NSF-CBMS Research Conference
Algorithmic Fractal Dimensions

Lecture 4
Dimensions in computational complexity

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Decision Problems

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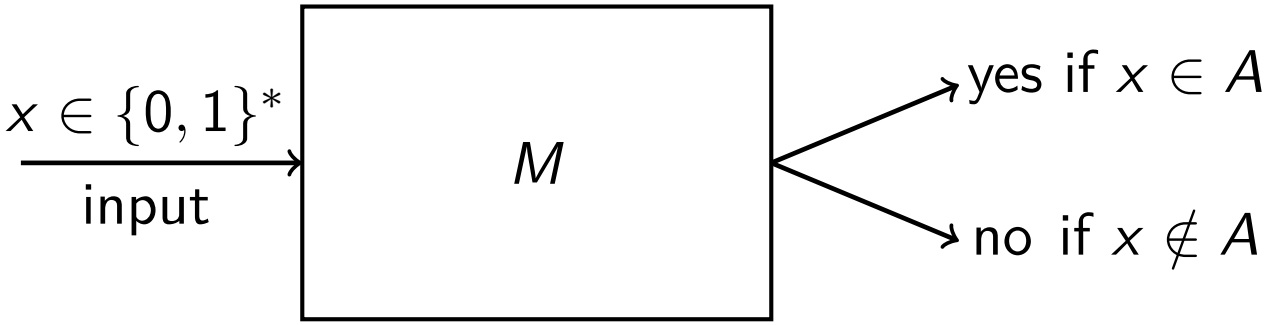

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does there exist a device M that, on every input $x \in \Sigma^*$, decides whether $x \in A$ using at most $t(|x|)$ of the resource?

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Convention

If $t(n) = n^2$, then complexity theorists write n^2 for the function t.

Thus, for example, TIME(n^2) is the class of all decision problems that can be solved in quadratic time.

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$$\mathsf{EXP} = \bigcup_{k=1}^{\infty} \mathsf{TIME}(2^{n^k}) \qquad \text{exponential time}$$

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Some things that we know:

$$\mathrm{P} \subsetneq \mathrm{E} \subsetneq \mathrm{EXP}$$

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We should not omit the famous class NP.

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If $B \subseteq \{0,1\}^*$ is a decision problem and $q : \mathbb{N} \to \mathbb{N}$, then we define the decision problem

$$\exists^q B = \left\{ x \in \{0,1\}^* | (\exists w \in \{0,1\}^{\leq q(|x|)}) \langle x, w \rangle \in B \right\}$$

Terminology: w is a witness that testifies that $x \in \exists^q B$.

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We know that $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE}$. We believe that these inclusions are proper.

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Because hundreds of very important problems in scientific computing can be formulated as decision problems $C \subseteq \{0,1\}^*$ that are NP-complete, meaning that

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We can prove this if we can find any intractable problem $D \in \mathrm{NP}$.

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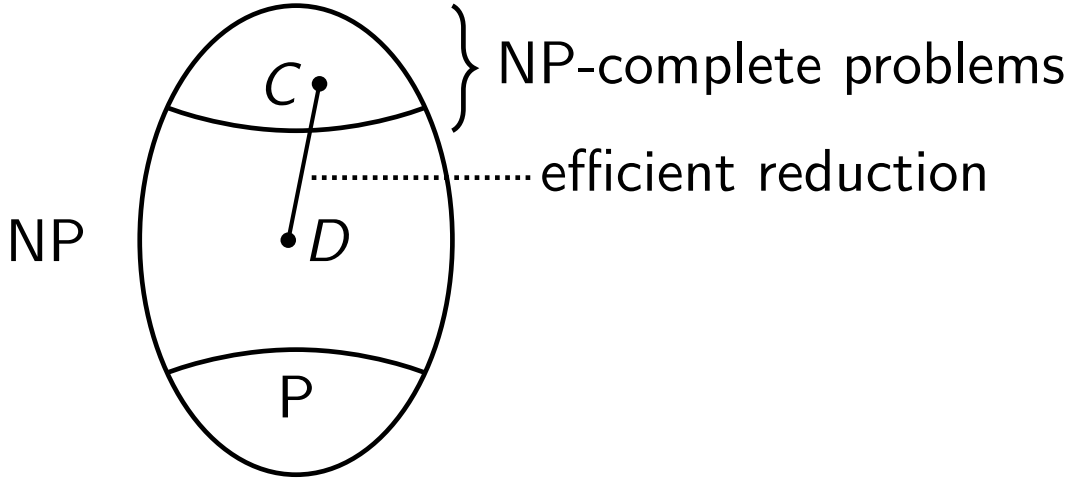

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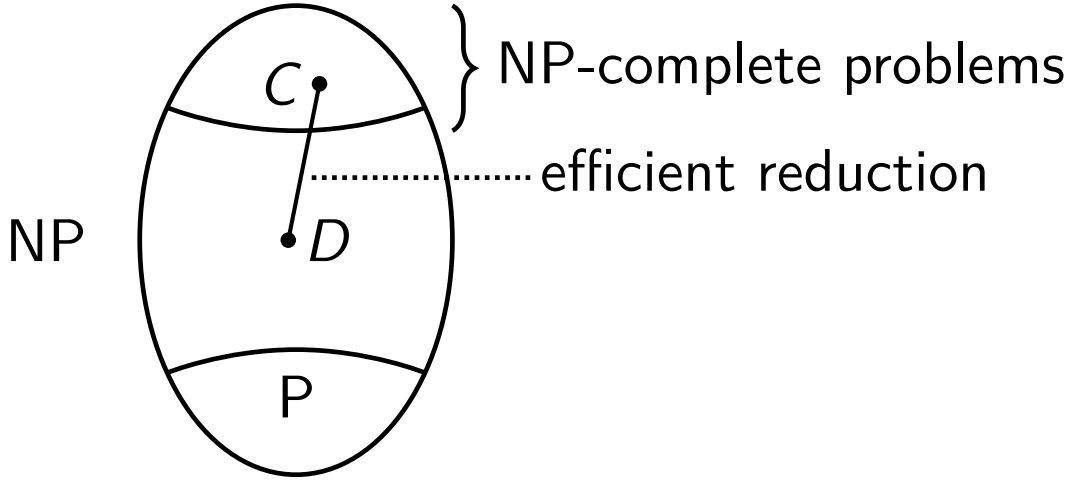

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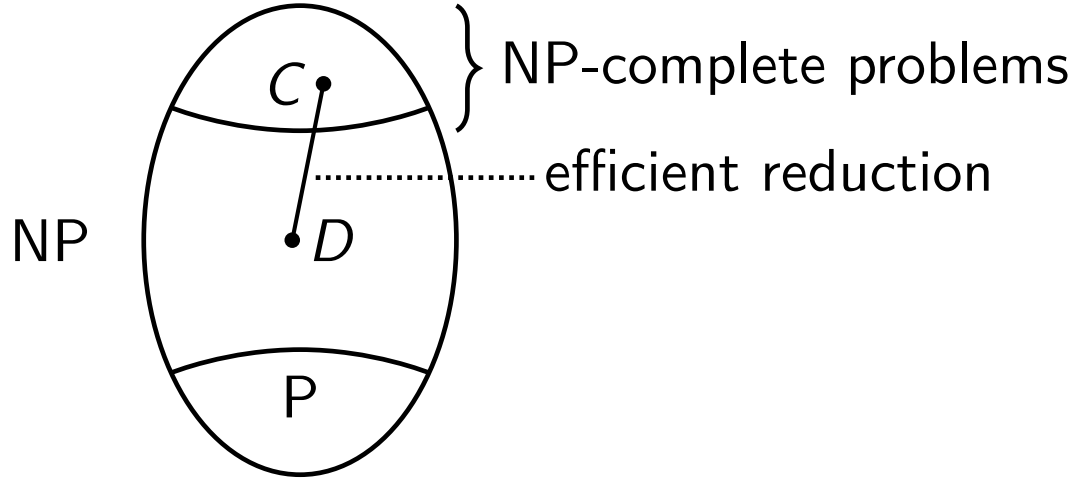

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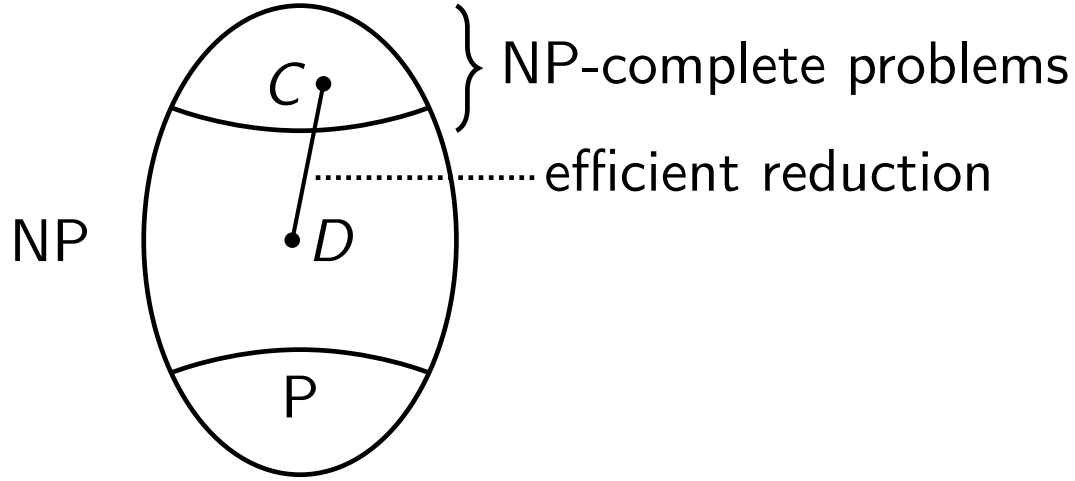

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$\therefore$ All the above complexity classes are countable subsets of Cantor space!

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The result class of a set Δ of functions is

$$R(\Delta) = \{R(\delta) | \delta \in \Delta \text{ is a constructor}\}.$$

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Note: If $x \in \{0,1\}^*$ *is the first string not decided by a prefix* w *of* χ_A*, then*

$$poly(|w|) = |w|^{O(1)} = \left(2^{|x|}\right)^{O(1)} = 2^{O(|x|)}.$$

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Constructors and their Results

Technical Note. Many of our functions will be of the form $f : D \to [0, \infty)$, where D is some discrete domain like $\{0,1\}^*$ or $\{0,1\}^* \times \mathbb{N}$. Such a function is Δ-computable if there is a function $\hat{f} : D \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ such that

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and

- $\hat{f} \in \Delta$, with r coded in unary and $\hat{f}(x, r)$ coded in binary.

Dimensions in Complexity Classes

Notation If Δ is a resource bound and $X \subseteq \{0,1\}^\omega$, then

$$\mathcal{G}_\Delta(X) = \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a } \Delta\text{-computable} \\ s\text{-gale } d \text{ such that } X \subseteq S^\infty[d] \end{array} \right\}$$

and

$$\mathcal{G}^{str}_\Delta(X) = \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a } \Delta\text{-computable} \\ s\text{-gale } d \text{ such that } X \subseteq S^\infty_{str}[d] \end{array} \right\}$$

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We saw yesterday that

$$\dim_H(X) = \inf \mathcal{G}_{\mathsf{all}}(X)$$

and

$$\dim_P(X) = \inf \mathcal{G}^{str}_{\mathsf{all}}(X).$$

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Definition (J. Lutz 2000, 2003).

Let Δ be a resource bound, and let $X \subseteq \{0,1\}^\omega$.

1. The Δ-dimension of X is

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2. Stability

$$\dim_{\Delta}(X \cup Y) = \max\{\dim_{\Delta}(X), \dim_{\Delta}(Y)\}.$$

In fact, for "Δ-countable unions",

$$\dim_{\Delta}(\bigcup_{k=0}^{\infty} X_k) = \sup\{\dim_{\Delta}(X_k)|\ k \in \mathbb{N}\}.$$

Dimensions in Complexity Classes

Observations (continued)

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$$\begin{array}{ccccccc} \dim_{\mathsf{p}}(X) & \geq & \dim_{\mathsf{qp}}(X) & \geq & \dim_{\mathsf{comp}}(X) & \geq & \dim_H(X) \\ \vee\!| & & \vee\!| & & \vee\!| & & \\ \dim(X|\mathsf{E}) & & \dim(X|\mathsf{EXP}) & & \dim(X|\mathsf{DEC}) & & \end{array}$$

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Definition (Selman 1979, adapting Jockusch 1968).

A decision problem $A \subseteq \{0,1\}^*$ is p-selective, and we write $A \in$p-SEL, if there is a polynomial-time algorithm that, given an ordered pair (x, y) of strings $x, y \in \{0,1\}^*$, outputs a string $z \in \{x, y\}$ such that

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The qp-selective sets are defined analogously.

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Hence,
$\dim(\mathrm{NP}|\mathrm{EXP}) > 0 \Rightarrow$ no qp-selective set can be $\leq_m^{\mathrm{p}}$-hard for NP.

Thank you!

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