NSF-CBMS Research Conference
Algorithmic Fractal Dimensions

Lecture 5
Finite-state dimensions

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Iowa State University

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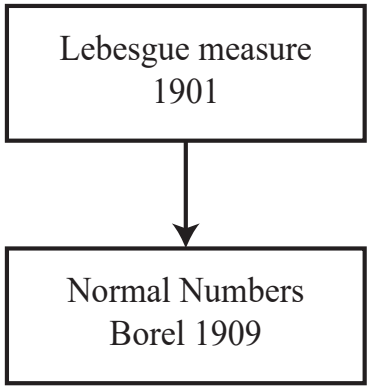
Drake University

Very Big Picture

Very Big Picture

Lebesgue measure
1901

Very Big Picture



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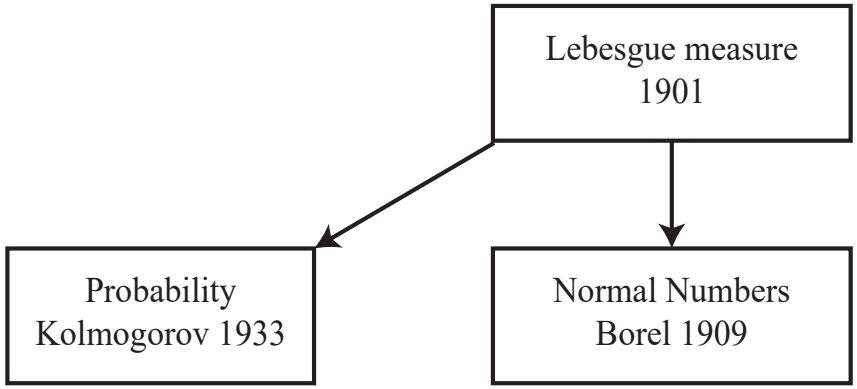

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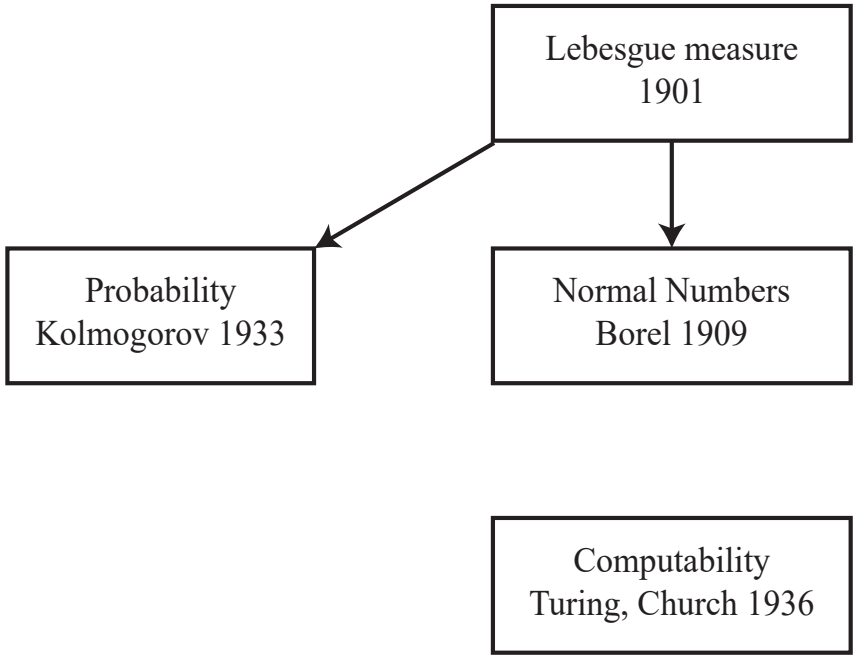

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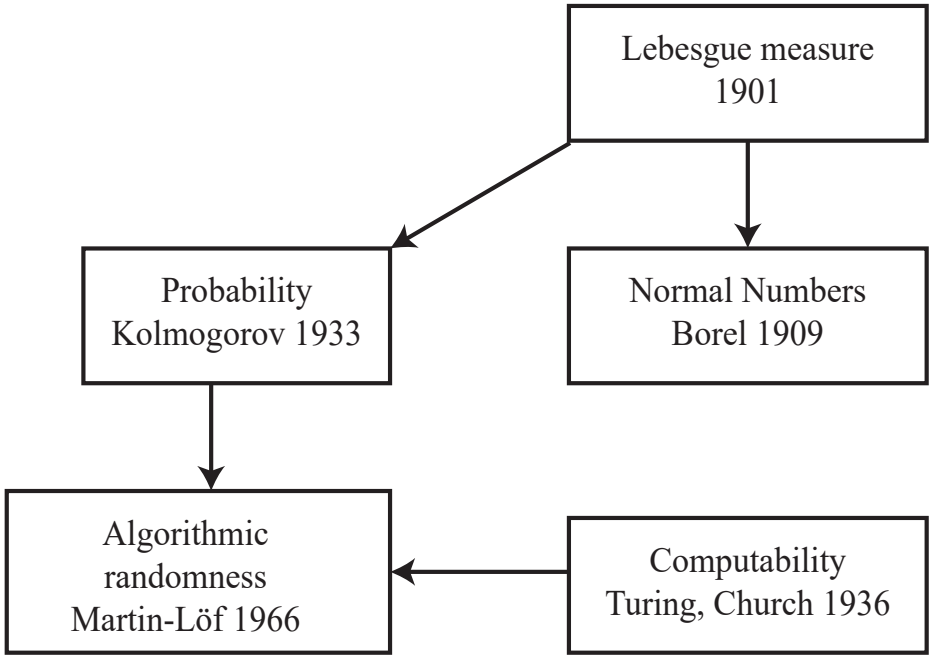

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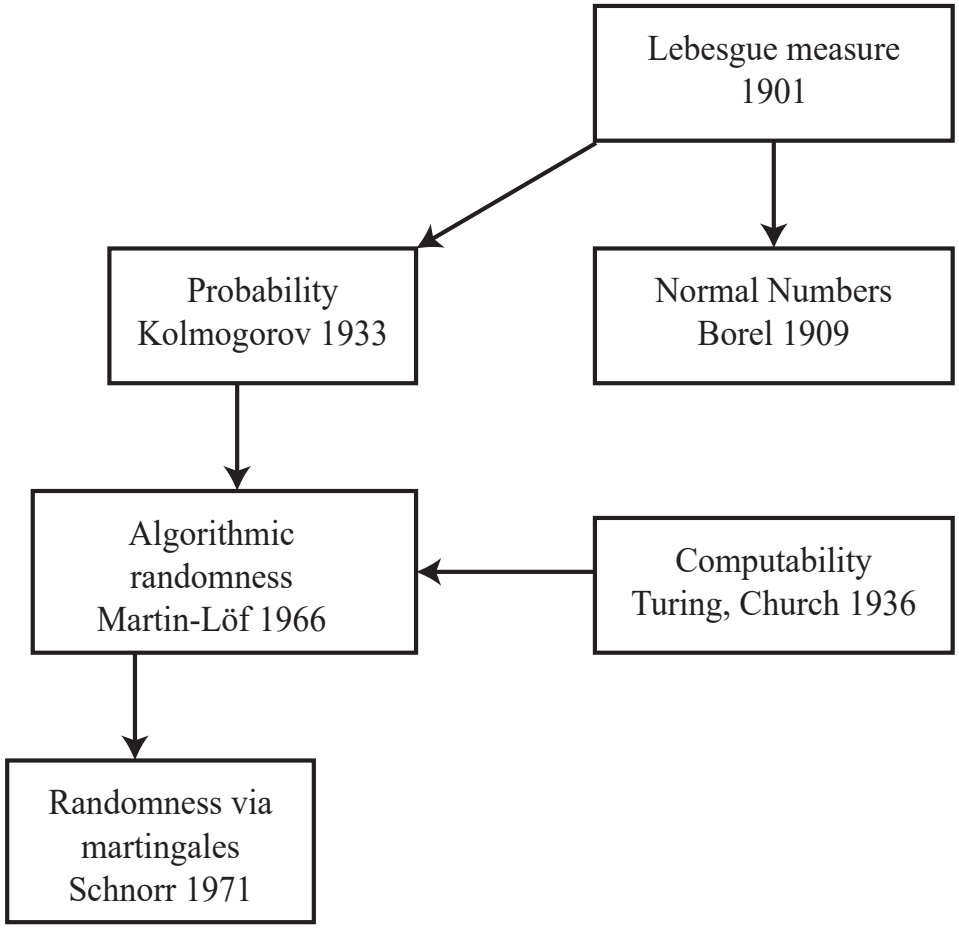

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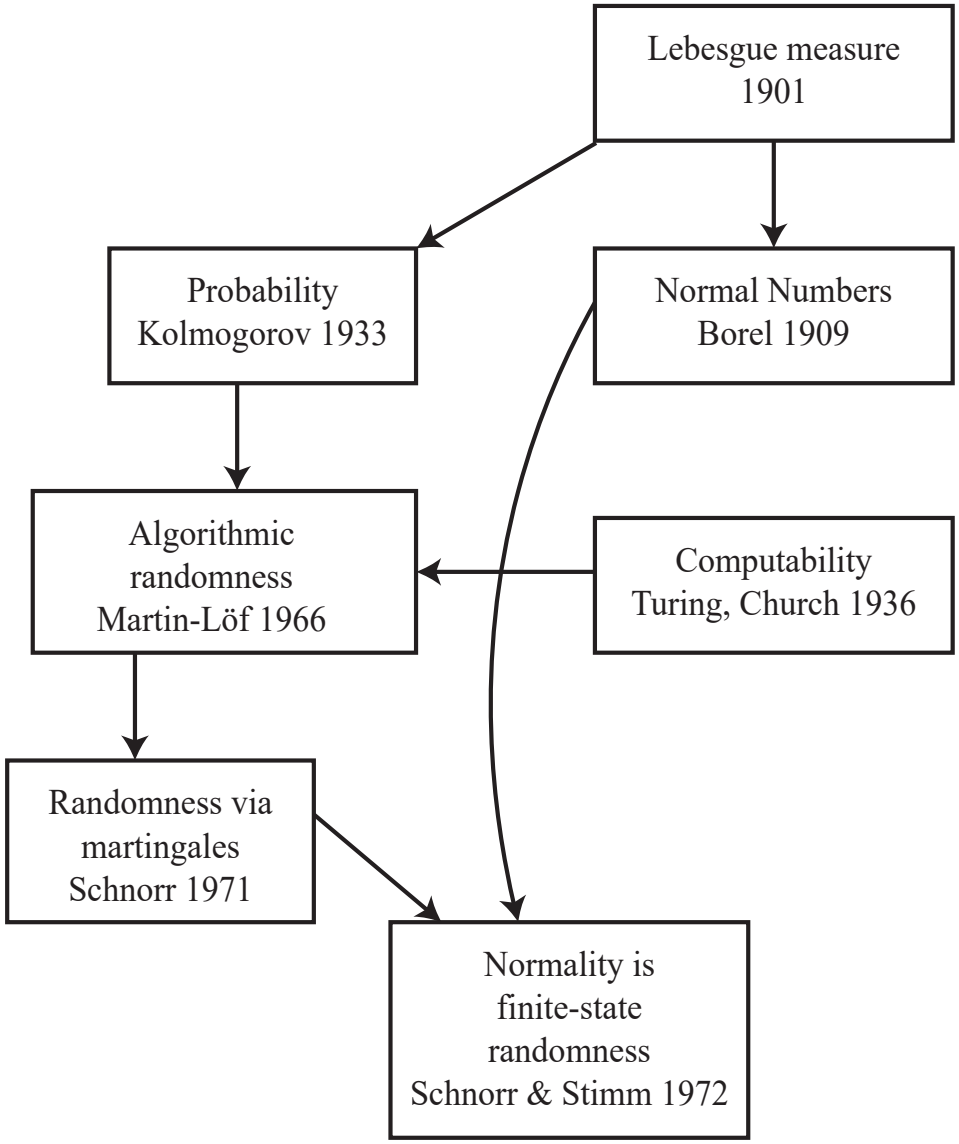

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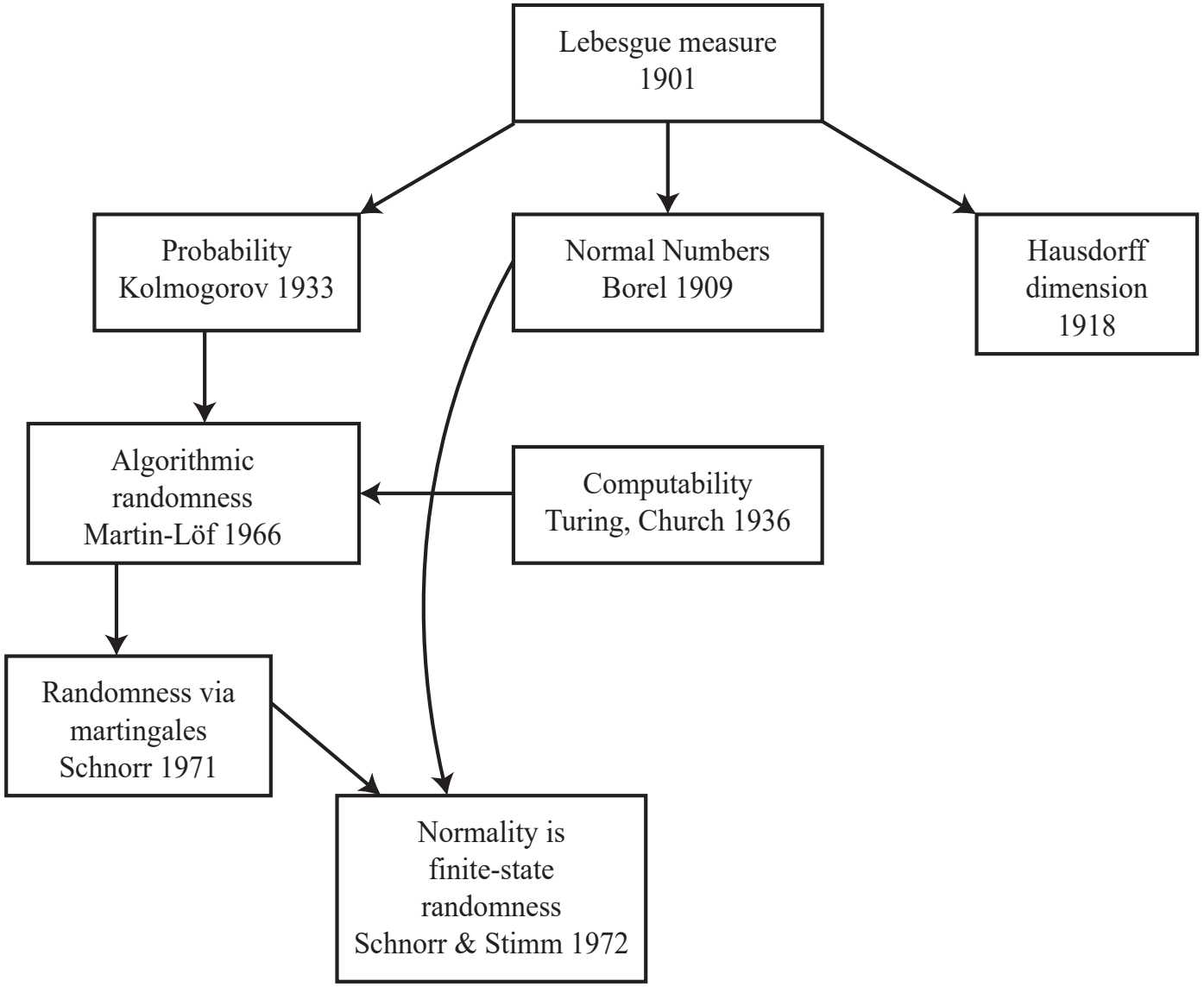

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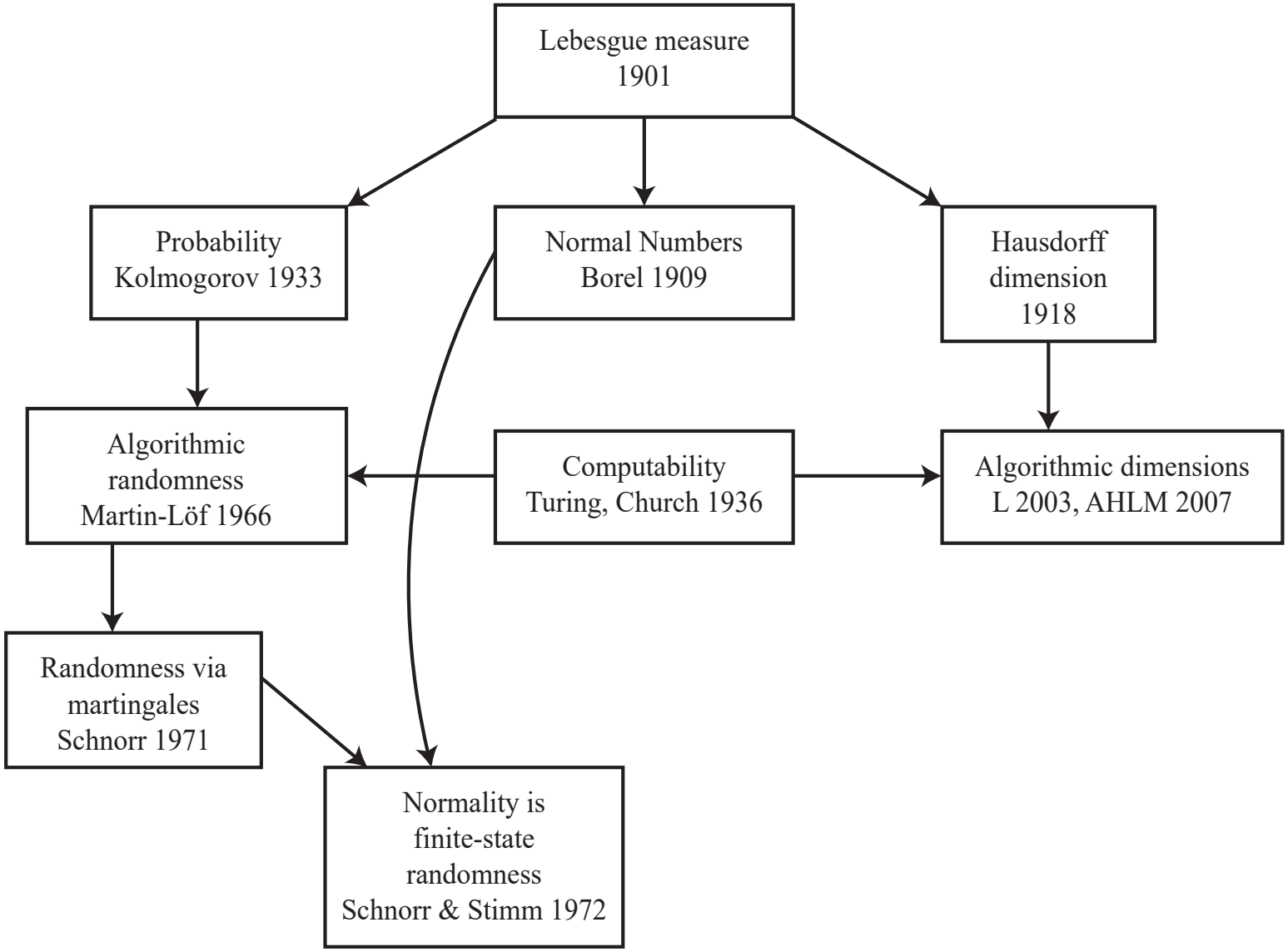

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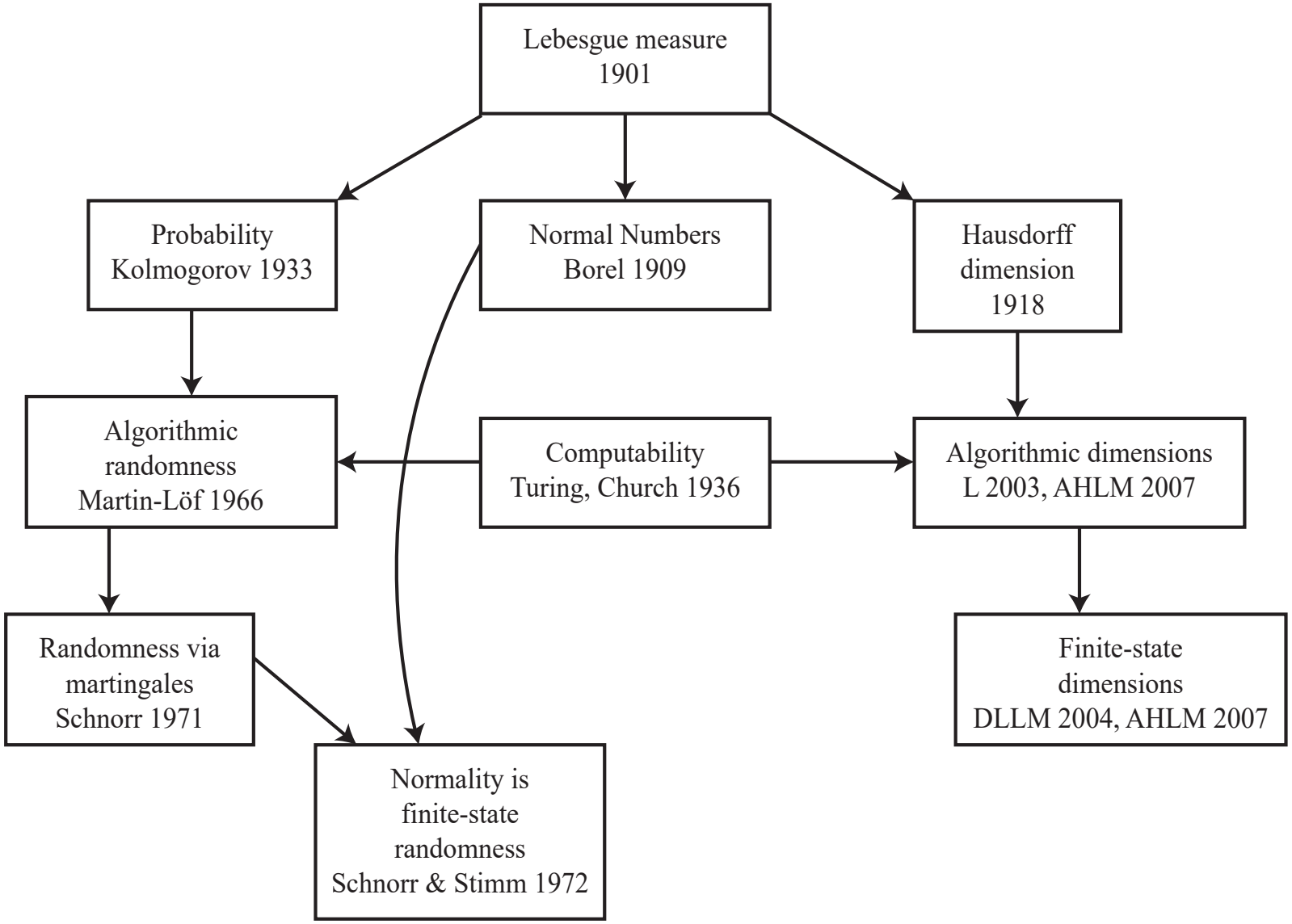

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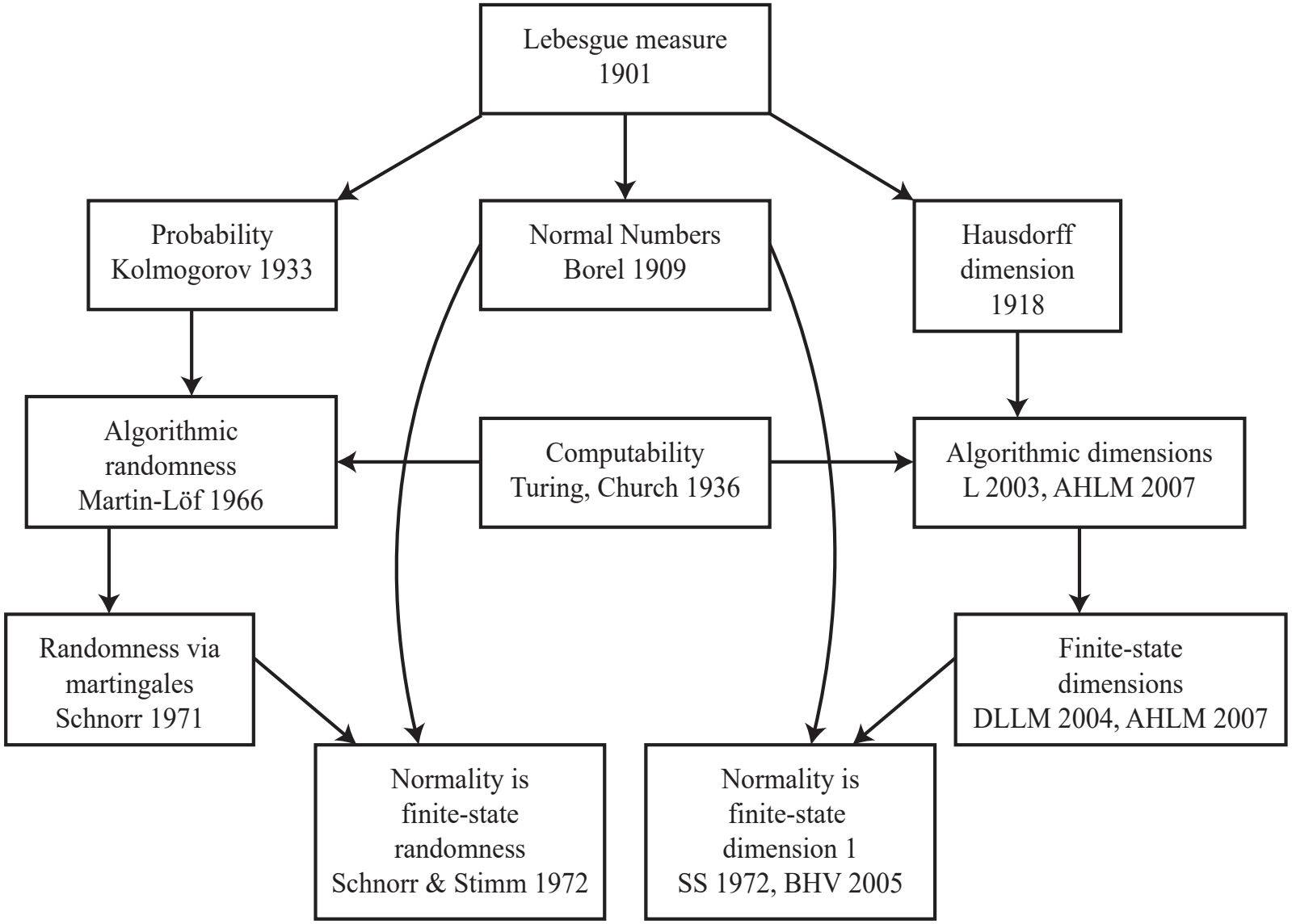

Normality

$\Sigma = \{0, 1, \cdots, b-1\} \quad (2 \leq b \in \mathbb{N})$
For $S \in \Sigma^\infty$, $w \in \Sigma^+$, and $n \in \mathbb{Z}^+$,

$$\begin{aligned}\mathrm{freq}_n(w, S) &= \frac{\left|\{i < n \mid S[i \,..\, i + |w| - 1] = w\}\right|}{n} \\ &= n\text{-th frequency of } w \text{ in S}\end{aligned}$$

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A sequence $S \in \Sigma^\infty$ is normal if

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A real number α is normal in base b if the base-b expansion of $\{\alpha\}$ is a normal sequence.

Martingales

Martin-Löf's 1966 definition of algorithmic randomness was compelling because it was based on measure theory. Specifically, he defined a sequence $S \in \{0,1\}^*$ to be random if $\{S\}$ does not have algorithmic (Σ_1^0) Lebesgue measure 0.

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Schnorr's characterization in terms of martingales is especially useful:

> Martingales have nice linearity properties.
>
> The martingale characterization works at all levels of effectivity: Σ_1^0, computable, p, pspace, finite-state, etc.

A martingale is a function $d : \{0,1\}^* \to [0,\infty)$ that satisfies

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

for all $w \in \{0,1\}^*$.

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$S_{str}[d]^\infty$ = the strong success set of $d = \{S \mid S \text{ succeeds strongly on} S\}$.

Theorem (Ville 1939)

A set $E \subseteq \{0,1\}^\omega$ has Lebesgue measure 0 if and only if there exists a martingale d such that $E \subseteq S^\infty[d]$.

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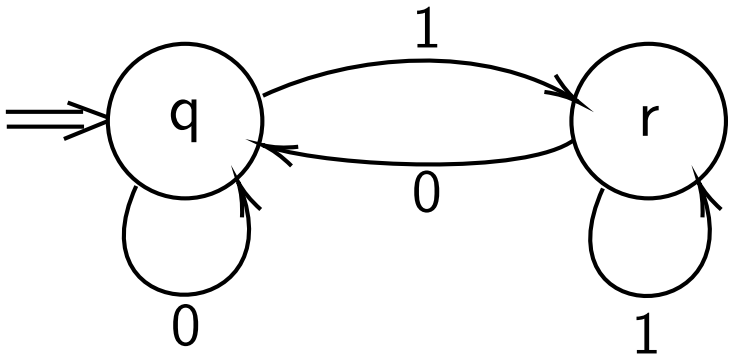
Note: d is lower semicomputable if there is a computable function $\widehat{d} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ satisfying the following two conditions for all $w \in \{0,1\}^*$.

1. For all $t \in \mathbb{N}$, $\widehat{d}(x,t) \leq \widehat{d}(x,t+1) \leq d(x)$.
2. $\lim_{t\to\infty} \widehat{d}(x,t) = d(x)$.

But today we want finite-state martingales!

Example



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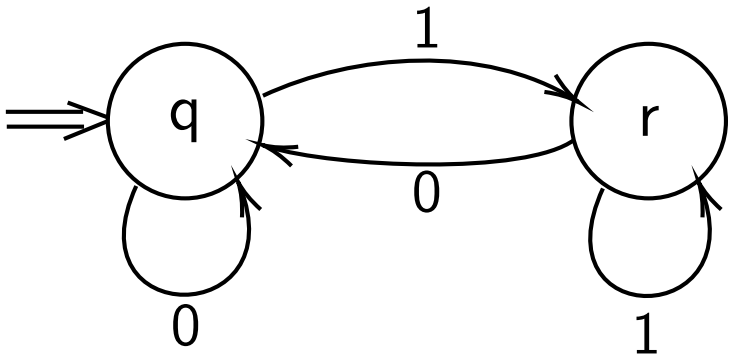

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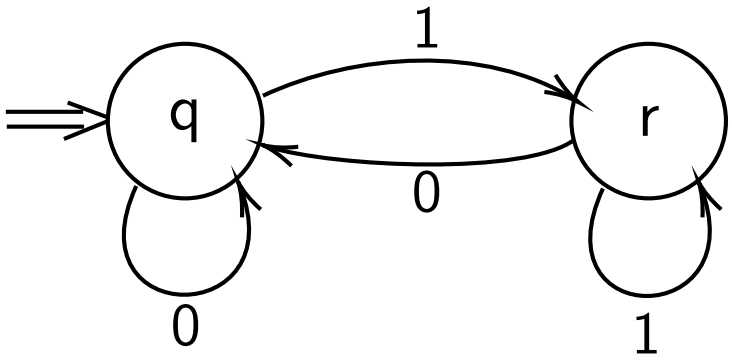

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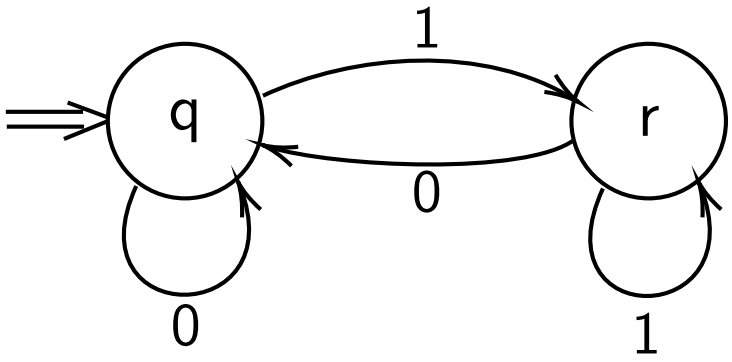

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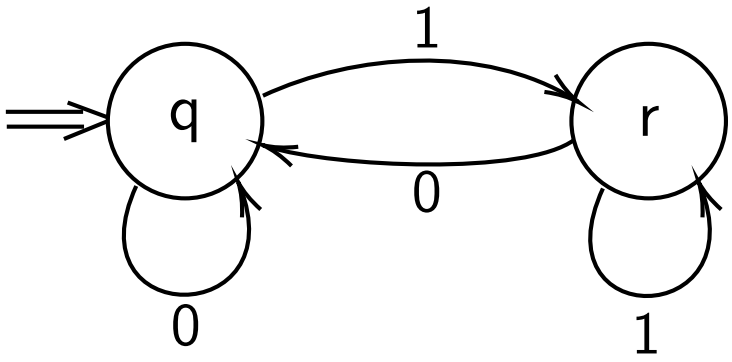

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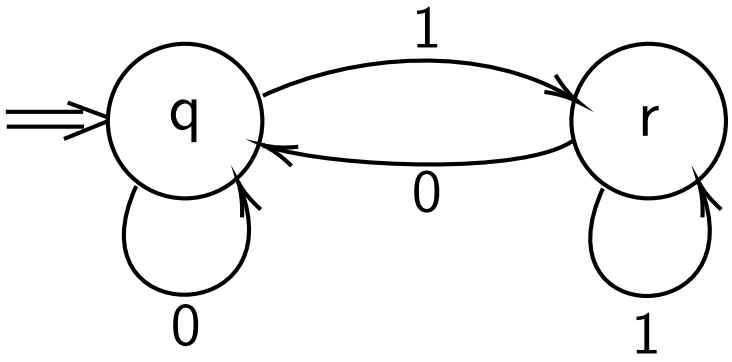

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A sequence $S \in \Sigma^\omega$ is normal in the sense of Borel (1909) if and only if there is no finite-state gambler that succeeds on S.

$\therefore$ Normality is finite-state randomness!

They proved even more!

The Schnorr-Stimm dichotomy:

1. If $S \in \Sigma^\omega$ is not normal, then there exist a finite-state gambler G and a real $\alpha > 1$ such that, for infinitely many prefixes $w \sqsubseteq S$,

$$d_G(w) > \alpha^{|w|}.$$

2. If $S \in \Sigma^\omega$, then, for every finite-state gambler G there is a real $\alpha < 1$ such that, for every prefix $w \sqsubseteq S$,

$$d_G(w) < \alpha^{\#bets(w)},$$

where $\#bets(w)$ is the number of times G actually bets on w.

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1. If S is not normal then there is a finite-state gambler that makes money at an infinitely-often exponential rate on S.

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2. If S is normal, then every finite-state gambler loses money at an exponential rate when it actually bets on S.

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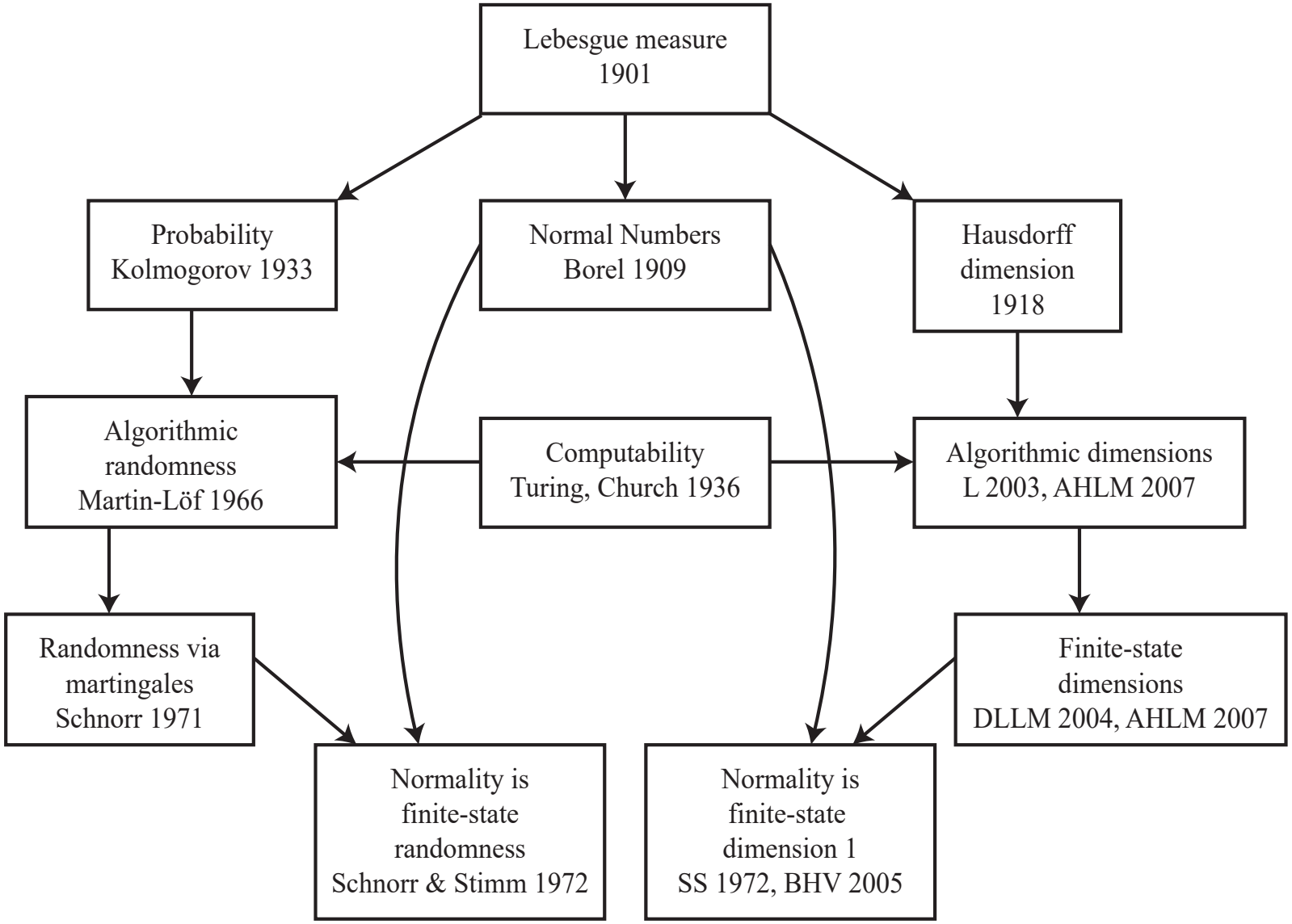

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A (discrete) probability measure on a nonempty finite set Ω is a function $\pi : \Omega \to [0, 1]$ satisfying

$$\sum_{w \in \Omega} \pi(w) = 1.$$

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Today Ω will be Σ^l for some $l \in \mathbb{Z}^+$.

We use the economists' notation $\Delta(\Omega)$ for the set of all probability measures on Ω.

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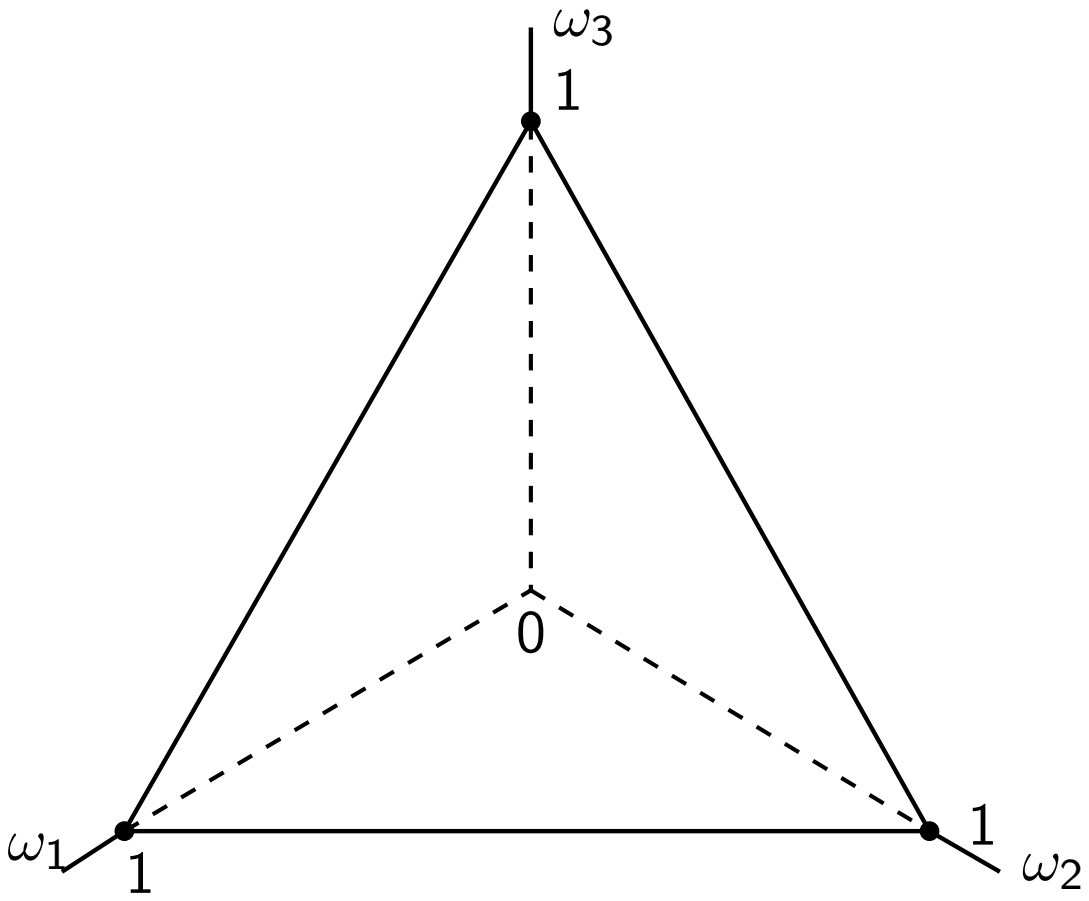
This is a great notation, because $\Delta(\Omega)$ is the set of all points on the $(\mid \Omega \mid -1)$-dimensional unit simplex in $\mid \Omega \mid$-dimensional Euclidean space.



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The number of block occurrences of w in x is
$\#_\square(w, x) = |\{m \leq \frac{|x|}{|w|} - 1 \mid x[m|w|...(m+1)|w| - 1] = w\}|$.
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For $S \in \Sigma^\omega$, $n \in \mathbb{Z}^+$, and $w \in \Sigma^+$, the nth block frequency of w in S is

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$\pi_{s,n}^{(l)} \stackrel{def}{=} \pi_{s,n} \restriction \Sigma^l \in \Delta(\Sigma^l)$ is the nth empirical probability measure on Σ^l given by S.

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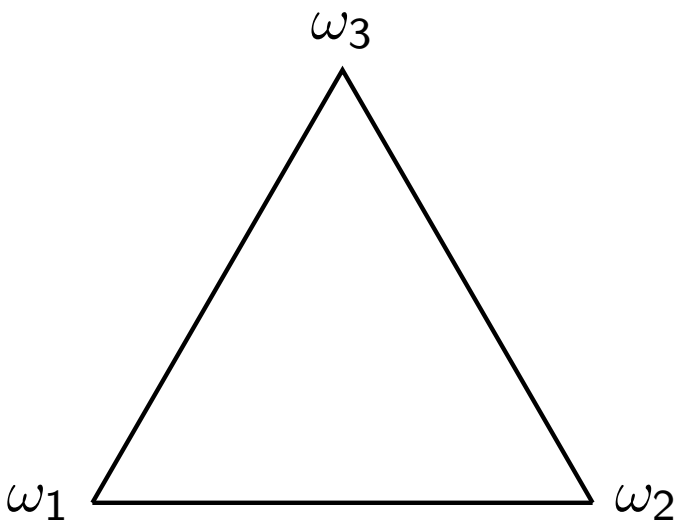
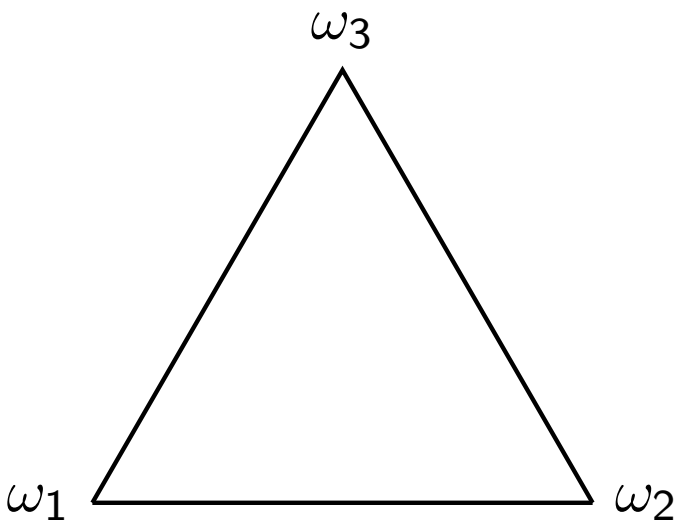
Theorem (Niven and Zuckerman 1951)

This is equivalent to the "sliding window" definition that we used earlier.



Let $\alpha, \beta \in \Delta(\Omega)$. The Kullback-Leibler divergence (KL-divergence) of β from α is

$$D(\alpha||\beta) = \mathbb{E}_\alpha \log \frac{\alpha}{\beta}$$



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$$= \sum_{w \in \Omega} \alpha(w) \log \frac{\alpha(w)}{\beta(w)},$$

where $\log = \log_2$.

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3. Entropy is divergence from certainty:

$$H(\alpha) = \mathbb{E}_\alpha D(\pi_w||\alpha).$$

4. Mutual information is divergence from independence:

$$I(\alpha;\beta) = D(\alpha\beta||\gamma),$$

where γ is the joint probability measure of α and β.

Properties of KL-Divergence.

1. $D(\alpha||\beta) \geq 0$, with equality if and only if $\alpha = \beta$.

2. But KL-divergence is <u>not</u> a metric!

3. Entropy is divergence from certainty:

$$H(\alpha) = \mathbb{E}_\alpha D(\pi_w||\alpha).$$

4. Mutual information is divergence from independence:

$$I(\alpha;\beta) = D(\alpha\beta||\gamma),$$

   where γ is the joint probability measure of α and β.

$\therefore$ KL-divergence is <u>fundamental</u> to Shannon information theory.

Asymptotic Divergences

Let $l \in \mathbb{Z}^+, S \in \Sigma^\omega, \alpha \in \Delta(\Sigma)$.

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$$Div(S||\alpha) = \sup_{l\in\mathbb{Z}^+} \frac{Div_l(S||\alpha)}{l}.$$

Theorem (Huang, J. Lutz, Mayordomo, and Stull 2021)

For all $\alpha \in \Delta(\Sigma)$ and $S \in \Sigma^\omega$, the following conditions are equivalent.

1. *S is α-normal.*
2. *$Div(S||\alpha) = 0$.*

A finite-state gambler is a 4-tuple

$$G = (Q, \delta, s, B),$$

where

Q is a finite set of states,
$\delta : Q \times \Sigma \to Q$ is the transition function
$s \in Q$ is the start state, and
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Given $\alpha \in \Delta(\Sigma)$, the α-payoffs give G the martingale (capital function)

$$d_{G,\alpha}(\lambda) = 1,$$

$$d_{G,\alpha}(wa) = d_{G,\alpha}(w)\frac{B(\delta(w))(a)}{\alpha(a)}.$$

If $\delta(w) = q$ is a state of G in which $B(q) = \alpha$, then $d_{G,\alpha}(wa) = d_{G,\alpha}(w)$ for all $a \in \Sigma$. In this case, we say that G does not bet in state q.

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$$risk_G(q) = D(\alpha || B(q)),$$

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i.e., the KL-divergence of $B(q)$ from not betting. The total risk that G takes along a string $w \in \Sigma^*$ is

$$Risk_G(w) = \sum_{u \sqsubsetneq w} risk_G(\delta(u))$$

Theorem (Huang, J. Lutz, Mayordomo, and Stull 2021)

1. *If S is not α-normal, then there is a finite-state gambler G such that, for infinitely many prefixes $w \sqsubseteq S$,*

$$d_{G,\alpha}(w) > 2^{0.99\ Div(S||\alpha)|w|}.$$

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2. *2. If S is α-normal, then, for every finite-state gambler G, for all but finitely many prefixes $w \sqsubseteq S$,*

$$d_{G,\alpha}(w) < 2^{-0.99\ Risk_G(w)}.$$

Let $G = (Q, \delta, s, B)$ be a finite-state gambler, let $\alpha \in \Delta(\Sigma)$, and let $s \in [0, \infty)$ be a "fairness parameter". The s-α-gale of G is the function

$$d_{G,\alpha}^{(s)} : \Sigma^* \to [0, \infty)$$

$$d_{G,\alpha}^{(s)}(w) = \frac{d_G(w)}{|\Sigma|^{|w|}\alpha(w)^s},$$

where d_G is the martingale of G.

For each $\alpha \in \Delta(\Sigma)$ and each $S \in \Sigma^\omega$, let

$$\mathcal{G}^{\alpha}_{FS}(S) = \{s \in [0, \infty) \mid (\exists \text{ FSG } G) d^{(s)}_{G,\alpha} \text{ succeeds on } S\},$$

$$\mathcal{G}^{\alpha,str}_{FS}(S) = \{s \in [0, \infty) \mid (\exists \text{ FSG } G) d^{(s)}_{G,\alpha} \text{ succeeds strongly on } S\}.$$

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Definition (Dai, Lathrop, J. Lutz, and Mayordomo 2004)

The finite-state α-dimension of S is

$$\dim^{\alpha}_{FS}(S) = \inf \mathcal{G}^{\alpha}_{FS}(S).$$

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Definition (Athreya, Hitchcock, J. Lutz, and Mayordomo 2007)

The strong finite-state α-dimension of S is

$$\mathrm{Dim}^{\alpha}_{FS}(S) = \inf \mathcal{G}^{\alpha,str}_{FS}(S).$$

Observations.

1. If G never bets in any state and $s > 1$, then $d^{(s)}_{G,\alpha}(w) = |\Sigma|^{(s-1)|w|}$ for all $w \in \Sigma^*$. This implies that, for all $S \in \Sigma^\infty$,

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3.

(Bourke, Hitchcock, and Vinodchandran 2005)

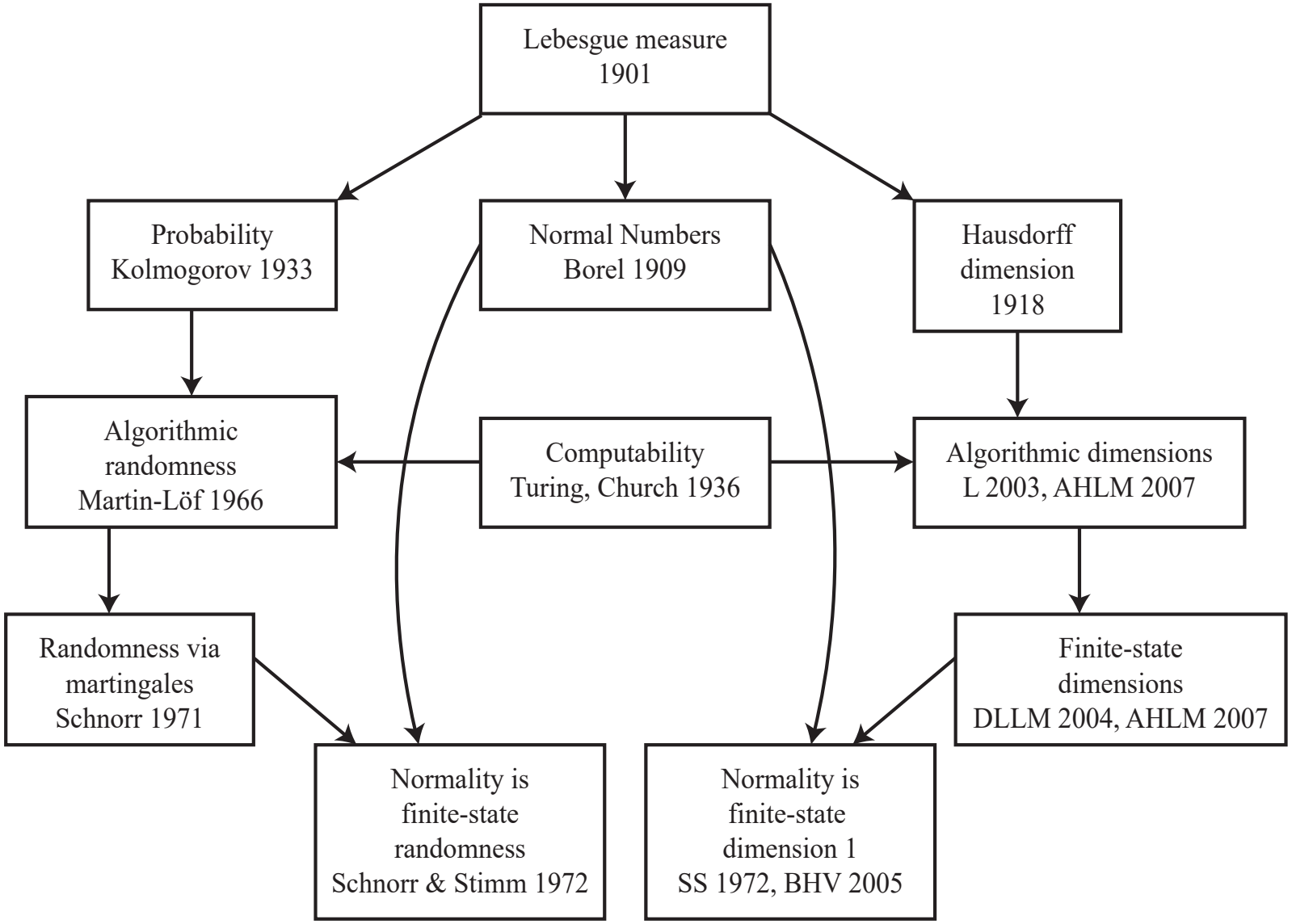
The Schnorr-Stimm dichotomy tells us that, if S is not α-normal, then $\dim^{\alpha}_{FS}(S) < 1$.

Hence a sequence $S \in \Sigma^\omega$ is α-normal if and only if $\dim^\alpha_{FS}(S) = 1$.

Hence a sequence $S \in \Sigma^\omega$ is α-normal if and only if $\dim_{FS}^{\alpha}(S) = 1$.

The finite-state world is the only world that we know of where dimension 1 implies randomness.

Very Big Picture



Thank you!

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