla_x f(x,v) + \sigma f(x,v) = Q(f)(x,v)$$

- $Q(f)$ - collision operator. Specific form of Q will be discussed later.
- σ - the absorption coefficient which reflects the rate of particles being absorbed by the media

Linear Boltzmann equation - Radiative transfer equation (RTE)

For example, consider the stationary case, let $f \equiv f(x, v)$ satisfy the **linear Boltzmann equation:**

$$v \cdot \nabla_x f(x, v) = - \underbrace{\sigma f(x, v)}_{Absorption+scattering\ Losses} + \underbrace{\int_{\mathbb{S}^{n-1}} \mu(x, v', v) f(x, v') dv'}_{Scattered\ into\ v}$$

where

$$Q(f) \equiv Kf(x, v) = \int \mu(x, v', v) f(x, v') dv'.$$

- $\mu(x, v', v)$: scattering phase function models the probability that the particle with an initial direction v' will being scattered into a direction v at the position x.

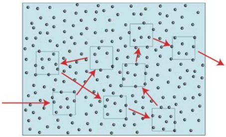


Figure: A path due to multiple scattering in tissue. Courtesy to "Diffuse optics: Fundamentals and tissue applications" by R. C. Mesquita and A. G. Yodh

Inverse problems for transport equations

Stationary RTE:

$$v \cdot \nabla_x f(x,v) + \sigma f(x,v) = \int_{\mathbb{S}^{n-1}} \mu(x,v',v) f(x,v') dv'.$$

Inverse Problem: Can we recover medium properties σ and μ ?

Inverse problems

- When $\boxed{\mu = 0}$, one gets Beer's law:
  $I(x)$ – intensity of X-ray along the beam line L at point x.

  $$\underline{\text{Beer's law}}: \qquad \frac{dI}{I} = -\sigma(x)dx$$

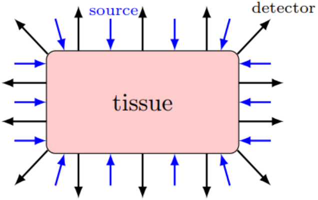
  $$-\int_L \sigma(x)\, dx = \ln I(x)\Big|_{\text{source}:x_0}^{\text{receiver}:x_1} = \underbrace{\ln(I_1/I_0)}_{\text{known!}}$$

  Thus, one can recover σ from inverting the X-ray transform.

- When $\boxed{\mu \neq 0}$,
  Question: How do we recover both σ and μ from boundary measurements?

Measurements

The incoming and outgoing measurements:



More precisely, we denote the **Albedo operator** $\mathcal{A}$, which maps the incoming conditions to the outgoing radiation:

$$\mathcal{A} : f|_{\Gamma_-} \mapsto f|_{\Gamma_+}.$$

Here $\Gamma_\pm = \{(x,v) \in \partial\Omega \times \mathbb{S}^{n-1} :\ \pm v \cdot n(x) > 0\}$, and $n(x)$ is the outward unit normal to Ω at point $x \in \partial\Omega$.

Inverse transport problems

Let Ω be an open bounded and convex domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary. Let $f \equiv f(x, v)$ satisfy

$$\begin{cases} v \cdot \nabla_x f + \sigma f &= Kf \quad \text{in } \Omega \times \mathbb{S}^{n-1}, \\ f &= f_- \quad \text{on } \Gamma_-. \end{cases}$$

- The forward problem is well-posed by assuming for example that $\int \mu(x, v', v) dv' < \sigma$.

Inverse Problem: Recover absorption coefficient $\sigma(x, v)$ and the scattering coefficient $\mu(x, v', v)$ from $\mathcal{A}$.

Previous results on RTE

The following is only a partial list of earlier results.

Uniqueness: ($\sigma = \tilde{\sigma},\ \mu = \tilde{\mu}$ if $\mathcal{A} = \tilde{\mathcal{A}}$)

- [Choulli-Stefanov 1996]: For $n \geq 3$, uniqueness for σ and μ and $n = 2$ uniqueness for σ.
- [Stefanov-Uhlmann 2003]: For $n = 2$, uniqueness for σ and μ provided that μ is small.

Stability: ($\|\sigma - \tilde{\sigma}\|,\ \|\mu - \tilde{\mu}\| \lesssim \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\beta, \beta > 0$)

- [Wang 1999]: $n \geq 3$, assume $\mu = \mu(v', v)$.
- [Bal-Jollivet 2008]: $n \geq 3$, consider general μ.
- [Stefanov-Uhlmann 2003]: $n = 2$.

Non uniqueness

For $\sigma \equiv \sigma(x, v)$, there is **no uniqueness**.

We let

$$\boxed{\tilde{\sigma}(x, v) = \sigma(x + p(x, v)v, v)}$$

for $(x + p(x, v)v, v) \in \Omega \times \mathbb{S}^{n-1}$ and $p(x, v)$ is a nontrivial continuous function. Then

$$(\sigma, 0) \ \text{ and } \ (\tilde{\sigma}, 0)$$

produce the same albedo operators. However, $\sigma \neq \tilde{\sigma}$.

To see this, when $\mu = 0$*, the solution to* $v \cdot \nabla_x f + \sigma f = 0$ *with boundary data* $f = f_-$ *on* Γ_- *is*

$$\boxed{f(x, v) = e^{-\int_0^{\tau_-(x,v)} \sigma(x - sv, v) ds} f_-}$$

for $(x, v) \in \Omega \times \mathbb{S}^{n-1}$.[2]

Therefore, it common to consider $\sigma(x)$ or $\sigma(x, |v|)$.

[2] $\tau_-(x, v)$ is the backward travel time at position x with velocity v, i.e.,

$$\tau_-(x, v) = \max\{s > 0 : \, x - sv \in \partial\Omega\}.$$

Inverse transport problems

Some approaches to solve inverse transport problems:

- **Singular decomposition** of the kernel of the albedo operator.
- **Carleman estimate**: an L^2 weighted estimate.

$$\vdots$$

Decomposition of transport solutions

Let $\phi \in C_c^\infty(\Gamma_-)$.

$$\begin{cases} v \cdot \nabla_x f + \sigma f = Kf & \text{in } \Omega \times \mathbb{S}^{n-1}, \\ f = \phi & \text{on } \Gamma_-. \end{cases}$$

Fix $(x_0, v_0) \in \Gamma_-$. Decompose the solution of the transport equation as

$$f(x,v;x_0,v_0) = f_1 + f_2 + f_3,$$

where $f_1 \equiv f_1(x,v;x_0,v_0)$ (**ballistic term**) as

$$f_1(x,v) = e^{-\int_0^{\tau_-(x,v)} \sigma(x-tv,v)dt} \phi(x - \tau_-(x,v)v, v)$$

solves

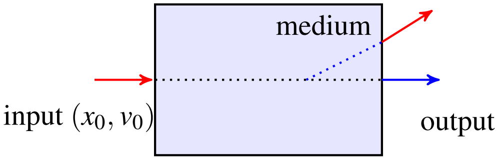
$$\begin{aligned} & v \cdot \nabla f_1 + \sigma f_1 = 0, && \Omega \times \mathbb{S}^{n-1}, \\ & f_1 = \phi, && \Gamma_-. \end{aligned}$$

And, $f_2 \equiv f_2(x, v; x_0, v_0)$ as

$$f_2(x, v) = \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma(x-tv)dt} \int_{\mathbb{S}^{n-1}} \mu(x - sv, v', v) f_1(x - sv, v') dv' ds$$

solves

$$\begin{aligned} &v \cdot \nabla f_2 + \sigma f_2 = \int_{\mathbb{S}^{n-1}} \mu(x, v', v) f_1(x, v') dv', && \Omega \times \mathbb{S}^{n-1}, \\ &f_2 = 0, && \Gamma_-. \end{aligned}$$



- Knowledge of $\mathcal{A}$ is equivalent to that of $f(x, v; x_0, v_0)$ for $(x, v) \in \Gamma_+$.

When $f|_{\Gamma_-} = \delta(x - x_0)\delta(v - v_0)$, f_1 is more singular than $f_2 + f_3$. In $n \geq 3$, f_2 is more singular than f_3.

Uniqueness results

Theorem (Choulli-Stefanov 1996)

In $n \geq 2$, knowledge of $\mathcal{A}$ implies $f_1(x, v; x_0, v_0)$ on $\Gamma_+ \times \Gamma_-$. Then $\sigma(x)$ is uniquely determined.
In $n \geq 3$, knowledge of $\mathcal{A}$ implies that of $f_2(x, v; x_0, v_0)$ on $\Gamma_+ \times \Gamma_-$. Then $\mu(x, v', v)$ can be uniquely determined.

In particular, σ is reconstructed from

$$f_1(x, v; x_0, v_0)|_{(x,v)\in\Gamma_+} = \int_0^{\tau_+(x_0,v_0)} e^{-\int_0^{\tau_-(x,v)} \sigma(x-sv)ds} \delta(x - x_0 - tv)\delta(v - v_0)dt,$$

and μ is reconstructed from

$$f_2(x, v; x_0, v_0)|_{(x,v)\in\Gamma_+} = \int_0^{\tau_-(x,v)} \int_0^{\tau_+(x_0,v_0)} e^{-\int_0^{\eta} \sigma(x-tv)dt - \int_0^{\tau_-(x-\eta v, v_0)} \sigma(x-\eta v - tv_0)dt} \mu(x - \eta v, v_0, v)\delta(x - x_0 - \eta v - tv_0)dtd\eta.$$

Carleman estimate: time-dependent RTE

We consider the following initial boundary value problem for $T > 0$ large.

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma f = K(f) & \text{in } (0,T) \times \Omega \times \mathbb{S}^{n-1}, \\ f = f_0 & \text{on } \{0\} \times \Omega \times \mathbb{S}^{n-1}, \\ f = f_- & \text{on } (0,T) \times \Gamma_-. \end{cases}$$

We denote the measurement operator by

$$\mathcal{A} : (f_0, f_-) \mapsto f|_{(0,T)\times\Gamma_+}.$$

Inverse Problem: Recover $\sigma(x, v)$ and $\mu(x, v', v)$ from $\mathcal{A}$.

Carleman estimate

Define the operator

$$P = \partial_t + v \cdot \nabla_x + \sigma - K.$$

Let φ be a weight function and V be some subset of $\mathbb{S}^{n-1}$.

Theorem (Carleman Estimate, Machida-Yamamoto 2014)

For some constant $c_0 > 0$ independent of s, then we have

$$s \int_{\Omega \times V} |f(0,x,v)|^2 e^{2s\varphi(0,x)} dxdv + s^2 \int_0^T \int_{\Omega \times V} |f|^2 e^{2s\varphi} dxdvdt$$
$$\leq c_0 s \int_0^T \int_{\Gamma_+} (v \cdot n(x)) |f|^2 e^{2s\varphi} d\sigma dt + c_0 \int_0^T \int_{\Omega \times V} |Pf|^2 e^{2s\varphi} dxdvdt,$$

provided that s is sufficiently large and $f|_{t=T} = 0$.

- The role of parameter s is used to control the nonessential terms.
- The initial data contains the to-be-recovered coefficients.

Carleman estimate: inverse source problem

[Inverse Source Problem: Recover source]
Let $\tilde{f}$ satisfies

$$\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} + \sigma \tilde{f} = K\tilde{f} + \underbrace{S_0(x,v) S_1(t,x,v)}_{source}, \quad \text{with } \tilde{f}(0,x,v) = 0.$$

We observe that

$$\partial_t \tilde{f}(0,x,v) = S_0(x,v) S_1(0,x,v).$$

The initial condition $\partial_t \tilde{f}(0,x,v)$ is connected with $S_0(x,v)$ (unknown) by S_1 (known).

If $S_1(0,x,v)$ is suitably controlled, then one expects to recover $S_0(x,v)$.

Applications of Carleman estimate

We choose smooth cut-off function χ in time t so that

$$\chi(t) = \begin{cases} 1, & 0 \le t \le T - 2\varepsilon, \\ 0, & T - \varepsilon \le t \le T. \end{cases}$$

We consider the function

$$F(t,x,v) = \chi(t)\partial_t \tilde{f}(t,x,v),$$

which satisfies $\partial_t F + v \cdot \nabla_x F + \sigma F = KF + S$ with $F|_{t=T} = 0$.
Applying the Carleman estimate to F gives

$$\boxed{s \int_{\Omega \times V} |\underbrace{\partial_t \tilde{f}(0,x,v)}_{S_0(x,v)S_1(0,x,v)}|^2 e^{2s\varphi(0,x)} dxdv \le Data + l.o.t.}$$

From this, one can derive the L^2 estimate of S_0:

$$\|S_0\|_{L^2(\Omega \times V)} \le C \|\partial_t \tilde{f}\|_{L^2((0,T)\times\Gamma_+)}.$$

For this case, “single measurement” is sufficient.

[Inverse Coefficient Problem: Recover σ]

Idea: Transform this problem into the inverse source problem.

To recover the unknown σ_j, suppose μ is given.
Let $f_j, j = 1, 2$, be the solution to

$$\partial_t f_j + v \cdot \nabla_x f_j + \sigma_j f_j = K f_j \quad \text{with } f_j|_{t=0} = f_0 \neq 0.$$

Let $\tilde{f} = f_1 - f_2$. We write

$$\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} + \sigma_1(x, v) \tilde{f} = K\tilde{f} - \underbrace{(\sigma_1 - \sigma_2)(x, v) f_2(t, x, v)}_{source}$$

Thus, Carleman estimate can give the desired stability estimate for $(\sigma_1 - \sigma_2)$.

- Similarly, we can also recover μ provided that σ is given.
- This approach can only recover **time-independent** coefficient.
- Compared to the singular decomposition method (only recover $\sigma(x, |v|)$), applying Carleman estimate can determine $\sigma(x, v)$ due to the extra knowledge of initial data.

Outline

Goal

We want to study the nonlinear transport equation:

$$\partial_t f + v \cdot \nabla_x f + \sigma f + \mathit{Nonlinearity} = K(f)$$

To understand what kind of nonlinear term can be recovered.

Inverse problems for semilinear transport equations

We consider the following initial boundary value problem for $T > 0$ large.

Nonlinear Transport Equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma f + N(x, v, f) & = & K(f) & \text{in } S\Omega_T, \\ f & = & f_0 & \text{on } \{0\} \times S\Omega, \\ f & = & f_- & \text{on } \partial_- S\Omega_T. \end{cases}$$

Here

$$S\Omega = \Omega \times \mathbb{S}^{n-1}, \quad S\Omega_T = (0, T) \times \Omega \times \mathbb{S}^{n-1}, \quad \partial_- S\Omega_T := (0, T) \times \Gamma_-.$$

The scattering operator $K(f)(t, x, v) = \int_{\mathbb{S}^{n-1}} \mu(x, v, v') f(t, x, v')\, dv'$. Suppose that the absorption and scattering coefficients σ, μ satisfy

$$0 \leq \sigma(x, v) \leq \sigma^0, \quad \text{and} \quad 0 \leq \mu(x, v, v') \leq \mu^0,$$

and

$$\int_{\mathbb{S}^{n-1}} \mu(x, v, v')\, dv' \leq \sigma(x, v), \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \mu(x, v', v)\, dv' \leq \sigma(x, v).$$

Nonlinearity

Case 1:

$$N(x,v,z) = \sum_{k=2}^{\infty} q^{(k)}(x,v)\frac{z^k}{k!},$$

which converges in the $L^\infty(S\Omega)$ topology with $q^{(k)}(x,v) \in L^\infty(S\Omega)$.

Case 2:

$$N(x,v,f) = q(x,v)N_0(f),$$

where N_0 satisfies

$$\|N_0(f)\|_{L^\infty(S\Omega_T)} \leq C_1 \|f\|^{\ell}_{L^\infty(S\Omega_T)},$$

and

$$\|\partial_z N_0(f)\|_{L^\infty(S\Omega_T)} \leq C_2 \|f\|^{\ell-1}_{L^\infty(S\Omega_T)}, \quad \ell \geq 2.$$

When $\ell = 2$, e.g. $N_0(f) = f\int f dv'$, *photoacoustic tomography with nonlinear absorption effect* [Ren-Zhang '18, Lai-Ren-Zhou '21, Stefanov-Zhong '22].

Well-posedness and inverse problems

Theorem (L.-Uhlmann-Zhou 2023)

Suppose that σ and μ satisfy the above conditions. Then there exists a small parameter $0 < \delta < 1$ such that for any $(f_0, f_-) \in L^\infty(S\Omega) \times L^\infty(\partial_- S\Omega_T)$ with $\|f_0\|_{L^\infty(S\Omega)} \leq \delta$, $\|f_-\|_{L^\infty(\partial_- S\Omega_T)} \leq \delta$, the IBVP problem has a unique small solution $f \in L^\infty(S\Omega_T)$ satisfying

$$\|f\|_{L^\infty(S\Omega_T)} \leq C(\|f_0\|_{L^\infty(S\Omega)} + \|f_-\|_{L^\infty(\partial_- S\Omega_T)}),$$

where the positive constant C is independent of f, f_0 and f_-.

We denote the measurement operator by

$$\mathcal{A}_{\sigma,\mu,N} : (f_0, f_-) \mapsto f|_{\partial_+ S\Omega_T} \in L^\infty(\partial_+ S\Omega_T).$$

Inverse Problem: Can one determine σ, μ, N from $\mathcal{A}_{\sigma,\mu,N}$?

Previous results for transport-type equations

Linear transport equation:

Uniqueness [Chen, Choulli, Kawagoe, Stefanov, Uhlmann ...]

Stability [Bal, Jollivet, Machida, Monard, Yamamoto, Wang, Zhao, Zhong ...]

Carleman estimate [Bugheim, Golgeleyen, Lai, Li, Klibanov, Machida, Pamyatnykh, Yamamoto ...]

Riemannian case [Assylbekov, McDowall, Stefanov, Tamasan, Yang ...]

Nonlinear transport equation:

Two photon absorption RTE [Ren-Zhong '21, Stefanov-Zhong '22, L.-Ren-Zhou '22]

Boltzmann equation [L.-Uhlmann-Yang '21, Li-Ouyang '22, Balehowsky-Kujanpaa-Lassas-Liimatainen '22, L.-Yan '23]

Inverse problems for semilinear equations

Theorem (L.-Uhlmann-Zhou 2023)

Let Ω be an open bounded and convex domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary. Suppose that σ_j, μ, N_j satisfy additional conditions regarding a fixed direction γ. If

$$\mathcal{A}_{\sigma_1,\mu,N_1}(h,0) = \mathcal{A}_{\sigma_2,\mu,N_2}(h,0)$$

for any $h \in L^\infty(S\Omega)$ with $\|h\|_{L^\infty(S\Omega)} \leq \delta$ for sufficiently small δ, then

$$\sigma_1(x,v) = \sigma_2(x,v) \quad \text{in } S\Omega, \quad N_1(x,v,z) = N_2(x,v,z) \quad \text{in } S\Omega \times \mathbb{R}.$$

- We will apply the **higher-order linearization** and **Carleman estimate**.
- Similarly, if σ is known, then μ can be recovered. Under additional assumptions, it's possible to recover both σ and μ, for instance, [Choulli-Stefanov '96, '98].

Linearization

- A classical method to study the inverse boundary value problem for **nonlinear** PDEs was introduced by [Isakov]. It consists of performing a **first order linearization** of the given nonlinear Dirichlet-to-Neumann map. This reduces the inverse problem to an inverse boundary problem for a **linear** equation, and thus one can employ the available results in this case.

- Recently, the **higher-order linearization** was introduced by [Kurylev-Lassas-Uhlmann] for nonlinear hyperbolic equations and [Lassas-Liimantainen-Lin-Salo], [Feizmohammadi-Oksanen] for nonlinear elliptic equations.

Higher-order linearization

This technique *employs nonlinearity as a tool* in solving inverse problems for nonlinear equations.
The idea is to bring several small parameters into the data and then differentiating the nonlinear equation with respect to these parameters to get a simpler linearized equation.

Some related works:
[Kurylev-Lassas-Uhlmann], [Lassas-Uhlmann-Wang], [Chen-Lassas-Oksanen-Paternain], [Lin-Liu-Liu-Zhang], [Lassas-Liimatainen-Lin-Salo], [Feizmohammadi- Oksanen], [Krupchyk-Uhlmann], [Kian-Krupchyk-Uhlmann], [L.-Zhou], [Assylbekov-Zhou], [Kang-Nakamura], [Cârstea-Nakamura-Vashisth], [Cârstea-Feizmohammadi-Kian-Krupchyk-Uhlmann], [L.-Uhlmann-Yang], [L.-Ren-Zhou], [Stefanov-Zhong]

Higher-order linearization

Idea: Differentiating the equation with respect to ε at $\varepsilon = 0$ leads to a simpler linear equation.

Let $f = f(t, x, v; \varepsilon h, 0)$ satisfy

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma f + N(x, v, f) &= K(f) \quad \text{in } S\Omega_T, \\ f &= \varepsilon h \quad \text{on } \{0\} \times S\Omega, \\ f &= 0 \quad \text{on } \partial_- S\Omega_T. \end{cases}$$

Denote $F^{(n)} = \partial_\varepsilon^n|_{\varepsilon=0} f(t, x, v; \varepsilon h, 0)$.

1^{st} **order linearization:** recover σ or/and μ

$$\begin{cases} \partial_t F^{(1)} + v \cdot \nabla_x F^{(1)} + \sigma F^{(1)} &= K(F^{(1)}) \quad \text{in } S\Omega_T, \\ F^{(1)} &= h \quad \text{on } \{0\} \times S\Omega, \\ F^{(1)} &= 0 \quad \text{on } \partial_- S\Omega_T. \end{cases}$$

Higher-order linearization

Higher order linearization: recover nonlinear term N with known σ and μ.

$$\begin{cases} \partial_t F^{(2)} + v\cdot\nabla_x F^{(2)} + \sigma F^{(2)} - K(F^{(2)}) = -\partial_\varepsilon^2|_{\varepsilon=0} N(f) & \text{in } S\Omega_T, \\ F^{(2)} = 0 & \text{on } \{0\}\times S\Omega, \\ F^{(2)} = 0 & \text{on } \partial_- S\Omega_T. \end{cases}$$

Key: View the nonlinear term as a source

Carleman estimates

Theorem (Machida-Yamamoto '14, L.-Uhlmann-Zhou '23)

Let $f \in H^1(0,T;L^2(S\Omega))$ *satisfy* $v \cdot \nabla_x f \in L^2(S\Omega_T)$ *and* $f(T,x,v) = 0$. *Denote* $V := \{v \in \mathbb{S}^{n-1} : \gamma \cdot v \geq \gamma_0 > 0\}$ *and* $B(v) = \gamma \cdot v - \beta$, $0 < \beta < \gamma_0$. *Then there exist positive constants* C *and* s_0 *so that for all* $s \geq s_0 > 0$, *we have*

$$s\|f(0,x,v)e^{s\varphi(0,x)}\|^2_{L^2(\Omega\times V)} + s^2\|Bfe^{s\varphi}\|^2_{L^2(S\Omega_T)} \leq C\|(\partial_t f + v \cdot \nabla_x f + \sigma f - Kf)e^{s\varphi}\|^2_{L^2(S\Omega_T)} + s\|fe^{s\varphi}\|^2_{L^2(\partial_+ S\Omega_T)}.$$

The Carleman weight takes the form $\varphi(t,x) = -\beta t + \gamma \cdot x$ for some fixed vector γ.

Sketch of proof

For simplicity, let's take

$$N(x,v,f) = q(x,v)f^2.$$

Assume that both σ and μ are known now. The 2^{nd} order linearization gives

$$\partial_t F^{(2)} + v\cdot\nabla_x F^{(2)} + \sigma F^{(2)} - K(F^{(2)}) = -2q(x,v)(F^{(1)})^2(t,x,v).$$

Applying Carleman estimate yields

$$\boxed{s\int_{\Omega\times V} |\underbrace{\partial_t F^{(2)}(0,x,v)}_{q(x,v)(F^{(1)})^2}|^2 e^{2s\varphi(0,x)}dxdv \leq Data.}$$

Hence the stability estimate of q in L^2 norm is derived, which implies the unique determination of q.

Thank you for your attention.